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# Inverse Problems Involving Generalized Axial-Symmetric Helmholtz Equation

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This paper deals with inverse problems related to the solutions of the Helmholtz equation

$$L_{0,k}u_0 = u_{0xx} + u_{0yy} + ku_0 = 0 \ \ (k = const. > 0)$$

for domains of two types. The generalized axial-symmetric Helmholtz equation

$$L_{\nu,k}u = u_{xx} + u_{yy} + \frac{2\nu}{y}u_y + ku = 0 \ \ (\nu, k = const. > 0)$$

is also considered. The inversion formulae obtained here may be used to reduce the boundary value problems for these equations to the boundary value problems of the theory of analytic functions. Solution of a radiation problem is given as an example.

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*Key Words*: Helmholtz equation, inversion formula, Bessel functions, analytic function, radiation problem

## 1. Introduction

The Helmholtz equation often arises in the study of physical problems involving partial differential equations in both space and time [6], [7]. The Helmholtz equation plays an important role in problems of electromagnetic radiation, seismology and acoustics [1], [2].

Let $G$ be a simply connected domain in the complex plane $z = x + iy$, and symmetrical with respect to the real axis. Further, let $f_m(z)$ $(m = \overline{0,1})$ be two arbitrary analytical functions in $G$, representing the initial condition. The imaginary parts of these functions are zero. Consider the Helmholtz equation

$$L_{0,k}u_0 = u_{0xx} + u_{0yy} + ku_0 = 0 \ \ (k > 0), \tag{1}$$

and the axial symmetric Helmholtz equation

$$L_{\nu,k}u = u_{xx} + u_{yy} + \frac{2\nu}{y}u_y + ku = 0 \quad (\nu \text{ and } k \text{ are positive constants}). \tag{2}$$

Then the function defined by the following formula

$$u_0(x,y) = Re\left[f_0(z) + \frac{\sqrt{k}}{2}(z-\bar{z})\int_a^z f_0(t)\frac{J_1\left(\sqrt{k(t-z)(t-\bar{z})}\right)}{\sqrt{(t-z)(t-\bar{z})}}dt\right]$$
$$+J_m\int_a^z f_1(t)J_0\left(\sqrt{k(t-z)(t-\bar{z})}\right)dt \tag{3}$$

is a regular solution of equation (1) in $G$; here $a$ is an arbitrary point on $d = G\cap\{y=0\}$, the integration takes place along the rectifiable contour $\Gamma \subset G\cap\{y=0\}$ or $\Gamma \subset G\bigcap\{y<0\}$ and $J_v(..)$ is the Bessel function of first kind of order $\nu$, [6].

Let us note that (3) gives the general integral representation of all regular solutions of the equation (1) in $G$, involving two arbitrary analytical functions.

**Definition 1.** The domain $G_1$ which is symmetrical with respect to the real axis belongs to class $A$ if it contains a segment connecting any two of its points with the same abscissa.

**Definition 2.** The domain $G_2$ which is symmetrical with respect to the real axis belongs to class $B$ if it contains a segment of line leading from $\infty$ to arbitrary point $z$ parallel to O$y$.

Let us consider the domain $G_1$ in (3) with contour - the vertical segment connecting the point $z = x$ with the point $z = x + iy$, then we get the integral representation of solutions of the Helmholtz equation (1) with the help of analytical functions $f_m(z) = \phi_m(x,y) + i\psi_m(x,y)$ $(m = \overline{0,1})$:

$$u_0(x,y) = \phi_0(x,y) - ky\int_0^y \frac{J_1\left(\sqrt{k(y^2-\tau^2)}\right)}{\sqrt{k(y^2-\tau^2)}}\phi_0(x,\tau)d\tau$$
$$+\int_0^y J_0\left(\sqrt{k(y^2-\tau^2)}\right)\phi_1(x,\tau)d\tau. \tag{4}$$

If $f_m(z)$ are analytical functions in the domain $G_2$ and

$$f_m(z)\cos\sqrt{k}z\cdot z^{-\frac{1}{2}} = \mathrm{O}\left(\frac{1}{|z|^{\varepsilon}}\right), \quad z\to\infty, \tag{5}$$

with contour of integration - the segment connecting the points $z$ and $\bar{z}$ and passing through $\infty$, we get the integral representation of the solutions of equation

