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Exact Solutions of Nonlocal BVPs for the Multidimensional Heat Equations

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In this paper a method for obtaining exact solutions of the multidimensional heat equations with nonlocal boundary value conditions in a finite space domain with time-nonlocal initial condition is developed. One half of the space conditions are local, and the other are nonlocal. Extensions of Duhamel principle are obtained. In the case when the initial value condition is a local one i.e. of the form $u(x_1, ..., x_n, 0) = f(x_1, ..., x_n)$ the problem reduces to n one-dimensional cases. In the Duhamel representations of the solution are used multidimensional non-classical convolutions. This explicit representation may be used both for theoretical study, and for numerical calculation of the solution.

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0. Introduction

In Gutterman [1], direct operational calculi of Mikusiński's type for functions of several real variables are proposed. These calculi are applicable only to Cauchy problems, but not to mixed initial-boundary value problems. According to Gutterman, such problems need new ideas and approaches. Here we propose direct operational calculi connected with linear nonlocal boundary value problems for a large class of heat equations with several space variables and one time variable in finite space domain. Our starting point is the class of linear nonlocal boundary value problems for PDEs of the form:

$$u_t - u_{x_1x_1} - ... - u_{x_nx_n} = F(x_1, ..., x_n, t), \quad 0 < t, \quad 0 < x_j < a_j, \tag{1}$$

determined by a time-nonlocal initial condition of the form

$$\chi_\tau\{u(x_1,...,x_n,\tau)\} = f(x_1,...,x_n), \tag{2}$$

with a given non-zero linear functional χ on $C[0,\infty)$, and n space-local boundary value conditions of the form

$$u(x_1,...,x_{j-1},0,x_{j+1},...,x_n,t) = g_j(x_1,...,x_{j-1},x_{j+1},...,x_n,t) \tag{3}$$

and n space-nonlocal boundary value conditions of the form

$$\Phi_{j,\xi}\{u(x_1,...,x_{j-1},\xi,x_{j+1},...,x_n,t)\} = h_j(x_1,...,x_{j-1},x_{j+1},...,x_n,t), \tag{4}$$

$j = 1,...,n$, where Φ_j are given non-zero linear functionals on $C^1[0,a_j]$. Here the given functions $F(x_1,...,x_n,t)$, $f(x_1,...,x_n)$, $g_j(x_1,...,x_{j-1},x_{j+1},...,x_n,t)$ and $h_j(x_1,...,x_{j-1},x_{j+1},...,x_n,t)$ are supposed to have corresponding degree of smoothness. We assume that each of the carriers of the functionals Φ_j, $j = 1,...,n$, contains at least one point, different from 0. This is the reason to name the corresponding BVCs nonlocal. In the next considerations we suppose also that χ and Φ_j satisfy the normalizing restrictions:

$$\chi\{1\} = 1, \qquad \Phi_{j,\xi}\{\xi\} = 1\ , \qquad j = 1,...,n. \tag{5}$$

These restrictions are made for the sake of simplification and could be ousted by some unessential technical involvements.

1. Weak solutions of BVP (1) - (4)

It is natural to look for a classical solution of the BVP (1)-(4), but, in general, the sufficient conditions for the existence of such solutions may happen to be too restrictive. That's why we introduce the notion of a *weak solution* of (1)-(4). In order to give an exact meaning of this notion, we introduce some auxiliary notations. We introduce the right inverse operator l of $\dfrac{\partial}{\partial t}$:

$$lu(x_1,...,x_n,t) = \int_0^t u(x_1,...,x_n,\tau)d\tau - \chi_\tau\left\{\int_0^\tau u(x_1,...,x_n,\sigma)d\sigma\right\}$$

in $C = C(D) = C([0,a_1]\times ... \times[0,a_n]\times[0,\infty))$.

Analogically, we introduce the right inverse operators L_j of $\dfrac{\partial^2}{\partial x_j^2}$:

$$L_j u(x_1,...,x_n,t) = \int_0^{x_j}(x_j-\xi)u(x_1,...x_{j-1},\xi,x_{j+1},...,x_n,t)d\xi$$

$$-x_j\Phi_{j,\xi}\left\{\int_0^\xi(\xi-\eta)u(x_1,...x_{j-1},\eta,x_{j+1},...,x_n,t)d\eta\right\},$$

$j=1,...,n$ in $C=C(D)=C([0,a_1]\times...\times[0,a_n]\times[0,\infty))$.

