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Strict L^p Solutions for Nonautonomous Fractional Evolution Equations

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Nonautonomous problems are important especially as a transient case between the linear and the nonlinear theory. We study the nonautonomous linear problem for the fractional evolution equation

$$D_t^\alpha u(t) + A(t)u(t) = f(t), \quad \text{a.a. } t \in (0,T),$$

where D_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, $\{A(t)\}_{t\in[0,T]}$ is a family of linear closed operators densely defined on a Banach space X and the forcing function $f(t) \in L^p(0,T;X)$. Strict L^p solvability of this problem is proved for a suitable class of operators $A(t)$. The proof is based on L^p regularity estimates for the corresponding autonomous problem.

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1. Introduction

The notion of maximal L^p regularity plays an important role in the functional analytic approach to parabolic partial differential equations. Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

$$u'(t) + Au(t) = f(t),\ t \in I, \quad u(0) = 0, \tag{1}$$

where $I = (0,T)$, $T > 0$, $-A$ generates a bounded analytic semigroup on a Banach space X and f and u are X-valued functions defined on I. It is well known that (1) has a strong solution for all locally Bochner integrable f, but in many applications we need that u' has the same "smoothness" as f. This property is called maximal regularity. In particular, one says that problem (1) has maximal L^p regularity on I if for every $f \in L^p(I;X)$ there exists one and only one $u \in L^p(I;D(A)) \cap W^{1,p}(I;X)$ satisfying (1). Here

$$W^{1,p}(I;X) = \{f|\ \exists\varphi \in L^p(I;X):\ f(t) = \int_0^t \varphi(\tau)d\tau,\ t \in I\}$$

are the Sobolev spaces. From the closed graph theorem it follows easily that if there is L^p regularity then there exists $C > 0$ such that

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} \le C\|f\|_{L^p}. \tag{2}$$

The theory of strongly continuous semigroups could suggest that it is more natural to study the continuous regularity of (1), i.e. the existence and uniqueness of a solution $u \in C(I;D(A)) \cap C^1(I;X)$ for any continuous f. But Baillon [1] proved that if there is continuous regularity for an unbounded operator A that generates a C_0 semigroup, then the space X must contain a subspace isomorphic to c_0, the space of sequences converging to 0. This fact implies that X cannot be reflexive. On the other hand there are good results of L^p regularity in some reflexive spaces. Because of this many authors choose to work with L^p functions instead of continuous functions when study regularity.

There is a vast amount of literature on sufficient conditions for maximal L^p regularity of (1) (see e.g. [10] for a survey). It appears that for most classical differential operators that may be of interest, there is maximal L^p regularity of this problem. Recently, necessary and sufficient conditions for maximal L^p regularity was obtained in terms of R-boundedness (see e.g. ([9]).

The maximal L^p regularity is an important tool in treating evolution equations more complex than the basic Cauchy problem (1), such as fractional order equations, Volterra equations, nonautonomous (A depends on time) and quasilinear (A depends on the unknown function u) equations.

In this article we apply maximal L^p regularity to study nonautonomous fractional order equations. More precisely, consider the following problem for the fractional differential equation of order $\alpha \in (0,1)$

$$\begin{aligned} &D_t^\alpha u(t) + A(t)u(t) = f(t),\ \text{a.a. } t \in I = (0,T),\\ &(J_t^{1-\alpha}u)(0) = x_0. \end{aligned} \tag{3}$$

Here D_t^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (0,1):$

$$D_t^\alpha = \frac{d}{dt}J_t^{1-\alpha}$$

and J_t^β is the Riemann-Liouville fractional integral:

$$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)\,ds, \quad \beta > 0,\ t > 0,$$

$\{A(t)\}_{t\in[0,T]}$ is a family of closed densely defined linear operators on a Banach space X and $f \in L^p(0,T;X)$, $1 < p < \infty$. We assume moreover that the domain of $A(t)$ does not depend on t: $D(A(t)) = D(A(0)) = X_1$ and $0 \in \rho(A(t))$. To avoid trivialities, we consider only unbounded operators $A(t)$.

Definition 1. We say that problem (3) has *maximal L^p-regularity* if for every $f \in L^p(I;X)$ there exists one and only one $u(t)$ such that

$$u \in L^p(I;X_1), \quad J_t^{1-\alpha}u \in W^{1,p}(I;X) \tag{4}$$

and (3) is satisfied with $x_0 = 0$. The function u with these properties (but with x_0 not necessarily zero) is said to be a *strict L^p solution* of (3).

