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α-Mellin Transform and One of Its Applications

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We consider a generalization of the classical Mellin transformation, called α-Mellin transformation, with an arbitrary (fractional) parameter $\alpha > 0$. Here we continue the presentation from the paper [5], where we have introduced the definition of the α-Mellin transform and some of its basic properties. Some examples of special cases are provided. Its operational properties as Theorem 1, Theorem 2 (Convolution theorem) and Theorem 3 (α-Mellin transform of fractional R-L derivatives) are presented, and the proofs can be found in [5]. Now we prove some further properties of this integral transform, useful for its application to solving some fractional order differential equations. An example of such application is proposed for the fractional order Bessel differential equation of the form

$$t^{\beta+1}{}_0D_t^{\beta+1}y(t) + t^{\beta}{}_0D_t^{\beta}y(t) = f(t)\ , \quad 0 < \beta < 1.$$

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1. Introduction

This paper deals with the theory and applications of the α-Mellin transform. We derive the α-Mellin transform and its inverse from the complex Fourier transformation. This is followed by several examples and basic operational properties of the α-Mellin transform. We discuss an application of the α-Mellin transform for solving a fractional differential equation. Historically, Riemann (1876) first recognized the Mellin transform in his famous memoir on the prime numbers. Its explicit formulation was given by Cahen (1894). Almost simultaneously Mellin (1896, 1902) gave an elaborate discussion of the Mellin transform and its inversion formula.

2. Definition of the α-Mellin transform

In [2] Luchko, Martinez and Trujillo introduced the fractional Fourier transformation (FRFT). A substitution $x = e^t$ in the FRFT leads to a generalization of the Mellin transform, called further α-Mellin transform. The reason for this name is explained by the following definitions.

Definition 2.1. Let $0 < \alpha \leq 1$ and the variable be a complex number, such that

$$p = \begin{cases} Rep - i|\omega|^{1/\alpha}, & \omega \leq 0 \\ Rep + i|\omega|^{1/\alpha}, & \omega > 0 \end{cases} .$$

The integral transform of the form

$$M_\alpha\{f(t);p\} = M(p\mathrm{sign}\omega) = \int_0^\infty t^{p\,\mathrm{sign}\omega - 1} f(t)dt \tag{1}$$

is called *α-Mellin transform* of the function $f(t)$.

Definition 2.2. Let $Re\alpha > 0$ and $f \in C$. Then for $a \in R$ and $x > a$ the integral operator

$$_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{2}$$

is called a *Riemann-Liouville fractional integral operator* of order α. The symbol $_aD_x^\beta$ with $\beta > 0$ is interpreted as a corresponding differintegral operator, called *Riemann-Lioville fractional derivative*. For details and definitions of these operators of Fractional Calculus (FC), see for example [7], [6].

3. Basic operational properties of the α-Mellin transforms

Like for the FRFT in [8], [6], [3] our interest in the α-Mellin transform is based on the possibilities of its applications for solving certain ordinary differential equations of fractional order.

Using Definition 2.1, we can prove some basic properties of the α-Mellin transforms. The following operational properties hold:

Theorem 3.1. ([5]) *If we denote $M_\alpha\{f(t);p\} = M(p\,sign\omega)$, then*:

a) $M_\alpha\{f(at);p\} = a^{-p\,\mathrm{sign}\omega} M(p\,\mathrm{sign}\omega)$, $a > 0$ (*Scaling Property*);
b) $M_\alpha\{t^a f(t);p\} = M(p\,\mathrm{sign}\omega + a)$ (*Shifting Property*).

Theorem 3.2. ([5]) (*Convolution Theorem*)
If $M_\alpha\{f(t);p\} = M(p\mathrm{sign}\omega)$ and $M_\alpha\{g(t);p\} = G(p\mathrm{sign}\omega)$, then

$$M_\alpha\{f(t) * g(t);p\} = M(p\,\mathrm{sign}\omega)G(1 - p\,\mathrm{sign}\omega),$$

where

$$f(t) * g(t) = \int_0^\infty f(t\tau)g(\tau)d\tau.$$

Corollary 3.1. ([5]) *If $n > 0$ and $f(t)$ is a function such that*

$$\lim_{t\to\infty} t^{\text{sign}\omega Rep-1}|f(t)| = 0\ , \quad \lim_{t\to 0} t^{\text{sign}\omega Rep-1}|f(t)| = 0\ ,$$

then

$$M_\alpha\{f^{(n)}(t);p\} = \frac{\Gamma(1 - p\text{sign}\omega + n)}{\Gamma(1 - p\text{sign}\omega)} M(p\text{sign}\omega - n).$$

Corollary 3.2. ([5]) *If $0 < \beta < 1$ and $f(t)$ is a function such that*

$$\lim_{t\to\infty} t^{\text{sign}\omega Rep-1}|f(t)| = 0\ , \quad \lim_{t\to 0} t^{\text{sign}\omega Rep-1}|f(t)| = 0\ ,$$

then

$$M_\alpha\{{}_0D_t^\beta f(t);p\} = \frac{\Gamma(1 - p\ \text{sign}\omega + \beta)}{\Gamma(1 - p\ \text{sign}\omega)} M(p\ \text{sign}\omega - \beta)\ ,$$

where ${}_0D_t^\beta$ is the Riemann-Liouville fractional derivative of order β.