Applying the product operator $lL_1...L_n$ to differential equation (1) and using initial and boundary value conditions (2)-(4) we get

$$\begin{aligned}L_1...L_nu-\sum_{j=1}^{n}lL_1...L_{j-1}L_{j+1}...L_nu \qquad (6)\\ =lL_1...L_nF(x_1,...,x_n,t)+L_1...L_nf(x_1,...,x_n)\\ +\sum_{j=1}^{n}(x_j\Phi_j\{1\}-1)lL_1...L_{j-1}L_{j+1}...L_ng_j(x_1,...x_{j-1},x_{j+1},...,x_n,t)\\ -\sum_{j=1}^{n}x_jlL_1...L_{j-1}L_{j+1}...L_nh_j(x_1,...x_{j-1},x_{j+1},...,x_n,t).\end{aligned}$$

Of course, here for consistence of the notations, we are to assume that under L_{n+1} one should understand the identity operator $L_{n+1}\equiv 1$.

Definition 1. A function $u(x_1,...,x_n,t)\in C^1(D)$ is said to be a *weak solution* of problem (1)-(4), iff it satisfies the integral relation (6).

It is easy to show that each classical solution of (1)-(4) is a weak solution too. If it happens $u\in C^2(D)$, then the converse is also true. Nevertheless, if u is only a weak solution but not necessarily classical, we can prove that it always satisfies the BVCs (2)-(4).

Lemma 1. *Let $u\in C^1(D)$ satisfies* (6). *Then u satisfies BVCs* (2)-(4).

P r o o f. Assume that u is a solution of (6) (weak, or classical). Applying the functional χ to (6), we find $L_1...L_n\chi_\tau\{u(x_1,...,x_n,\tau)\}=L_1...L_nf(x_1,...,x_n)$. Hence $\chi_\tau\{u(x_1,...,x_n,\tau)\}=f(x_1,...,x_n)$. For $x_j=0$ we find

$$-lL_1...L_{j-1}L_{j+1}...L_nu(x_1,...,x_{j-1},0,x_{j+1},...,x_n,t)$$

$$=-lL_1...L_{j-1}L_{j+1}...L_ng_j(x_1,...x_{j-1},x_{j+1},...,x_n,t),$$

and hence $u(x_1,...x_{j-1},0,x_{j+1},...,x_n,t)=g_j(x_1,...x_{j-1},x_{j+1},...,x_n,t)$. Next, applying Φ_j to (6), we get

$$-lL_1...L_{j-1}L_{j+1}...L_n\Phi_{j,\xi}\{u(x_1,...x_{j-1},\xi,x_{j+1},...,x_n,t)\}$$

$$= -lL_1...L_{j-1}L_{j+1}...L_nh_j(x_1,...x_{j-1},x_{j+1},...,x_n,t).$$

It remains to apply $\frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}...\frac{\partial^2}{\partial x_{j-1}^2}\frac{\partial^2}{\partial x_{j+1}^2}...\frac{\partial^2}{\partial x_n^2}$, we get

$$\Phi_{j,\xi}\{u(x_1,...x_{j-1},\xi,x_{j+1},...,x_n,t)\} = h_j(x_1,...x_{j-1},x_{j+1},...,x_n,t).$$

In a similar way, one can verify the fulfillment of all others BVCs. ∎

Lemma 2. *Assume that u is a solution of* (6) *with continuous partial derivatives u_t, $u_{x_jx_j}$, $j = 1,...,n$. Then u is a classical solution of* (2)-(4).

P r o o f. Applying the operator $\frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}...\frac{\partial^2}{\partial x_n^2}$ to (6), we get $u_t - u_{x_1x_1} - ... - u_{x_nx_n} = F(x_1,...,x_n,t)$. The fulfilment of the initial and boundary value conditions (2)-(4) follows from Lemma 1. ∎

Our final aim is to reduce the solution of BVP (1)-(4) in the case $\chi_\tau\{f(\tau)\} = f(0)$, to the following n nonlocal one-dimensional BVPs:

$$\frac{\partial v_j}{\partial t} - \frac{\partial^2 v_j}{\partial x_j^2} = 0, \quad v_j(x_j,0) = f_j(x_j), \quad v_j(0,t) = 0, \quad \Phi_{j,\xi}\{v_j(\xi,t)\} = 0, \qquad (7)$$

$j = 1,...,n$.