Our goal is to prove the existence of unique strict L^p solution of (3) under suitable assumptions on the operators $A(t)$ and on the space, to which the initial condition belongs.

2. Preliminaries

First we give briefly some notations and definitions. Let X, Y are Banach spaces. By $\mathcal{B}(X,Y)$ we denote the Banach space of all bounded linear operators from X to Y; $\mathcal{B}(X) := \mathcal{B}(X,X)$ for short.

We call an operator $A : D(A) \subset X \to X$ *sectorial* if $D(A)$ and $R(A)$ are dense in X, $(-\infty, 0) \subset \rho(A)$ and

$$\sup_{t>0} \|t(t+A)^{-1}\|_{\mathcal{B}(X)} < \infty.$$

For a sectorial operator A denote

$$\phi_A := \sup\{\phi \in [0,\pi] | \sup_{|\arg\lambda| \le \phi, \lambda \neq 0} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{B}(X)} < \infty\},$$

$$K_A(\phi) := \sup_{|\arg\lambda| \le \phi, \lambda \neq 0} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{B}(X)}, \ \phi < \phi_A.$$

The number $\omega_A := \pi - \phi_A$ is called *spectral angle* of A.

A family of operators $\tau \subset \mathcal{B}(X)$ is called *randomized bounded* or *R-bounded* (see e.g. [9]), if there exists a constant $M < \infty$ such that for all $\{T_j\}_{j=1}^N \subset \tau$, $\{x_j\}_{j=1}^N \subset X$, $N \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=0}^N r_j(s) T_j x_j \right\|_X ds \le M \int_0^1 \left\| \sum_{j=0}^N r_j(s) x_j \right\|_X ds,$$

where r_j is a sequence of independent, symmetric $\{1,-1\}$valued random variables on $[0,1]$, e.g. $r_j(t) = \mathrm{sgn}(\sin(2^j \pi t))$ - the Rademacher function. The smallest constant M, for which this inequality holds, is called R-bound of τ and is denoted by $\mathcal{R}\{\tau\}$.

This definition can be considered as a strengthening of the property of uniform boundedness of the family τ. Note that in a Hilbert space every norm-bounded set τ is R-bounded.

Next, define a notion which combines the two concepts: of R-boundedness and sectoriality. A sectorial operator A is called *R-sectorial*, if it satisfies

$$\mathcal{R}\{t(tI+A)^{-1}|\ t>0\}<\infty.$$

In analogy to sectorial operators define

$$R_A(\phi):=\mathcal{R}\{\lambda(\lambda I+A)^{-1}|\ |\arg\lambda|\le\phi,\lambda\neq 0\}.$$

Then the *R-angle* of A is defined as

$$\omega_A^R:=\inf\{\phi\in(0,\pi)|\ R_A(\pi-\phi)<\infty\}.$$

It is immediate that $\omega_A<\omega_A^R$.

For sectorial operators A in X with $0\in\rho(A)$, $\gamma\in(0,1)$, $p\in(0,\infty)$ define the spaces

$$D_A(\gamma,p):=\{x\in X|\ \|x\|_{D_A(\gamma,p)}<\infty\},$$

where

$$\|x\|_{D_A(\gamma,p)}:=\left\{\int_0^\infty(t^\gamma\|A(tI+A)^{-1}x\|_X)^p\frac{dt}{t}\right\}^{\frac{1}{p}}.$$

These spaces coincide up to the equivalence of norms with the real interpolation spaces and the inclusions hold (see [5]):

$$D(A)\hookrightarrow D_A(\gamma,p)\hookrightarrow D_A(\gamma',p)\hookrightarrow X,\quad 0<\gamma'<\gamma<1.$$

Recall that a Banach space X is said to belong to the class $\mathcal{H}T$ if the Hilbert transform H defined by

$$(Hf)(t)=\lim_{\varepsilon\to0+}\int_{|s|\ge\varepsilon}f(t-s)\frac{ds}{\pi s},\quad t\in\mathbb{R},\ f\in C_0^\infty(\mathbb{R};X),$$

where $C_0^\infty(\mathbb{R};X)$ is the space of rapidly decreasing functions, extends to a bounded linear operator on $L^p(\mathbb{R};X)$ for $p\in(1,\infty)$. Examples of $\mathcal{H}T$ spaces are all closed subspaces and quotient spaces of a $L^q(\Omega,\mu)$ space with $1<q<\infty$ (see e. g. [4]).

