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Commutants of the Square of Differentiation on the Half-Line

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Let C^1 denote the space of the smooth functions on the real half-line $\mathbb{R}_{\geq 0} = [0, \infty)$ and C_h^1 is the subspace of C^1 consisting of its functions $f(x)$ which satisfy the initial value condition $f'(0) - hf(0) = 0$ with a fixed real h.

The aim of the paper is to characterize all continuous linear operators $M : C^1 \to C^1$ which has the subspace $C_h^1 = \{f : f \in C^1, f'(0) - hf(0) = 0\}$ as an invariant subspace and commuting with the square D^2 of the differentiation operator $D = \dfrac{d}{dx}$ on C_h^2.The set of all such operators is said to be the *commutant* of D^2 under the constraints considered and is denoted by $\mathrm{Comm}(D^2; h)$. We prove that each operator M from $\mathrm{Comm}(D^2; h)$ has an explicit representation $Mf(x) = \Phi_y\{T^y f(x)\}$, where

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt$$

is a generalized translation operator in the sense of B. Levitan [10] and Φ is a linear functional on C^1.

Next, for a fixed h we prove that $\mathrm{Comm}(D^2; h)$ is a commutative algebra. The kernel space of each of the operators from the commutant, denoted by MP_Φ, forms a space of mean-periodic functions for D^2 in the sense of K. Trimeche [11]. A convolution structure $* : C \times C \to C$ is introduced, such that MP_Φ is an ideal in the convolution algebra $(C, *)$. This result can be used for effective solution in mean-periodic functions of ordinary differential equations of the form $P(D^2)y = f$ with a polynomial P.

By means of this convolution structure, we characterize the commutant of D^2 in C^1, subjected to a local constraint of the form $f'(0) - hf(0) = 0$ and to an additional non-local one of the form $\Phi\{f\} = 0$ with Φ being a linear functional on C^1. It consists of all linear operators $M : C^1 \to C^1$ of the form

$$Mf(x) = \mu f(x) + (m * f)(x)$$

with $\mu = \mathrm{const}$ and $m \in C^1$.

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0. Introduction

Till recently, not too many investigations could be pointed out on the problem for characterizing of commutants of the square of differentiation. Ch. Kahane [9] announced that the commutant of D^2 on $C[a,b]$ for a finite interval $[a,b]$, without any additional constraints, consists of the operators of the form

$$Mf(x) = Af(x) + Bf(a+b-x) + C \int_x^{a+b-x} f(t)dt,$$

where A, B, C are arbitrary constants.

Starting with J. Delsarte [2] and ending with B. Levitan [10], the family of the generalized translation operators

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt$$

is introduced as consisting of operators $C^1 \to C^1$, commuting with D^2 in their invariant subspace

$$C_h^1 = C_h^1(\mathbb{R}_{\geq 0}) = \{f(x) : f \in C^1(\mathbb{R}_{\geq 0}), f'(0) - hf(0) = 0\}.$$

But they do not exhaust all such operators. It happens that all continuous linear operators $M : C^1 \to C^1$, having C_h^1 as an invariant subspace and commuting with D^2 in C_h^2, are exhausted by the operators of the form

$$Mf(x) = \Phi_y\{T^y f(x)\} = \Phi_y\left\{\frac{f(x+y) + f(|x-y|)}{2} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt\right\}$$

with an arbitrary linear functional Φ on C^1 (Theorem 1).

This representation resembles the description of the commutant of the operator of differentiation $D = \dfrac{d}{dx}$ in $C(\mathbb{R})$ published in Bourbaki [1]:

$$Mf(x) = \Phi_y\{f(x+y)\}.$$

with a linear functional Φ on $C(\mathbb{R})$.

Following this pattern the authors have also found the commutants of several operators, e.g. the Pommiez operator [4], the Euler operator ([5] and [6]), and the Dunkl operators [7].

1. A family of operators commuting with $D^2 = \dfrac{d^2}{dx^2}$

Let C_h^1 be the subspace of the space C^1 of the smooth functions f on $\mathbb{R}_{\geq 0} = [0, \infty)$ satisfying the boundary value condition

$$f'(0) - hf(0) = 0 \tag{1}$$

with a fixed $h \in \mathbb{R}$. By C_h^2 we denote the subspace of twice continuously differentiable functions of C_h^1. We are looking for the linear operators $M : C^1 \to C^1$ with an invariant subspace C_h^1, commuting with D^2 in C_h^2.