Lemma 3. *Let $v_j(x_j,t) \in C^1([0,a_j] \times [0,\infty))$, $j = 1,...,n$ be weak solutions of problems* (7). *Then $u(x_1,...,x_n,t) = v_1(x_1,t)...v_n(x_n,t) \in C(D)$ is a weak solution of the problem*

$$u_t - u_{x_1x_1} - ... - u_{x_nx_n} = 0, \quad u(x_1,...,x_n,0) = f_1(x_1)...f_n(x_n), \qquad (8)$$

$$u(x_1,...,x_{j-1},0,x_{j+1},...,x_n,t) = 0, \quad \Phi_{j,\xi}\{u(x_1,...,x_{j-1},\xi,x_{j+1},...,x_n,t)\} = 0,$$

in the sense of Definition 1.

Remark 1. If $v_j(x_j,t)$, $j = 1,...,n$, are classical solutions of (7), then we may assert that $u(x_1,...,x_n,t) = v_1(x_1,t)...v_n(x_n,t)$ is a classical solution of (8) too.

P r o o f. For the simplicity sake, we consider only the case $n = 2$. Consider the two-dimensional problem

$$\begin{aligned} u_t = u_{xx} + u_{yy}, \quad & u(x,y,0) = f(x)g(y), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < t \qquad (9)\\ & u(0,y,t) = 0, \quad \Phi_\xi\{u(\xi,y,t)\} = 0, \\ & u(x,0,t) = 0, \quad \Psi_\eta\{u(x,\eta,t)\} = 0, \end{aligned}$$

and the one-dimensional problems

$$v_t = v_{xx},\ v(x,0) = f(x), v(0,t) = 0,\ \Phi_\xi\{v(\xi,t)\} = 0,\quad 0 < x < a,\ 0 < t, \tag{10}$$

and

$$w_t = w_{yy},\ w(y,0) = g(y), w(0,t) = 0,\ \Psi_\eta\{w(\eta,t)\} = 0,\ 0 < y < b,\ 0 < t. \tag{11}$$

Assume that $u = u(x,t)$ and $v = v(y,t)$ are weak solutions of (10) and (11). Then

$$L_x v = lv + L_x f(x),\quad L_y w = lw + L_y g(y), \tag{12}$$

and we are to prove that:

$$L_x L_y vw - lL_y vw - lL_x vw = L_x L_y f(x)g(y). \tag{13}$$

Using (12) for the left side, we find

$$L_x L_y vw - lL_y vw - lL_x vw$$
$$= (lv)(lw) - l(v(lw)) - l(w(lv)) + (L_x f(x))(L_y g(y)).$$

In order to prove the assertion of Lemma 3, it remains to show that $(lv)(lw) - l(v(lw)) - l(w(lv)) = 0$. Indeed,

$$(lv)(lw)\ -\ l(v(lw))\ -\ l(w(lv)) = \Big(\int_0^t v(x,\tau)d\tau\Big)\Big(\int_0^t w(y,\tau)d\tau\Big)$$
$$-\int_0^t v(x,\tau)\Big(\int_0^\tau w(y,\theta)d\theta\Big)d\tau - \int_0^t w(y,\tau)\Big(\int_0^\tau v(x,\theta)d\theta\Big)d\tau = 0.$$