Consider first the autonomous problem, that is problem (3) where the operator A does not depend on time:

$$\begin{aligned}&D_t^\alpha u(t)+Au(t)=f(t),\ \text{a.a. } t\in(0,T),\\&(J_t^{1-\alpha}u)(0)=x_0.\end{aligned}\tag{5}$$

The study of the existence of unique strict L^p solution of this problem can be divided in two parts. For $x_0=0$, $f(t)\neq0$, recent generalizations of the Michlin multiplier theorems (see [7], [12]) can be applied to prove maximal L^p regularity. For the case of zero forcing function direct estimates are obtained provided x_0 belong to appropriate interpolation spaces (see [3]). Combining these two techniques the following result is proven in [2], Theorem 4.16:

Theorem 1. *Suppose that $\alpha \in (0,1)$, $1 < p < \infty$, X is a Banach space of class $\mathcal{HT}$, A is an R-sectorial operator in X with $0 \in \rho(A)$ and with R-angle ω_A^R, satisfying*

$$\omega_A^R < \pi(1 - \alpha/2) \tag{6}$$

and $f \in L^p(I; X)$. Then the following statements hold:

(*a*) *if $1 < p < \frac{1}{1-\alpha}$, then there is a unique strict L^p solution u of (5) iff $x_0 \in D_A(\frac{p-1}{\alpha p}, p)$;*

(*b*) *if $p \geq \frac{1}{1-\alpha}$ then (5) has a unique strict L^p solution iff $x_0 = 0$.*

In both cases the following estimate is satisfied (for (b) we set $x_0 = 0$):

$$\|u\|_{L^p(I;X)} + \|D_t^\alpha u\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leq M(\|x_0\|_{D_A(\frac{p-1}{\alpha p}, p)} + \|f\|_{L^p(I;X)}). \tag{7}$$

The constant M depends on X, α, p, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_A)$ and on $R_A(\alpha\pi/2)$, but does not depend on T and on the individual operator A.

3. The nonautonomous problem

Now we are ready to study the nonautonomous problem (3). First we suppose that the corresponding autonomous problems with $A = A(s)$, where $s \in [0,T]$ is fixed, are strictly solvable in $L^p(0,T;X)$ with estimates, uniform on $s \in [0,T]$. On the base of these assumptions we will solve (3) inductively, dividing the interval $[0,T]$ in sufficiently small intervals. This approach was introduced in [8] in the case $\alpha = 1$ and it is used in e.g. [6], [11] for studying of some fractional equations.

Let us suppose that the family of operators $A(t)$ satisfies the following properties:

(*A1*) $D(A(t)) = D(A(0)) =: X_1$ and $0 \in \rho(A(t))$ for any $t \in [0,T]$; $A(.) \in C(0,T;\mathcal{B}(X_1,X))$.

We equip X_1 with the graph norm $\|x\|_{X_1} := \|Ax\|_X$. It follows from (A1) and the compactness of $[0,T]$ that the graph norms of the operators $A(t)$ are uniformly equivalent, i.e. there exist constants a_1 and a_2 such that for each $x \in X_1$ and $t \in [0,T]$ we have

$$a_1\|x\|_{X_1} \leq \|A(t)x\|_X \leq a_2\|x\|_{X_1}. \tag{8}$$

Denote by $\rho_{A,T}(s)$ the modulus of continuity of the continuous function $A(t)$, that is

$$\rho_{A,T}(s) := \sup_{t_1,t_2\in[0,T],\ |t_1-t_2|\leq s} \|A(t_1) - A(t_2)\|_{\mathcal{B}(X_1,X)}.$$

(*A2*) For any $t \in [0,T]$ $A(t)$ is sectorial with spectral angle

$$\omega_{A(t)} < (1-\alpha/2)\pi. \tag{9}$$

(*A3*) There exist a subspace $Z_0 \hookrightarrow X$ such that for any $x_0 \in Z_0$, $f \in L^p(0,T;X)$ and for any fixed $s \in [0,T]$ the problem

$$\begin{aligned} &D_t^\alpha u(t) + A(s)u(t) = f(t), \text{ a.a. } t \in (0,T),\\ &(J_t^{1-\alpha}u)(0) = x_0, \end{aligned} \tag{10}$$

has a strict L^p solution $u(t)$, satisfying the estimate:

$$\|u\|_{L^p(0,T;X_1)} \le M(\|f\|_{L^p(0,T;X)} + \|x_0\|_{Z_0}), \tag{11}$$

where the constant M does not change for different values of $s \in [0,T]$.