Lemma 1. *The operators*

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt \tag{2}$$

map C into C and have the following properties:

(i) *C_h^1 is an invariant subspace for T^y;*

(ii) *$T^y f(x) = T^x f(y)$;*

(iii) *$T^0 f(x) = f(x)$;*

(iv) *$D^2 T^y = T^y D^2$ on C_h^2;*

(v) *$T^y T^z = T^z T^y$.*

P r o o f. (i) It is seen directly that $(T^y f)'(0) - h(T^y f)(0) = 0$ for arbitrary $f \in C^1(\mathbb{R}_{\geq 0})$ and hence $T^y : C_h^1 \to C_h^1$.

(ii) and (iii) are obvious.

(iv) We verify it first for $y \leq x$ and then for $x < y$. If $y \leq x$, then

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(x-y)\} + \frac{h}{2}\int_{x-y}^{x+y} f(t)dt$$

and

$$\frac{d^2}{dx^2} T^y f(x) = \frac{1}{2}[f''(x+y) + f''(x-y)] + \frac{h}{2}[f'(x+y) - f'(x-y)] = T^y f''(x).$$

If $x < y$, then the verification of $\frac{d^2}{dx^2} T^y f(x) = T^y f''(x)$ goes in the same way.

(v) We verify it first for even powers of x, i.e. for $f(x) = x^{2n}$, and then proceed by approximation of an arbitrary function $f \in C$ by polynomials of the form $P(x^2)$.

Since the operators (2) are very special case of the generalized translation operators of B. M. Levitan (see [10]), one may rely also on the general proof in this book. ∎

2. Characterization of the operators $M : C^1 \to C^1$ commuting with D^2 in the invariant subspace C_h^1

Theorem 1. *Let $M : C^1 \to C^1$ be a continuous linear operator with C_h^1 as an invariant subspace and such that $M : C^2 \to C^2$. Then the following assertions are equivalent:*

(i) $MD^2 = D^2M$ *on* C_h^2;

(ii) $MT^y = T^yM$ *for each* $y \geq 0$;

(iii) M *has the explicit representation*

$$Mf(x) = \Phi_y\{T^y f(x)\} = \Phi_y\left\{\frac{f(x+y)+f(|x-y|)}{2} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt\right\} \quad (3)$$

with a linear functional Φ *in* C^1.

P r o o f. (i)⇒(ii) Let $f(x)$ be an even polynomial. Then the Maclaurin expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} D^{2n} f(0)$$

gives the following representation of the "translated" function:

$$T^y f(x) = T^x f(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^x D^{2n} f(0)$$

$$= \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^0 D^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} f(x).$$

Now (ii) will follow if we apply M to both sides and use the identity $MD^{2n}f(x) = D^{2n}Mf(x)$ which follows immediately from (i) for each $n \in \mathbb{N}$:

$$MT^y f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} MD^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} Mf(x) = T^y Mf(x).$$

(ii)⇒(iii) Let us define a continuous linear functional Φ on C^1 by $\Phi\{f\} = (Mf)(0)$. Substituting $y = 0$ in

$$T^y Mf(x) = MT^y f(x) = MT^x f(y),$$

we obtain

$$T^0 Mf(x) = MT^x f(0).$$

The left hand side is $Mf(x)$ and the right hand side is the value of the functional Φ for the function $T^x f$. Hence

$$Mf(x) = \Phi_y\{T^x f(y)\} = \Phi_y\{T^y f(x)\}.$$

Thus the implication is proved using y as the "dumb" variable of the functional.

(iii)⇒(i) Let $Mf(x) = \Phi_y\{T^y f(x)\}$. Then $D^2Mf(x) = \Phi_y\{D^2T^y f(x)\}$ for $f \in C_h^2$. Using $D^2T^y = T^yD^2$ from Lemma 1, we obtain

$$D^2 M f(x) = \Phi_y\{T^y D^2 f(x)\} = M D^2 f(x).$$

Hence (iii)⇒(i). ■

Theorem 2. *The commutant of* $D^2 = \dfrac{d^2}{dx^2}$ *in* C^1_h *is a commutative ring.* Proof. Let the operators $M : C^1_h \to C^1_h$ and $N : C^1_h \to C^1_h$ commute with $D^2 = \dfrac{d^2}{dx^2}$ in C^2_h.