■

Example. Let $\Phi_\xi\{f(\xi)\} = \dfrac{2}{a}\displaystyle\int_0^a f(\xi)d\xi$ and $\Psi_\eta\{g(\eta)\} = \dfrac{2}{b}\displaystyle\int_0^b g(\eta)d\eta$. The weak solutions $V = V(x,t)$ and $W = W(y,t)$ of (10) and (11) for

$$f(x) = L_x\{x\} = \Big(\frac{x^3}{6} - \frac{x}{6}\Phi_\xi\{\xi^3\}\Big) = \frac{x^3}{6} - \frac{a^2\ x}{12}$$

and

$$g(y) = L_y\{y\} = \Big(\frac{y^3}{6} - \frac{y}{6}\Psi_\eta\{\eta^3\}\Big) = \frac{y^3}{6} - \frac{b^2\ y}{12}$$

are (see Dimovski [2]):

$$V(x,t) = -\frac{1}{2}\sum_{k=1}^{\infty} e^{-\frac{4k^2\pi^2}{a^2}t}\Bigg(\Big(\frac{a^3}{k^3\pi^3} + \frac{2a}{k\pi}t\Big)\sin\frac{2k\pi}{a}x - \frac{a^2}{k^2\pi^2}x\cos\frac{2k\pi}{a}x\Bigg) \tag{14}$$

and

$$W(y,t) = -\frac{1}{2}\sum_{k=1}^{\infty} e^{-\frac{4k^2\pi^2}{b^2}t}\left(\Big(\frac{b^3}{k^3\pi^3} + \frac{2b}{k\pi}t\Big)\sin\frac{2k\pi}{b}y - \frac{b^2}{k^2\pi^2}y\cos\frac{2k\pi}{b}y\right) \tag{15}$$

respectively. Then, according to Lemma 3,

$$U(x,y,t) = V(x,t)W(y,t) \tag{16}$$

is a weak solution of BVP (9)

2. Convolutions

2.1. One-dimensional convolutions

Definition 2. (Dimovski [2], p.52; [3]) For $\varphi, \psi \in C[0,\infty)$ define

$$(\varphi \overset{t}{*} \psi)(t) = \chi_\tau\left\{\int_\tau^t \varphi(t+\tau-\sigma)\psi(\sigma)d\sigma\right\}, \tag{17}$$

where the subscript τ to χ means that χ acts to the variable τ only.

Theorem 1. (Dimovski [2], p.52; [3]) *The operation* $(\varphi \overset{t}{*} \psi)(t)$ *is bilinear, commutative and associative in* $C[0,\infty)$*, and such that* $l\varphi = \{1\} \overset{t}{*} \varphi(t)$*, i.e.* $l = \{1\} \overset{t}{*}$.

Definition 3. (Dimovski [2], p.119) Let $f, g \in C[0,a_j]$. Then

$$(f \overset{x_j}{*} g)(x_j) = -\frac{1}{2}\tilde{\Phi}_{j,\xi}\left\{h(x_j,\xi)\right\}, \tag{18}$$

where $\tilde{\Phi}_{j,\xi} = \Phi_{j,\xi} \circ l_\xi$ with $l_{x_j} f(x_j) = \displaystyle\int_0^{x_j} f(\sigma)d\sigma$ and

$$h(x_j,\eta) = \int_{x_j}^{\eta} f(x_j+\eta-\xi)g(\xi)d\xi - \int_{-x_j}^{\eta} f(|\eta-x_j-\xi|)g(|\xi|)\ \mathrm{sgn}\ \xi(\eta-x_j-\xi)d\xi.$$

Theorem 2. (Dimovski [2], p.119) *The operation* $(f \overset{x_j}{*} g)(x_j)$ *is bilinear, commutative and associative in* $C[0,a_j]$*, and such that* $L_j f = \{x_j\} \overset{x_j}{*} f$*, i.e.* $L_j = \{x_j\} \overset{x_j}{*}$.

2.2. Higher dimensional convolutions

Important for applications to BVPs (1)-(4) with n space variables $x_1, ..., x_n$ and a time variable t are the following k-dimensional convolutions $\overset{x_1,...,x_k}{*}$ in $C([0,a_1] \times ... \times [0,a_k])$, $k = 2,3,...,n$. We are looking for a bilinear, commutative and associative operation in $C([0,a_1] \times ... \times [0,a_k])$ such that for $u = f_1(x_1)f_2(x_2)...f_k(x_k)$ and $v = g_1(x_1)g_2(x_2)...g_k(x_k)$, to have $u \overset{x_1,...,x_k}{*} v = (f_1 \overset{x_1}{*} g_1)(f_2 \overset{x_2}{*} g_2)...(f_k \overset{x_k}{*} g_k)$.