Under these assumptions we have the following result on strict L^p solvability of the nonautonomous problem (3):

Proposition 1. *Let $\alpha \in (0,1)$ and assume that (A1), (A2) and (A3) hold. Then for any $x_0 \in Z_0$, $f \in L^p(0,T;X)$, there exists a unique strict L^p solution of (3) and the following estimate holds:*

$$\|u\|_{L^p(0,T;X_1)} \le N(\|f\|_{L^p(0,T;X)} + \|x_0\|_{Z_0}), \tag{12}$$

where N depends only on M.

Proof. First note that (A3) holds also with T replaced by an arbitrary $\hat{T} \in [0,T]$ and the same constant M in (11). To see this, consider the equation

$$D_t^\alpha u(t) + A(s)u(t) = f(t), \quad \text{a.a. } t \in [0,\hat{T}], \tag{13}$$

where $f \in L^p(0,\hat{T};X)$ and define $f_0(t) = f(t)$ a.a. $t \in [0,\hat{T}]$ and $f_0(t) = 0$ a.a. $t \in [\hat{T},T]$. If $u_0(t)$ is the unique solution of (10) with f replaced by f_0, then $u(t) = u_0(t)$, a.a. $t \in [0,\hat{T}]$, is the unique solution of (13) and by the definition of L^p norms we conclude that we in fact have our claim:

$$\|u\|_{L^p(0,\hat{T};X_1)} \le M(\|f\|_{L^p(0,\hat{T};X)} + \|x_0\|_{Z_0}). \tag{14}$$

Take T_ε such that

$$\rho_{A,T}(T_\varepsilon) \le \varepsilon = \frac{a_1}{2Ma_2}. \tag{15}$$

We solve first (3) for $t \in [0,T_\varepsilon]$. Let $v \in V_\varepsilon := L^p(0,T_\varepsilon;X)$ and consider the equation

$$D_t^\alpha u(t) + A(0)u(t) = f(t) + (A(0)-A(t))A(0)^{-1}v(t), \quad \text{a.a. } t \in [0,T_\varepsilon] \tag{16}$$

with initial condition $(J_t^{1-\alpha}u)(0) = x_0$. Since the right-hand side is an element of V_ε we obtain by the strict solvability of (12) that there is a unique solution $u \in V_\varepsilon$ such that $A(0)u \in V_\varepsilon$. Therefore the mapping $\gamma : v \to A(0)u$ maps V_ε into itself and from the linearity of the equation, (14) and (8), we have

$$\|\gamma(v_1)-\gamma(v_2)\|_{L^p(0,T_\varepsilon;X)} \leq a_2 M\|(A(0)-A(t))A(0)^{-1}(v_1-v_2)\|_{L^p(0,T_\varepsilon;X)}$$
$$\leq \frac{a_2}{a_1}M \sup_{t\in[0,T_\varepsilon]} \|A(0)-A(t)\|_{\mathcal{B}(X_1,X)}\|v_1-v_2\|_{L^p(0,T_\varepsilon;X)}$$
$$= \frac{a_2}{a_1}M\rho_{A,T}(T_\varepsilon)\|v_1-v_2\|_{L^p(0,T_\varepsilon;X)}$$

According to (15) the mapping γ is a contraction and there is a unique fixed point. Thus, $v = A(0)u$ and we get a solution of (3) on the interval $[0, T_\varepsilon]$. Again by (14) and (8) we have for the solution u of (16) the following estimate

$$\|u\|_{L^p(0,T_\varepsilon;X_1)} \leq M(\|f+(A(0)-A(t))A(0)^{-1}v\|_{L^p(0,T_\varepsilon;X)} + \|x_0\|_{Z_0})$$
$$\leq M(\|f\|_{L^p(0,T_\varepsilon;X)} + \|x_0\|_{Z_0}) + M\rho_{A,T}(T_\varepsilon)\|u\|_{L^p(0,T_\varepsilon;X_1)}.$$