According to (iii) from Theorem 1, there are linear functionals Φ and Ψ in C^1, such that

$$Mf(x) = \Phi_y\{T^y f(x)\} \quad \text{and} \quad Nf(x) = \Psi_z\{T^z f(x)\}.$$

Then

$$MNf(x) = \Phi_y \Psi_z\{T^y T^z f(x)\} \quad \text{and} \quad NMf(x) = \Psi_z \Phi_y\{T^z T^y f(x)\}.$$

By (iv) from Lemma 1, $T^z T^y = T^y T^z$, and hence

$$NMf(x) = \Psi_z \Phi_y\{T^z T^y f(x)\} = \Psi_z \Phi_y\{T^y T^z f(x)\}.$$

It remains to use the Fubini property $\Psi_z \Phi_y\{g(y,z)\} = \Phi_y \Psi_z\{g(y,z)\}$ for functions $g(y,z) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ in order to assert that $MN = NM$. ■

3. Characterization of the operators $M : C^1 \to C^1$ commuting with D^2 in the invariant subspaces C^1_h and $C^1_\Phi = \{f \in C^1, \Phi\{f\} = 0\}$

Let Φ be a continuous linear functional on $C^1 = C^1(\mathbb{R}_{\geq 0}) = C^1[0,\infty)$. We are looking for the set of the linear operators $M : C^1 \to C^1$ with invariant subspaces C^1_h and $C^1_\Phi = \{f \in C^1, \Phi\{f\} = 0\}$ and commuting with D^2 in them with the notation $\mathrm{Comm}(D^2, h, \Phi)$.

In Dimovski and Petrova [8] a convolution structure $* : C^1 \times C^1 \to C^1$ is introduced with the following properties:

$$f * g \in C^1_h \quad \text{and} \quad \Phi\{(f * g)\} = 0 \tag{4}$$

for arbitrary $f, g \in C^1$.

Our aim is to show that each operator $M : C^1 \to C^1$ of the commutant $\mathrm{Comm}(D^2, h, \Phi)$, we are interested in, has the explicit form

$$Mf = \mu f + m * f$$

with $\mu = \text{const}$ and $m \in C^1$.

There is no need to know the explicit form of the convolution $f * g$. We will use only the fact that the resolvent operator $R_{-\lambda^2}$ of D^2 under the constraints

$y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$ is a convolution operator $R_{-\lambda^2} = \varphi_\lambda *$ with $\varphi_\lambda \in C^1_h$.

Let us remind that the resolvent operator $R_{-\lambda^2} f = y$ of the operator $D^2 = \dfrac{d^2}{dx^2}$ is defined as the solution of the differential equation $y'' + \lambda^2 y = f(x)$ with the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$.

In Dimovski and Petrova [8] it is shown that

$$R_{-\lambda^2} f(x) = \left\{ \frac{\lambda \cos \lambda x + h \sin \lambda x}{\lambda E(\lambda)} \right\} * f(x)$$

with $E(\lambda) = \Phi_\xi \left\{ \dfrac{\lambda \cos \lambda\xi + h \sin \lambda\xi}{\lambda E(\lambda)} \right\}$.

Theorem 3. *Let $M : C^1 \to C^1$ be continuous linear operator with invariant subspaces C^1_h and C^1_Φ which maps C^2 into C^2. Then the following assertions are equivalent:*

(i) *M commutes with D^2 in C^2_h and C^2_Φ;*

(ii) *M commutes with $R_{-\lambda^2}$ in C^1 for a fixed λ;*

(iii) *M is a multiplier of the convolution algebra $(C^1, *)$;*

(iv) *M admits a representation of the form*

$$Mf = \mu f + m * f \tag{5}$$

with $\mu = \text{const}$ and $m \in C^1$.

P r o o f. (i) $\Rightarrow$ (ii) Assume that $MD^2 = D^2 M$ in C^1_h and C^1_Φ. Let $f \in C^1$ be arbitrary. Consider the function

$$\psi(x) = MR_{-\lambda^2} f(x) - R_{-\lambda^2} Mf(x).$$

We obtain

$$(D^2 + \lambda^2)\psi = (D^2 + \lambda^2) MR_{-\lambda^2} f - Mf = M(D^2 + \lambda^2) R_{-\lambda^2} f - Mf = 0.$$

It is easy to verify that the function ψ satisfies the boundary value conditions $\psi'(0) - h\psi(0) = 0$ and $\Phi\{\psi\} = 0$. Hence, $\psi(x) \equiv 0$ since λ is not an eigenvalue, i.e.

$$MR_{-\lambda^2} f = R_{-\lambda^2} Mf, \qquad f \in C^1.$$

(ii) $\Rightarrow$ (iii) This follows from Theorem 1.3.11 in Dimovski [3], p. 53. According to this theorem, the commuting of M with $R_{-\lambda^2}$ implies that M is a multiplier of the convolution algebra $(C^1, *)$.