Also we use $(k+1)$-dimensional convolutions $\overset{x_1,...,x_k,t}{*}$ in $C([0,a_1] \times ... \times [0,a_k] \times [0,\infty))$, $k = 1,2,3,...,n$. Here we are looking for a bilinear, commutative and associative operation in $C([0,a_1] \times ... \times [0,a_k] \times [0,\infty))$ such that if $u = f_1(x_1)f_2(x_2)...f_k(x_k)\varphi(t)$ and $v = g_1(x_1)g_2(x_2)...g_k(x_k)\psi(t)$, then $u \overset{x_1,...,x_k,t}{*} v = (f_1 \overset{x_1}{*} g_1)(f_2 \overset{x_2}{*} g_2)...(f_k \overset{x_k}{*} g_k)(\varphi \overset{t}{*} \psi)$.

Inductively, we define such higher-dimensional convolutions $\overset{x_1,...,x_k}{*}$ in the spaces $C([0,a_1] \times ... \times [0,a_k])$, $k = 2,3,...,n$ and then $\overset{x_1,...,x_k,t}{*}$ in $C([0,a_1] \times ... \times [0,a_k] \times [0,\infty))$. For the whole space of the continuous functions in $D = [0,a_1] \times ... \times [0,a_n] \times [0,\infty)$ we denote the corresponding convolution $\overset{x_1,...,x_n,t}{*}$ simply by $*$.

Definition 4. For $u, v \in C[0,a_1]$ take the convolution product $u \overset{x_1}{*} v$ as it is defined by (18). Let $u, v \in C([0,a_1] \times ... \times [0,a_k])$, $k = 2,...,n$. Then

$$u(x_1,...,x_k) \overset{x_1,...,x_k}{*} v(x_1,...,x_k) = -\frac{1}{2}\tilde{\Phi}_{k,\xi}\{h_{k-1}(x_1,...,x_k,\xi)\} \tag{19}$$

with

$$h_{k-1}(x_1,...,x_k,\xi) = \int_{x_k}^{\xi} u(x_1,...,x_{k-1},\xi + x_k - \eta) \overset{x_1,...,x_{k-1}}{*} v(x_1,...,x_{k-1},\eta)d\eta -$$

$$- \int_{-x_k}^{\xi} u(x_1,...,x_{k-1},|\xi - x_k - \eta|) \overset{x_1,...,x_{k-1}}{*} v(x_1,...,x_{k-1},|\eta|) \operatorname{sgn} \eta(\xi - x_k - \eta)d\eta.$$

Theorem 3. *The operation* $u(x_1,...,x_k) \overset{x_1,...,x_k}{*} v(x_1,...,x_k)$*, defined by* (19) *is bilinear, commutative and associative in* $C([0,a_1] \times ... \times [0,a_k])$*, and such that*

$$L_{x_1}...L_{x_k}u(x_1,...,x_k) = \{x_1...x_k\} \overset{x_1,...,x_k}{*} u(x_1,...,x_k). \tag{20}$$

Definition 5. Let $u, v \in C([0,a_1] \times ... \times [0,a_k] \times [0,\infty))$. Then

$$(u \overset{x_1,...,x_k,t}{*} v)(x_1,...,x_k,t) = \tag{21}$$
$$= \chi_\tau \left\{ \int_\tau^t u(x_1,...,x_k,t+\tau-\sigma) \overset{x_1,...,x_k}{*} v(x_1,...,x_k,\sigma) d\sigma \right\},$$

$k = 1, 2, ..., n$.

Theorem 4. *Operation* $u(x_1,...,x_k,t) \overset{x_1,...,x_k,t}{*} v(x_1,...,x_k,t)$*, defined by* (21) *is bilinear, commutative and associative in* $C([0,a_1] \times ... \times [0,a_k] \times [0,\infty))$*, and such that*

$$lL_1...L_k u(x_1,...,x_k) = \{x_1...x_k\} \overset{x_1,...,x_k,t}{*} u(x_1,...,x_k,t). \tag{22}$$

Outline of the proofs (of Theorem 3 and 4). They are to be verified, first, for product functions and then one should use approximation argument, based on the multi-dimensional Stone-Weierstrass theorem [5].

3. Multipliers of $(C, *)$

Further, we introduce the ring of the multipliers of the convolution algebra $(C, *)$.