Theorem 3.3. *If $0 < \beta < 1$ and $M_\alpha\{f(t);p\} = M(p\text{sign}\omega)$, then*

$$M_\alpha\left\{\sum_{k=0}^n a_k t^{\beta+k} {}_0D_t^{\beta+k} f(t);p\right\}$$
$$= M(p\ \text{sign}\omega)\frac{\Gamma(1-p\ \text{sign}\omega)}{\Gamma(1-p\ \text{sign}\omega-\beta)} \sum_{k=0}^n (-1)^k a_k \prod_{j=0}^{k-1} (p\ \text{sign}\omega + \beta + j)\ ,$$

where ${}_0D_t^\beta$ is the Riemann-Liouville fractional derivative operator of order β.

P r o o f. Under (1) and Theorem 3.1 (b),

$$M_\alpha\left\{\sum_{k=0}^n a_k t^{\beta+k} {}_0D_t^{\beta+k} f(t);p\right\} = \sum_{k=0}^n a_k M_\alpha\left\{{}_0D_t^{\beta+k} f(t); p\text{sign}\omega + \beta + k\right\}\ .$$

On the other hand, under Corollary 3.2,

$$\sum_{k=0}^n a_k M_\alpha\left\{{}_0D_t^{\beta+k} f(t); p\text{sign}\omega + \beta + k\right\}$$

$$= \sum_{k=0}^n a_k \frac{\Gamma(1 - p\ \text{sign}\omega)}{\Gamma(1 - p\ \text{sign}\omega - \beta - k)} M(p\ \text{sign}\omega)$$

$$= M(p\ \mathrm{sign}\omega)\Gamma(1 - p\mathrm{sign}\omega)\sum_{k=0}^{n} \frac{a_k}{\Gamma(1 - p\ \mathrm{sign}\omega - \beta - k)} .$$

Applying the property of the $\Gamma-$function, we obtain

$$\sum_{k=0}^{n} \frac{a_k}{\Gamma(1 - p\ \mathrm{sign}\omega - \beta - k)} = \sum_{k=0}^{n} \frac{(-1)^k a_k \prod\limits_{j=0}^{k-1} (p\ \mathrm{sign}\omega + \beta + j)}{\Gamma(1 - p\ \mathrm{sign}\omega - \beta)}$$

and this proves the theorem. ■

Corollary 3.3. *If* $0 < \beta < 1$ *and* $M_\alpha\{f(t);p\} = M(p\mathrm{sign}\omega)$, *then*

$$M_\alpha\{t^{\beta+1}{}_0D_t^{\beta+1} f(t) + t^\beta {}_0D_t^\beta f(t);p\} = \frac{(1-p\ \mathrm{sign}\omega-\beta)\Gamma(1-p\ \mathrm{sign}\omega)}{\Gamma(1-p\ \mathrm{sign}\omega-\beta)} M(p\ \mathrm{sign}\omega) .$$

4. Generalized Bessel fractional equation

The fractional order differential equation of the form

$$t^{\beta+1}{}_0D_t^{\beta+1} y(t) + t^\beta {}_0D_t^\beta y(t) = f(t) , \quad 0 < \beta < 1, \tag{3}$$

we call *generalized Bessel fractional equation.*

Theorem 4.1. *The solution of the boundary value problem for the Bessel fractional equation (3) with the conditions*

$$y(0) = y'(0) = 0 , \quad y(\infty) = y'(\infty) = 0, \tag{4}$$

has the form

$$y(t) = \int_0^\infty f(t\tau)g(\tau)d\tau, \tag{5}$$

where

$$g(t) = M_\alpha^{-1}\{G(p\mathrm{sign}\omega);t\}.$$

P r o o f. Applying of the α-Mellin transform (1) to both sides of (3) and condition (4), by Corollary 3.1 and Corollary 3.3, we obtain

$$\frac{(1 - p\mathrm{sign}\omega - \beta)\Gamma(1 - p\mathrm{sign}\omega)}{\Gamma(1 - p\mathrm{sign}\omega - \beta)} Y(p\mathrm{sign}\omega) = M(p\mathrm{sign}\omega) ,$$

where

$$M_\alpha\{y(t);p\} = Y(p\mathrm{sign}\omega) , \quad M_\alpha\{f(t);p\} = M(p\mathrm{sign}\omega) .$$

Rewriting the above equality in the following form:

$$Y(p\mathrm{sign}\omega) \ = M(p\mathrm{sign}\omega)G(1-p\mathrm{sign}\omega) \ ,$$

this leads to

$$G(p\mathrm{sign}\omega) = \frac{\Gamma(p\mathrm{sign}\omega - \beta)}{(p\mathrm{sign}\omega-\beta)\Gamma(p\mathrm{sign}\omega)} \ .$$

If the inverse α-Mellin ransform for $G(p\mathrm{sign}\omega)$ is

$$M_\alpha^{-1}\{G(p\mathrm{sign}\omega);t\} = g(t) \ ,$$

then, according to Theorem 3.2, we get the solution in the form

$$y(t) = \int\limits_0^\infty f(t\tau)g(\tau)d\tau \ .$$

This proves the theorem. ■

The solution is of the form $y(t) = \int\limits_0^1 f(t\tau)g(\tau)d\tau$, if $g(t)$ is zero for $t > 1$.

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