(1) in the following form:

$$u_0(x,y)=\phi_0(x,y)-ky\int_y^\infty \phi_o(x,\tau)\frac{\mathrm{I}_1\left(\sqrt{k(\tau^2-y^2)}\right)}{\sqrt{k(\tau^2-y^2)}}d\tau+\int_y^\infty\left(\frac{\partial\phi_1(x,\tau)}{\partial x}\right.$$

$$\left.+\sqrt{k}\phi_1(x,\tau)\ \mathrm{I}_0\left(\sqrt{k\left(\tau^2-y^2\right)}\right)\right)d\tau,\quad \phi_0=Ref_0(z). \tag{6}$$

Let us suppose $f_1(z)=0$ in (4) and (6). If the domain $G$ belongs simultaneously to the class $A$ and to the class $B$ (e.g., semi-plane), $f_0(z)$ is analytical function in $G$ as

$$J_m\ f_0(z)|_{z=x}=0,$$

the condition (5) is valid. Then the solutions of equation (1), defined by (4) and (6) respectively, coincide. This fact is important for solving boundary value problems.

## 2. Solution of integral equations

Now we consider the inverse problems related to the solutions of the Helmholtz equations (4) and (6).

**Theorem 1.** *The integral representation (4) is an integral equation of Volterra type, and its solution can be expressed as follows:*

$$\phi_0(x,y)=u_0(x,y)+ky\int_0^y u_o(x,\tau)\frac{\mathrm{I}_1\left(\sqrt{k(y^2-\tau^2)}\right)}{\sqrt{k(y^2-\tau^2)}}d\tau, \tag{7}$$

*where $I_\nu\ (x)$ is the modified Bessel function of the first kind.*

Using the Carson-Laplace transform [8] and the convolution theorem, we get the above result.

**Theorem 2.** *Let $f_0(z)$ satisfies the following condition*

$$f_0(z)\cos\sqrt{k}zz^{-\frac{1}{2}}=\mathrm{O}\left(|z|^{-\varepsilon}\right),\quad |z|\to\infty, \tag{8}$$

*then the solution of the integral equation (6) has the following form:*

$$\phi_0(x,y)=u_0(x,y)-ky\int_y^\infty u_o(x,\tau)\frac{J_1\left(\sqrt{k(\tau^2-y^2)}\right)}{\sqrt{k(\tau^2-y^2)}}d\tau, \tag{9}$$

$$\left(\phi_0(x,y)=Ref_0(z)\right).$$

The above result can be easily derived by using the Carson-Laplace transform [8] and the convolution theorem

$$\int_x^\infty f(y)g(y-x)dy \overset{\bullet}{\underset{\bullet}{\longleftrightarrow}} -\frac{\tilde{f}(p)\tilde{g}(-p)}{p}\,.$$

**3. Inverse problems related to the generalized Helmholtz equation**

Let $G$ be a simply connected domain in the complex plane $z = x + iy$, $G$ be symmetrical with respect to the real axis, $\tilde{f}_0(z)$ be an arbitrary analytical function in $G$, then the function

$$u(x,y) = -iC_\nu \left(\frac{z-\bar{z}}{2i}\right)^{1-2\nu} \int_{\bar{z}}^{z} \tilde{f}_0(\sigma)\, {}_0F_1\left(v; -\frac{k}{4}(z-\sigma)(\bar{z}-\sigma)\right)$$

$$\times \left[(z-\delta)(\bar{z}-\delta)\right]^{v-1} d\sigma \tag{10}$$

is a regular solution of the Helmholtz equation (2) and satisfies the condition

$$u(x,0) = \tilde{f}_0(x),$$

where

$$C_v = \frac{\Gamma\left(v+\frac{1}{2}\right)}{\Gamma(v)\Gamma\left(\frac{1}{2}\right)},\ {}_0F_1(v;z) = \Gamma(v)(i\sqrt{z})^{-v+1} J_{v-1}(2i\sqrt{z}).$$

The formula (10) establishes the one-to-one correspondence between the regular solutions of equation (2) as $k > 0$ and the analytical functions in $G$. The integrand involves the hypergeometric function, [5], [6].

**Theorem 3.** *Let the integral representation of the solutions of equation (3) in $G_1$ has the form*

$$u(x,y) = \frac{\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}} y^{1-2v} k^{\frac{1-v}{2}} 2^v \int_0^y Re\left[\tilde{f}_0(x+i\tau)\right]$$

$$\times J_{v-1}\left(\sqrt{k(y^2-\tau^2)}\right)\left(y^2-\tau^2\right)^{\frac{v-1}{2}} d\tau, \tag{11}$$