Definition 6. [7] A linear operator $M : C \to C$ is said to be a multiplier of the algebra $(C, *)$, iff the relation

$$M(u * v) = (Mu) * v \tag{23}$$

holds for all $u, v \in C$.

Important for the next consideration are the so-called partial numerical multipliers. Let $F = F(x_1,...,x_{j-1},x_{j+1},...,x_n,t) \in C([0,a_1] \times ... \times [0,a_{j-1}] \times [0,a_{j+1}]...\times[0,a_n]\times[0,\infty))$ be a function of the variables $x_1,...,x_{j-1},x_{j+1},...,x_n,t$ only and $G = G(x_1,...,x_n) \in C([0,a_1]\times...\times[0,a_n])$ be a function of the variables $x_1,...,x_n$ only, but both considered as functions of $C(D)$. The operators $[F]_{x_j}$ and $[G]_t$ defined by $[F]_{x_j} u = F \overset{x_1,...,x_{j-1},x_{j+1},...,x_n,t}{*} u$ and $[G]_t u = G \overset{x_1,...,x_n}{*} u$ are said to be *partial numerical operators* with respect to x_j and t, correspondingly.

The set of all multipliers of the convolution algebra $(C, *)$ is a commutative ring $\mathfrak{M}$ (see [7]). As a rule, in $\mathfrak{M}$ there are elements, which are divisors of zero. Nevertheless, in $\mathfrak{M}$ surely there are elements, which are non-divisors of zero. Such elements are e.g. the multipliers $\{x_j\} \overset{x_j}{*}$, i.e. the operators L_j, $j = 1, ..., n$ and also $\{1\} \overset{t}{*}$ i.e. the operator l.

Denote by $\mathfrak{N}$ the set of the non-zero non-divisors of zero on $\mathfrak{M}$. The set $\mathfrak{N}$ is a multiplicative subset on $\mathfrak{M}$, i.e. such that $p, q \in \mathfrak{N}$ implies $p\, q \in \mathfrak{N}$.

Further, we consider the multiplier fractions of the form $\dfrac{M}{N}$ with $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$. They are introduced in a standard manner, using the well-known method of "localization" from the general algebra [6]. The set of all multiplier fractions of $(C, *)$, denoted by $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$, is a commutative ring. Basic for our construction are the algebraic inverses $S_j = \dfrac{1}{L_j}$ and $s = \dfrac{1}{l}$ of the multipliers L_j and l in $\mathcal{M}$, correspondingly. If $u \in C^2(D)$, then, in general $S_j u$ and su are different from $u_{x_j x_j}$ and u_t, but they are connected with them.

Theorem 5. *Let $u \in C$ be such that it has continuous partial derivatives u_t and $u_{x_j x_j}$ in $C(D)$. Then*

$$\frac{\partial^2 u}{\partial x_j^2} = S_j u + S_j\{(x_j \Phi_j\{1\} - 1)u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n, t)\} \\ -[\Phi_{j,\xi}\{u(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n, t)\}]_{x_j}, \tag{24}$$

and

$$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x_1, ..., x_n, \tau)\}]_t. \tag{25}$$

The verification of (24) and (25) is straightforward by application of the operators L_j to (24) and l to (25)

4. Algebraization and formal solution of (1)-(4)

Let us consider problem (1)-(4). The equation (1) together with the initial and boundary conditions (2)-(4) can be reduced to a single algebraic equation for u in $\mathcal{M}$. Indeed, by Theorem 5, using (2)-(4), we get:

$$\frac{\partial^2 u}{\partial x_j^2} = S_j u + S_j\{(x_j \Phi_j\{1\} - 1)g_j\} - [h_j]_{x_j}, \tag{26}$$

$$\frac{\partial u}{\partial t} = su - [f]_t. \tag{27}$$

Then, BVP (1)-(4) reduces to the following algebraic equation in $\mathcal{M}$:

$$(s - S_1 - ... - S_n)u = F + [f]_t + \sum_{j=1}^{n} \left(S_j\{(x_j \Phi_j\{1\} - 1)g_j\} - [h_j]_{x_j}\right). \tag{28}$$

If $s - S_1 - ... - S_n$ is an non-divisor of zero in $\mathcal{M}$, then equation (28) has the following solution in $\mathcal{M}$:

$$u = \frac{1}{s - S_1 - ... - S_n}\Big(F + [f]_t + \sum_{j=1}^{n}\big(S_j\{(x_j\Phi_j\{1\} - 1)g_j\} - [h_j]_{x_j}\big)\Big). \quad (29)$$

It may be called a *formal solution* of BVP (1)-(4). In order to obtain exact solution (weak, or classical) of the BVP (1)-(4) we need to interpret (29) as a function of $C(D)$.