(iii) $\Rightarrow$ (iv) The identity $R_{-\lambda^2} f = \varphi_\lambda * f$ with $\varphi_\lambda = \left\{ \dfrac{\lambda \cos \lambda x + h \sin \lambda x}{\lambda E(\lambda)} \right\}$ $\in C^2_h$ implies $MR_{-\lambda^2} f = R_{-\lambda^2} Mf = (M\varphi_\lambda) * f$. Denoting $\psi_\lambda = M\varphi_\lambda \in C^2_h$, we get

$$Mf = (D^2 + \lambda^2)(\psi_\lambda * f) = D^2(\psi_\lambda * f) + (\lambda^2 \psi_\lambda * f)$$
$$= D^2(\psi_\lambda) * f + \Phi\{\psi_\lambda\} f + \lambda^2 \psi_\lambda * f = \Phi\{\psi_\lambda\} f + [(D^2 + \lambda^2)\psi_\lambda] * f,$$

which is the representation (5) with $\mu = \Phi\{\psi_\lambda\}$ and $m = (D^2 + \lambda^2)\psi_\lambda$.

(iv) $\Rightarrow$ (i) From the properties (4) of the convolution structure $f * g$ it follows that C_h^1 and C_Φ^1 are invariant subspaces of C^1. From (5) it follows that $M : C^2 \to C^2$ and M commutes with $R_{-\lambda^2}$ in C^1, i.e.

$$MR_{-\lambda^2} f = R_{-\lambda^2} M f, \qquad f \in C^1.$$

Multiplying with $(D^2 + \lambda^2)$ we obtain

$$(D^2 + \lambda^2) M R_{-\lambda^2} f = Mf.$$

Taking $f = (D^2 + \lambda^2)g$ with $g \in C_h^2 \cap C_\Phi^2$, we get

$$(D^2 + \lambda^2) M R_{-\lambda^2} (D^2 + \lambda^2) g = M(D^2 + \lambda^2) g,$$

but $R_{-\lambda^2}(D^2 + \lambda^2) g = g$ for $g \in C_h^2 \cap C_\Phi^2$. Hence $(D^2 + \lambda^2)M = M(D^2 + \lambda^2)$ on $C_h^2 \cap C_\Phi^2$ which is equivalent to $MD^2 = D^2 M$. ■

4. Mean-periodic functions for $D^2 = \dfrac{d^2}{dx^2}$ in C_h^1

Definition 1. The kernel space $\ker M$ of an operator $Mf(x) = \Phi_y\{T^y f(x)\}$ from $\mathrm{Comm}(D^2, h)$ is called the space of the mean-periodic functions for D^2, associated with the linear functional Φ.

We use the notation $MP_\Phi = \ker M = \{f \in C_h^1 : \Phi_y\{T^y f(x)\} = 0\}$.

Lemma 4. *$R_{-\lambda^2}$ maps MP_Φ into itself, i.e. $R_{-\lambda^2}(MP_\Phi) \subset MP_\Phi$.*

Proof. Let $f \in MP_\Phi$, i.e. $\Phi_y\{T^y f(x)\} = 0$. We are to prove that $\varphi(x) = \Phi_y\{T^y R_{-\lambda^2} f(x)\} \equiv 0$. Indeed, we have

$$(D^2 + \lambda^2)\varphi(x) = \Phi_y\{(D^2 + \lambda^2) T^y R_{-\lambda^2} f(x)\}$$
$$= \Phi_y\{T^y (D^2 + \lambda^2) R_{-\lambda^2} f(x)\} = \Phi_y\{T^y f(x)\} \equiv 0,$$

since $(D^2 + \lambda^2) R_{-\lambda^2} f(x) = f(x)$. Hence $\varphi(x)$ belongs to the kernel space of $D^2 + \lambda^2$, i.e. $\varphi(x) = A \cos \lambda x + B \sin \lambda x$ with constants A and B. φ satisfies the condition $\varphi'(0) - h\varphi(0) = 0$ and hence $B\lambda - hA = 0$. In other words, $\varphi(x)$ is a function of the form $\varphi(x) = A\left(\cos \lambda x + \dfrac{h \sin \lambda x}{\lambda}\right)$. Using the boundary value condition $\Phi\{f\} = 0$, we obtain

$$0 = A\Phi_t\left\{\cos \lambda t + \frac{h \sin \lambda t}{\lambda}\right\} = AE(\lambda).$$

But $E(\lambda) \neq 0$ and hence $A = 0$. Thus we proved that $\varphi(x) \equiv 0$. ■

For the sake of simplicity, from now on we restrict our considerations to the case $h = 0$, i.e. to the space

$$C_0^1 = \{f \in C^1(\mathbb{R}_{\geq 0}), f'(0) = 0\}.$$

This is possible due to an explicit isomorphism between C_h^1 and C_0^1.