According to (15) $M\rho_{A,T}(T_\varepsilon) \leq \frac{a_1}{2a_2} \leq \frac{1}{2}$. Therefore,

$$\|u\|_{L^p(0,T_\varepsilon;X_1)} \leq 2M(\|f\|_{L^p(0,T;X)} + \|x_0\|_{Z_0}).$$

Next we work by induction. Suppose that $T_0 \in [0, T]$ and that we have found a solution u of (3) on $[0, T_0]$ which satisfies the inequality

$$\|u\|_{L^p(0,T_0;X_1)} \leq N(T_0)(\|f\|_{L^p(0,T;X)} + \|x_0\|_{Z_0}). \tag{17}$$

Let $\hat{T} = \min\{T, T_0 + T_\varepsilon\}$ and define the set

$$V := \{v \in L^p(0,\hat{T};X) |\ v(t) = A(T_0)u(t), \quad \text{a.a. } t \in [0, T_0]\}.$$

For each $v \in V$ we proceed to find a solution w of the equation

$$D_t^\alpha w(t) + A(T_0)w(t) = f(t) + (A(T_0)-A(t))A(T_0)^{-1}v(t) \tag{18}$$

with initial condition $(J_t^{1-\alpha}w)(0) = x_0$. Since the right-hand side of (18) is an element of $L^p(0,\hat{T};X)$, from the strict solvability of (12) there is a unique solution $w \in L^p(0,\hat{T};X)$ such that $A(T_0)w \in L^p(0,\hat{T};X)$ and the uniqueness guarantees $A(T_0)w \in V$. Define the mapping $G : V \to V$ with $Gv = A(T_0)w$. By the linearity of (18) and applying (14) and (8), we obtain

$$\|G(v_1)-G(v_2)\|_{L^p(0,\hat{T};X)} \leq a_2M\|(A(T_0)-A(t))A(T_0)^{-1}(v_1-v_2)\|_{L^p(0,\hat{T};X)}.$$

Since $v_1, v_2 \in V$ then $v_1 - v_2 = 0$ a.e. on $[0, T_0]$ and the last inequality implies

$$\|G(v_1)-G(v_2)\|_{L^p(0,\hat{T};X)} \leq \frac{a_2}{a_1}M \sup_{t\in[T_0,\hat{T}]} \|A(T_0)-A(t)\|_{\mathcal{B}(X_1,X)}\|v_1-v_2\|_{L^p(0,\hat{T};X)}$$
$$\leq \frac{a_2}{a_1}M\rho_{A,T}(T_\varepsilon)\|v_1-v_2\|_{L^p(0,\hat{T};X)}.$$

Hence, by (15) the mapping G is a contraction and has a unique fixed point, thus $A(T_0)w = v$ on $[0, \hat{T}]$ and equation (18) reduces to (3). Thus we get a solution of (3) on the interval $[0, \hat{T}]$. If we take $v_0 \in V$ to be such that $v_0(t) = 0$ a.e. on $[T_0, \hat{T}]$, then $\|v_0\|_{L^p(0,\hat{T};X)} = \|A(T_0)u\|_{L^p(0,T_0;X)}$. Using inequalities (14), (17) and (8) we obtain

$$\|w\|_{L^p(0,\hat{T};X_1)} \leq M(\|f + (A(T_0) - A(t))A(T_0)^{-1}v_0\|_{L^p(0,\hat{T};X)} + \|x_0\|_{Z_0})$$

$$\leq M(1 + \frac{a_2}{a_1}\rho_{A,T}(T_\varepsilon)N(T_0))(\|f\|_{L^p(0,T;X)} + \|x_0\|_{Z_0})$$

and (17) holds with T_0 replaced by $\hat{T}$ and $N(T_0)$ replaced by

$$N(\hat{T}) = M(1 + \frac{a_2}{a_1}\rho_{A,T}(T_\varepsilon)N(T_0)).$$

Since this procedure can be repeated with the same T_ε, we find a solution on $[0, T]$ that satisfies the bound (12), where we have obtained by induction:

$$N = M\sum_{i=0}^{k}(\frac{a_2}{a_1}\rho_{A,T}(T_\varepsilon)M)^i,$$

where $k \in \mathbb{N}$ is the necessary number of steps $k = [T/T_\varepsilon] + 1$. Applying (15) it follows

$$N \leq M\sum_{i=0}^{k} 2^{-i} < M\sum_{i=0}^{\infty} 2^{-i} = 2M.$$

■

Suppose (A1) and (A2) are fulfilled. If the conditions of Theorem 1 are satisfied, then problems (10) have strict L^p solutions and (11) are satisfied, where the constant M depends on $\omega_{A(t)}$ and $K_{A(t)}(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_{A(t)})$ and on $R_{A(t)}(\alpha\pi/2)$. Therefore (A3) will hold if these quantities are uniformly bounded for $s \in [0, T]$.

Clearly (A1) and (A2) imply that given $s \in [0, T]$, we have

$$\|(\lambda I + A(s))^{-1}\|_{\mathcal{B}(X)} \leq \frac{K(s)}{1 + |\lambda|}, \ |\arg\lambda| \leq \phi_s,$$

where $\alpha\pi/2 < \phi_s < \phi_{A(s)} = \pi - \omega_{A(s)} \leq \pi$. Moreover,

$$(\lambda I + A(t))^{-1} = (\lambda I + A(s))^{-1}[I + (A(t) - A(s))A(0)^{-1}A(0)(\lambda I + A(s))^{-1}]^{-1}.$$

So, due to the compactness of $[0, T]$ there are constants $K \geq 0$ and $\phi \in (\alpha\pi/2, \pi]$ such that

$$\|(\lambda I + A(t))^{-1}\|_{\mathcal{B}(X)} \leq \frac{K}{1 + |\lambda|}, \ \ |\arg\lambda| \leq \phi, \ t \in [0, T].$$

Set $\omega = \pi - \phi$. In this way we proved that there exist constants ω and K such that $\omega_{A(t)} < \omega < \pi(1 - \alpha/2)$ and $K_{A(t)}(\pi - \omega) \leq K$ for any $t \in [0, T]$.

According to Theorem 1, the constant of maximal regularity M depends also on $R_{A(t)}(\alpha\pi/2)$. So, in order to obtain (A3) we have to require the uniform boundedness of this quantity. Since in the Hilbert space case the R-boundedness is equivalent to the uniform boundedness, then the constant of maximal regularity depends only on $K_{A(t)}(\theta)$.

Note also that $D_{A(s)}(\delta,p) = D_{A(0)}(\delta,p)$ because $D(A(s)) = D(A(0))$.

Now if we apply Proposition 1 and Theorem 1 to the problem (3), having in mind the above remarks, we obtain our main result:

Theorem 2. *Suppose that $\alpha \in (0,1)$, $1 < p < \infty$, X is a Banach space of class $\mathcal{H}T$, $\{A(t)\}_{t\in[0,T]}$ is a family of operators satisfying (A1) and $f \in L^p(I;X)$. Let moreover $A(t)$ be R-sectorial operators with R-angles*

$$\omega^R_{A(t)} < \pi(1-\alpha/2) \tag{19}$$

and $R_{A(t)}(\alpha\pi/2) \le R$ for $t \in [0,T]$. Let one of the following conditions be satisfied

(*a*) $1 < p < \frac{1}{1-\alpha}$ *and* $x_0 \in D_{A(0)}(\frac{p-1}{\alpha p}, p)$*;*

(*b*) $p \ge \frac{1}{1-\alpha}$ *and* $x_0 = 0$.

Then problem (3) has a unique strict L^p solution u satisfying

$$\|u\|_{L^p(I;X_1)} \le N(\|x_0\|_{D_{A(0)}(\frac{p-1}{\alpha p},p)} + \|f\|_{L^p(I;X)}).$$

The constant N depends on X, α, p, R, ω and K, but does not depend on T and on the individual operators $A(t)$.

Remark 1. The case $\alpha \in (1,2)$ can be considered in a similar way.

Remark 2. Strict L^p solvability of nonautonomous problems is an important tool in studying quasilinear problems:

$$D^\alpha_t u + A(u)u = f,$$

where $A(u)$ is a linear operator for any fixed u. A natural way of solving such problems is by applying a fixed point argument to the equation

$$D^\alpha_t u + A(v)u = f$$

with v fixed, which is in fact a nonautonomous linear problem.

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