5. Interpretation of the formal solution as a function

Our next task is to interpret (29) as a function of $C(D)$. To this end, we consider (1)-(4) for $f(x_1, ..., x_n) = L_1\{x_1\}...L_n\{x_n\}$, $F(x_1, ..., x_n, t) \equiv 0$, $h_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n, t) \equiv 0$, $g_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n, t) \equiv 0$, $j = 1, ..., n$. We denote its weak solution, if it exists, by $\Omega = \Omega(x_1, ..., x_n, t)$. We have the following algebraic representation of this *formal* solution:

$$\begin{aligned}\Omega &= \frac{1}{s - S_1 - ... - S_n}[L_1\{x_1\}...L_n\{x_n\}]_t \qquad (30)\\ &= \frac{1}{s - S_1 - ... - S_n}L_1^2...L_n^2 = \frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)}.\end{aligned}$$

Next, without any loss of generality we propose

$$g_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n, t) \equiv 0.$$

The formal solution (29) for arbitrary $F(x_1, ..., x_n, t)$, $f(x_1, ..., x_n)$ and $h_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n, t)$, $j = 1, ..., n$ can by represented in the form:

$$\begin{aligned}u = S_1^2...S_n^2\Bigg(&\frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)}F \qquad (31)\\ &+\frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)}[f]_t - \sum_{j=1}^{n}\Big(\frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)}[h_j]_{x_j}\Big)\Bigg).\end{aligned}$$

As a function it, takes the form

$$u = \frac{\partial^4}{\partial x_1^4}...\frac{\partial^4}{\partial x_n^4}\Bigg(\Omega * F + \Omega \overset{x_1,...,x_n}{*} f - \sum_{j=1}^{n}\Big(\Omega \overset{x_1,...,x_{j-1},x_{j+1},...,x_n,t}{*} h_j\Big)\Bigg), \quad (32)$$

provided the functions F, f and h_j are sufficiently smooth.

6. Reducing of the solution of BVP (1)-(4) for $\chi\{f\} = f(0)$ to the one-dimensional case

In the case $\chi\{f\} = f(0)$, the solution (32) can be represented by the product of solutions of one-dimensional BVPs. Next, for the simplicity sake we consider only the following BVP:

$$u_t - u_{x_1x_1} - ... - u_{x_nx_n} = 0, \quad u(x_1, ..., x_n, 0) = f(x_1, ..., x_n), \tag{33}$$

$$u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n, t) = 0, \quad \Phi_{j,\xi}\{u(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n, t)\} = 0,$$
$j = 1, ..., n$.

Consider the representation of the solution of BVP (33) in the form (32). Denote the weak solution of BVP (33) for $f(x_1, ..., x_n) = L_1\{x_1\}...L_n\{x_n\}$, if it exists by $U = U(x_1, ..., x_n, t)$. We have the following algebraic representation of this solution in $\mathcal{M}$:

$$U = \frac{1}{s - S_1 - ... - S_n}[L_1\{x_1\}...L_n\{x_n\}]_t = \frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)}. \tag{34}$$

Analogically, we denote by $U_j = U_j(x_j, t)$, $j = 1, ..., n$ the weak solutions of the problems

$$\frac{\partial v_j}{\partial t} - \frac{\partial^2 v_j}{\partial x_j^2} = 0, \quad v_j(x_j, 0) = f_j(x_j), \tag{35}$$
$$v_j(0, t) = 0, \quad \Phi_{j,\xi}\{v_j(\xi, t)\} = 0, \quad j = 1, ..., n,$$

for $f_j(x_j) = L_j\{x_j\}$, if they exist. The algebraic representations of these solutions are