*then the solution of the integral equation (11) has the following form:*

$$Re\left(f_0(x+iy)\right)=\begin{cases}\frac{2^{m-v}k^{\frac{v-m}{2}}}{\Gamma(v)C_v}\frac{d}{dy}\int_0^y\frac{d^m\left[\tau^{2v-1}u(x,\tau)\right]}{(d\tau^2)^m}\left(y^2-\tau^2\right)^{\frac{m-v}{2}}\tau\\ \times\mathrm{I}_{m-v}\left(\sqrt{k(y^2-\tau^2)}\right)d\tau,\quad m=[v],\quad v-1\neq n;\\ \frac{1}{n!}C_v y\frac{d^{n+1}\left[y^{2v-1}u(x,y)\right]}{(dy^2)^{n+1}}+\frac{\sqrt{k}y}{n!C_v}\int_0^y\frac{d^{n+1}\left[\tau^{2v-1}u(x,\tau)\right]}{(d\tau^2)^{n+1}}\\ \times\frac{\mathrm{I}_1\left(\sqrt{k(y^2-\tau^2)}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau,\quad v-1=n.\end{cases}\tag{12}$$

*Here* $v>0,\ u(x,y)\in C^2(y\neq 0),\ u(x,\pm 0)$ *is a restricted function, and as* $0<v<\frac{1}{2}$ *we have*

$$\lim_{y\to 0}|y|^{2v}u_y(x,y)=0.$$

P r o o f. Using the fact, that $u_0(x,y)$ is the solution of equation (1) and the fact that

$$u(x,y)=2C_v y^{1-2v}\int_0^y u_0(x,y)\left(y^2-\tau^2\right)^{v-1}d\tau \tag{13}$$

is the solution of equation (2), we get (12). ∎

**Theorem 4.** *If the domain G belongs to the class B, the function* $\tilde{f}_0(z)$ *satisfies the condition*

$$\tilde{f}_0(z)z^{\frac{2v-1}{2}}\cos\left(\sqrt{k}\left(z-\frac{v\pi}{2}+\frac{\pi}{4}\right)\right)=\mathrm{O}\left(\frac{1}{|z|^{\varepsilon}}\right),\ |z|\to\infty, \tag{14}$$

*and (10) has the following form:*

$$u(x,y)=\frac{\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}}2^v y^{1-2v}k^{\frac{1-v}{2}}\int_{\infty}^{y}Re\left[\tilde{f}_0(x+i\tau)e^{-i\pi(v-1)}\right]$$
$$\times\mathrm{I}_{v-1}\left(\sqrt{k(\tau^2-y^2)}\right)\left(\tau^2-y^2\right)^{\frac{1-v}{2}}d\tau, \tag{15}$$

*then the solution of equation (15) has the following form:*

$$Re\left[\left(\tilde{f}_0(x+iy)\right)e^{-i\pi(v-1)}\right]=$$
$$=\begin{cases}\frac{(-1)^m 2^{m-v}k^{\frac{v-m}{2}}}{\Gamma(v)C_v}\frac{d}{dy}\int_y^{\infty}\frac{d^m\left[\tau^{2v-1}u(x,\tau)\right]}{(d\tau^2)^m}\left(\tau^2-y^2\right)^{\frac{m-v}{2}}\tau\\ \times\mathrm{I}_{m-v}\left(\sqrt{k(\tau^2-y^2)}\right)d\tau,\ m=[v],\ v-1\neq n;\\ \frac{(-1)^n}{n!C_v}y\frac{d^{n+1}\left[y^{2v-1}u(x,y)\right]}{(dy^2)^{n+1}}+\frac{(-1)^n\sqrt{k}y}{n!C_v}\int_y^{\infty}\frac{d^{n+1}\left[\tau^{2v-1}u(x,\tau)\right]}{(d\tau^2)^{n+1}}\\ \times\frac{\mathrm{I}_1\left(\sqrt{k(\tau^2-y^2)}\right)}{\sqrt{\tau^2-y^2}}\tau d\tau,\ v-1=n.\end{cases}\tag{16}$$

P r o o f. The formula (16) is established by an appeal to the relation (9), and also by the help of an assertion that when $u_0(x,y)$ is the solution of equation (1), then

$$u(x,y) = 2C_v y^{1-2v} \int_y^\infty u_0(x,\tau)\left(\tau^2 - y^2\right)^{v-1} d\tau$$

is the solution of equation (2). ■

**4. Particular cases**

If we replace $\sqrt{k}$ by $i\sqrt{k}$ in (2), and set $v = \frac{1}{2}$, then we get the following equation

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{1}{y}\frac{\partial \tilde{u}}{\partial y} - k\tilde{u} = 0, \quad k > 0. \tag{17}$$