Lemma 5. *The linear operator*

$$\tau f(x) = f(x) + h \int_0^x e^{-h(x-t)} f(t)dt \tag{6}$$

maps C_h^1 onto C_0^1 and its inverse is

$$\tau^{-1} f(x) = f(x) + h \int_0^x f(t)dt. \tag{7}$$

If $f \in C_h^2$, then $\tau f \in C_0^2$ and $(\tau f)'' = \tau f''$.

The proof is a matter of simple check (see Dimovski [3], p.153).

Due to Lemma 6, instead of the resolvent operator $R_{-\lambda^2}$ of D^2 with boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$, we may consider the resolvent operator $\widetilde{R_0}$ of D^2, defined by the boundary value conditions $y'(0) = 0$ and $\widetilde{\Phi}\{y\} = 0$, where $\widetilde{\Phi} = \Phi \circ \tau^{-1}$.

From now on we will use the notation Φ instead of $\widetilde{\Phi}$, assuming that we are all the time in the case $h = 0$.

For a further simplification we assume that $\lambda = 0$ is not an eigenvalue of the eigenvalue problem $y'' + \lambda^2 y = 0,\ y'(0) = 0,\ \Phi\{y\} = 0$. This means that there exists a right inverse operator R of D^2, such that $(Rf)'(0) = 0,\ \Phi\{Rf\} = 0$ which is possible when $\Phi\{1\} \neq 0$. If so, we may assume additionally that $\Phi\{1\} = 1$ without any loss of generality. Then the right inverse of D^2 has the form

$$Rf(x) = \int_0^x (x-t)f(t)dt - \Phi_y \left\{ \int_0^y (y-t)f(t)dt \right\}.$$

In Dimovski [3], pp. 148-151, the following theorem is proved:

Theorem 4. *The operation*

$$(f * g)(x) = \int_0^x dt \int_0^t f(t-\tau)g(\tau)d\tau + \frac{1}{2}\Phi_t \left\{ \int_0^t \psi(x,\tau)d\tau \right\}, \tag{8}$$

where

$$\psi(x,t) = \int_x^t f(t+x-\tau)g(\tau)dz + \int_{-x}^t f(|t-x-\tau|)g(|\tau|)d\tau,$$

is an inner operation in C, which is bilinear, commutative, and associative, and R is the convolution operator $R = \{1\}*$, i.e. $Rf = \{1\} * f$.

Theorem 5. *The subspace MP_Φ of mean-periodic functions for D^2 associated with the linear functional Φ forms an ideal in the convolution algebra $(C, *)$.*

P r o o f. By Lemma 4, if $f \in MP_\Phi$, then $Rf \in MP_\Phi$. But from Theorem 2 we have $Rf = \{1\} * f$ and $R^k f = \{Q_k(x^2)\} * f$, where Q_k is a polynomial of degree k. Next, choose a polynomial sequence $\{P_n(x)\}_{n=1}^{\infty}$ converging to $g(\sqrt{x})$ uniformly on each segment $[a, b] \subset [0, \infty)$. Then $\{P_n(x^2)\}_{n=1}^{\infty}$ converges to $g(x)$ in C_0^1. But $P_n(x^2) = \sum_{k=0}^{n} \alpha_k Q_k(x^2)$ with some constants $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\{P_n(x^2)\}*f \in MP_\Phi$ since $\{Q_k(x^2)\}*f \in MP_\Phi$, $k = 0, 1, 2, \ldots, n$. Obviously the limit $\lim_{n\to\infty} \{P_n(x^2)\}*f = g*f$ of the sequence $\{\{P_n(x^2)\}*f\}_{n=1}^{\infty}$ of mean-periodic functions is also mean-periodic.

Hence $g * f \in MP_\Phi$ for arbitrary $g \in C$ and therefore MP_Φ is an ideal in $(C, *)$. ■

Theorem 5 may be used to study the problem of solution of ordinary differential equations with constant coefficients of the form

$$P\left(\frac{d^2}{dx^2}\right) y = f(x)$$

in mean-periodic functions of the space MP_Φ and to extend the Heaviside algorithm for obtaining such solutions in explicit form. This will be left for a subsequent publication, but analogous considerations for the Dunkl operator D_k instead of D^2 can be seen in Dimovski, Hristov, and Sifi [7].

5. Open problem

Characterize all continuous linear operators $M : C \to C$ with an invariant subspace $\{f \in C, f(0) = 0\}$ and commuting with D^2.

Conjecture: $Mf(x) = \mu f(x) + \Phi_y \left\{ \int_{|x-y|}^{x+y} f(t)dt \right\}$ with $\mu = \text{const}$ and a linear functional Φ on C.

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