$$U_j = \frac{1}{s - S_j}[L_j\{x_j\}]_t = \frac{1}{S_j^2(s - S_j)}.$$

Theorem 6. *Assume that* $U_j = \dfrac{1}{S_j^2(s - S_j)}$, $j = 1, ..., n$, *are weak solutions of BVPs* (35) *for* $f_j(x_j) = L_j\{x_j\}$. *Then*

$$U = \frac{1}{S_1^2...S_n^2(s - S_1 - ... - S_n)} = \prod_{j=1}^{n} U_j(x_j, t),$$

where $\prod_{j=1}^{n}$ *denotes the ordinary product, is a weak solution of* (33) *for* $f(x_1, ..., x_n) = L_1\{x_1\}...L_n\{x_n\}$.

The proof follows immediately from Lemma 3.

Now, using the general representation (32) we can write the explicit solution of problem (33) for arbitrary $f(x_1, ..., x_n)$, in the following "nested" form

$$u = \frac{\partial^4}{\partial x_1^4} \cdots \frac{\partial^4}{\partial x_n^4} \left(U \overset{x_1,\dots,x_n}{*} f \right) \tag{36}$$

$$= \frac{\partial^4}{\partial x_1^4} \cdots \frac{\partial^4}{\partial x_n^4} \left(U_1 \overset{x_1}{*} \left(U_2 \overset{x_2}{*} \left(U_3 \overset{x_3}{*} \dots \overset{x_{n-1}}{*} \left(U_n \overset{x_n}{*} f \right) \dots \right) \right) \right).$$

If $f(x_1, ..., x_n) = f_1(x_1)...f_n(x_n)$, then the solution of the problem (33) simplifies to the product

$$u = \prod_{j=1}^{n} \frac{\partial^4}{\partial x_j^4} \left(U_j \overset{x_j}{*} f_j \right). \tag{37}$$

Example. Solve the boundary value problem (a two-dimensional generalization of Ionkin's problem [4]):

$$\begin{gathered} u_t = u_{xx} + u_{yy}, \quad 0 < x < a, \quad 0 < y < b, \quad 0 < t, \\ u(x, y, 0) = f(x, y), \\ u(0, y, t) = 0, \quad \int_0^a u(\xi, y, t) d\xi = 0, \\ u(x, 0, t) = 0, \quad \int_0^b u(x, \eta, t) d\eta = 0. \end{gathered} \tag{38}$$

Theorem 8. *Let $f \in C(D)$ be such that f_x and f_y are continuous, $f(0, y) = f(x, 0) = 0$ and $\int_0^a f(\xi, y) d\xi = \int_0^b f(x, \eta) d\eta = 0$. Then*

$$u = \frac{\partial^4}{\partial x^4} \frac{\partial^4}{\partial y^4} \left(V \overset{x}{*} \left(W \overset{y}{*} f \right) \right), \tag{39}$$

is a weak solution of (38). Here $V = V(x, t)$ and $W = W(y, t)$ are the solutions (14) and (15) of the corresponding one-dimensional Ionkin's problems. If suppose additionally that f has continuous second derivative f_{xx}, f_{yy}, then (39) is a classical solution of (38).

In the special case $f(x,y) = f_1(x)f_2(y)$, then the solution of (38) is:

$$u = \frac{\partial^4}{\partial x^4}\Big(V \overset{x}{*} f_1\Big)\frac{\partial^4}{\partial y^4}\Big(W \overset{y}{*} f_2\Big) \tag{40}$$

$$= \frac{1}{a^2b^2}\left(\int_x^a f_1'(\xi)V_x(a+x-\xi,t)d\xi - \int_{-x}^a f_1'(|\xi|)V_x(|a-x-\xi|,t)d\xi \right.$$
$$\left. +2\int_0^x f_1'(\xi)V_x(x-\xi,t)d\xi\right)$$
$$\times\left(\int_y^b f_2'(\eta)W_y(b+y-\eta,t)d\eta - \int_{-y}^b f_2'(|\eta|)W_y(|b-y-\eta|,t)d\eta + \right.$$
$$\left. +2\int_0^y f_2'(\eta)W_y(y-\eta,t)d\eta\right),$$

where $V_x = \dfrac{\partial}{\partial x}V$ and $W_y = \dfrac{\partial}{\partial y}W$.

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