Then formulae (11) and (12) will take the following form as solution of the equation (17) and its inversion formula:

$$\tilde{u}(x,y) = \frac{2}{\pi}\int_0^y Re\left[\tilde{f}_0(x+i\tau)\right]\frac{ch\left(\sqrt{k\left(y^2-\tau^2\right)}\right)}{\sqrt{y^2-\tau^2}}d\tau, \tag{18}$$

$$Re\left[\tilde{f}_0(x+iy)\right] = \frac{\partial}{\partial y}\int_0^y \tilde{u}(x,\tau)\frac{\cos\left(\sqrt{k\left(y^2-\tau^2\right)}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau. \tag{19}$$

Similarly, (15) and (16) reduce to the following, as solution of equation (17) and its inverse:

$$\tilde{u}(x,y) = \frac{2}{\pi}\int_y^\infty J_m\left[\tilde{f}_0(x+i\tau)\right]\frac{\cos\left(\sqrt{k\left(\tau^2-y^2\right)}\right)}{\sqrt{\tau^2-y^2}}d\tau, \tag{20}$$

$$J_m\left[\tilde{f}_0(x+iy)\right] = -\frac{\partial}{\partial y}\int_y^\infty \tilde{u}(x,\tau)\frac{ch\left(\sqrt{k\left(\tau^2-y^2\right)}\right)}{\sqrt{\tau^2-y^2}}\tau d\tau. \tag{21}$$

Here

$$ch\left(\sqrt{k}z\right)\tilde{f}_0(z) = \mathrm{O}\left(\frac{1}{|z|^\varepsilon}\right), \quad |z| \to \infty. \tag{22}$$

Let us consider the following

**Example.** Find the concentration of the (radioactive) radiation in the semi-space X > 0. This radiation is due to the distribution of the radioactive substance with constant concentration $H_0$ in the circle of radius $a$ on the plane $X = 0$, with the condition that the remainder of plane is impenetrable. The radiation measurement is very important in several fields, see e.g. [1]–[4].

Here we have the Helmholtz equation (17), where $\tilde{u}(x, y)$ represents the radiation field and $k$ is distribution constant.

Taking into account the axis symmetry, we have the following boundary value problem:

*Find the solution of the Helmholtz equation (17) in the semi-plane $x > 0$ with the boundary conditions:*

$$\tilde{u}(0, y) = H_0, \qquad |y| \le a, \tag{23}$$

$$\frac{\partial \tilde{u}(0, y)}{\partial x} = 0, \qquad |y| > a. \tag{24}$$

The solution of this problem can be derived from (18).

Let us introduce the function

$$\Omega(z) = u_0(x, y) + iv_0(x, y) = \tilde{f}_0(z) ch\sqrt{k}z.$$

Since $\tilde{f}_0(z)$ satisfies the condition (22), then by help of the formulae (19) and (21) we obtain

$$u_0(x, y)|_{x=0} = \begin{cases} H_0 \cos^2\left(\sqrt{k}y\right), & |y| \le a, \\ 0 & , \quad |y| > a. \end{cases}$$

Taking into account that

$$u_0(0, -y) = u_0(0, y),$$

and assume the validity of equality

$$|u_0(0, y)| \le \frac{M}{|y|^{\varepsilon}}, \qquad (\varepsilon, M = const. > 0),$$

using the K. Schwarz formula [9], we get

$$\Omega(z) = u_0(x, y) + iV_0(x, y) = -\frac{H_0}{\pi i} \int_{-a}^{a} \frac{\cos^2\left(\sqrt{k}t\right)}{t + iz} dt$$

$$= \frac{H_0}{\pi} x \int_0^a \cos^2\left(\sqrt{k}t\right) \left[\frac{1}{(t - y)^2 + x^2} + \frac{1}{(t - y)^2 + x^2}\right] dt$$

$$+i\frac{H_0}{\pi}\int_0^a \cos^2\left(\sqrt{k}t\right)\left[\frac{t-y}{(t-y)^2+x^2}-\frac{t+y}{(t+y)^2+x^2}\right]dt.$$

Finally, the solution of the problem has the following form:

$$\tilde{u}(x,y)=\frac{2}{\pi^2}H_0x\int_0^y \frac{ch\left(\sqrt{k}x\right)\cos\left(\sqrt{k}\tau\right)}{ch^2\sqrt{k}x\cos^2\sqrt{k}\tau+sh^2\sqrt{k}x\sin^2\sqrt{k}\tau}$$

$$\times\frac{ch\sqrt{k\left(y^2-\tau^2\right)}}{\sqrt{y^2-\tau^2}}d\tau\int_0^a\cos^2\sqrt{k}t\left[\frac{1}{(t-\tau)^2+x^2}+\frac{1}{(t+\tau)^2+x^2}\right]dt$$

$$-\frac{2}{\pi^2}H_0\int_0^y\frac{sh\sqrt{k}x\sin\sqrt{k}\tau}{ch^2\sqrt{k}x\cos^2\sqrt{k}\tau+sh^2\sqrt{k}x\sin^2\sqrt{k}\tau} \tag{25}$$

$$\times\frac{ch\sqrt{k\left(y^2-\tau^2\right)}}{\sqrt{y^2-\tau^2}}d\tau\int_0^a\cos^2\sqrt{k}t\left[\frac{t-\tau}{(\tau-t)^2+x^2}-\frac{t+\tau}{(t+\tau)^2+x^2}\right]dt.$$

It is interesting to observe the behavior of the solution $\tilde{u}(x,y)$ on the singular line $y=0$. Let us notice that $\tilde{u}(x,y)$ has extremum (maximum) at the point $y=0$ (with $x$); $\tilde{u}(x,y)$ as the function of $y$ has no extremum.

The formula (25) for $y=0$ is:

$$\tilde{u}_0(x,y)=H_0\frac{x}{\pi ch\sqrt{k}x}\int_0^a\frac{\cos^2\sqrt{k}t}{t^2+x^2}dt. \tag{26}$$

It is evident that the distribution of the radioactive emanation depends on the height and decreases exponentially. Indeed:

$$\tilde{u}_0(x,y)=\frac{2H_0}{x\pi ch\sqrt{k}x}\int_0^a\cos^2\sqrt{k}t\left[1-\left(\frac{t}{x}\right)^2+0\left(\frac{t}{x}\right)^2\right]dt$$

$$=\frac{\tilde{H}_0}{xch(\sqrt{k}x)},$$

where

$$\tilde{H}_0=\frac{2H_0}{\pi}\left[\frac{a}{2}+\frac{1}{4\sqrt{k}}\sin 2a\sqrt{k}\right].$$

Let $G$ be an arbitrary starlike region and $z=0, z^*\in G^*=\{x-iy|\,x+iy\in G\}$, $\tau$ be a real or complex variable, $\tau\in T, f(z,\tau)$ be analytical in $G$ and continuous function in $\bar{G}$.

Let us consider the following differential equation:

$$u_{xx}+u_{yy}+\frac{2v}{y}u_y-su=0, \tag{27}$$

where $u = u(x, y, \tau)$, and $s$ depends only on $\tau$.

According to (10), the solution of equation (27) can be expressed as

$$u(z, z^*, \tau) = -iC_k \left(\frac{z - z^*}{2i}\right)^{1-2v} \int_{z^*}^{z} f(\sigma, \tau)\, {}_0F_1 \left[v; \frac{s}{4}(z - \sigma)(z^* - \sigma)\right]$$

$$\times \left[(z - \sigma)(z^* - \sigma)\right]^{v-1} d\sigma. \tag{28}$$

Let $z^* = \bar{z},\ \ \sigma = x + iy \cos t$, i.e.

$$(z - \sigma)(\bar{z} - \sigma) = y^2 \sin^2 t,$$

then we obtain

$$u(x, y, \tau) = u(z, \bar{z}, \tau) = C_v \int_0^{\pi} f(x + iy \cos t, \tau) \tag{29}$$

$$\times\, {}_0F_1 \left[v; \frac{3}{4} y^2 \sin^2 t\right] \sin^{2v-1} t dt.$$

This can be written as:

$$u(z, \bar{z}, \tau) = C_v \Gamma(v) \int_0^{\pi} \sum_{n=0}^{\infty} \frac{(y \sin t)^{2n} \sin^{2v-1} t}{(2n)!\Gamma(v + n)} \tag{30}$$

$$\times s^n f(x + iy \cos t, \tau) dt.$$

Thus, we have the following theorem.

**Theorem 5.** *For all functions which are analytic in $G$, and if the series in (30) converges uniformly for $\forall z \in G,\ \tau \in T_0 \subset T$, then the formula*

$$u(z, \bar{z}, \tau) = \frac{\Gamma\left(\frac{2v+1}{2}\right)}{\sqrt{\pi}} \int_0^{\pi} \sin^{2v-1} t \sum_{n=0}^{\infty} \frac{(y \sin t)^{2n}}{(2n)!\Gamma(v + n)}$$

$$\times\, s^n f(x + iy \cos t, \tau) dt$$

*gives the solution of equation (27) for $\forall \tau \in T_0;\ z,\ \bar{z}$ belongs to the neighborhood of $z = 0,\ \bar{z} = 0$.*

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