Mathematica
Balkanica

New Series Vol. 26, 2012, Fasc. 1-2

Inequalities and Asymptotic Formulae for the Three Parametric Mittag-Leffler Functions

Jordanka Paneva-Konovska

Presented at 6^{th} International Conference "TMSF' 2011"

We consider some families of 3-index generalizations of the classical Mittag-Leffler functions and study the behaviour of these functions in domains of the complex plane. First, some inequalities in the complex plane and on its compact subsets are obtained. We also prove an asymptotic formula for the case of "large" values of the indices of these functions. Similar results have also been obtained by the author for the classical Bessel functions and their Wright's generalizations with 2, 3 and 4 parameters, as well as for the classical and multi-index Mittag-Leffler functions.

MSC 2010: 33E12, 30A10, 30D15, 30E15

Key Words: special functions, Mittag-Leffler function and its generalizations, entire functions, inequalities, asymptotic formulae

1. Introduction

The special functions, defined in the whole complex plane $\mathbb{C}$ by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}\,, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}\,, \tag{1.1}$$

with $\alpha,\ \beta \in \mathbb{C}$, $Re(\alpha) > 0$, are known as Mittag-Leffler (M-L) functions ([5], Section 18.1). The first was introduced by Mittag-Leffler (1902-1905) who investigated some of its properties, while the latter appeared in a paper by Wiman (1905). The main results in the classical theory of these functions are presented in the handbook by Erdélyi et al. ([5], Section 18.1), further results are given in the books by Dzherbashyan [3], [4]: asymptotic formulae in different parts of the complex plane, distribution of the zeros, kernel functions of inverse Borel type integral transforms, various relations and representations. Detailed accounts of the properties of these functions can be found in the contemporary books of Kilbas et al. [7] and Podlubny [18], see also [8], [9], [10]. Recently the interest to Mittag-Leffler functions and their generalizations has grown up due to their applications in some evolution problems [2] and appearence in the solutions of fractional order differential and integral equations.

Prabhakar [19] generalized (1.1) by introducing a 3-parameter function $E^{\gamma}_{\alpha,\beta}$ of the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \quad \alpha,\ \beta,\ \gamma \in \mathbb{C},\ \ Re(\alpha) > 0, \tag{1.2}$$

where $(\gamma)_k$ is the Pochhammer symbol ([5], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1).$$

For $\gamma = 1$ this function coincides with $E_{\alpha,\beta}$, while for $\gamma = \beta = 1$ with E_α:

$$E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) \quad E^1_{\alpha,1}(z) = E_\alpha(z).$$

In fact, Prabhakar introduced this function for positive γ, and in this case it is an entire function of z of order $\rho = 1/Re(\alpha)$, as mentioned in [7] and [11], and type $\sigma = 1$. Unfortunately, the author could not find a proof of this fact in the literature.

Prabhakar studied some properties of the generalized 3-parametric Mittag-Leffler function (1.2) and of an integral operator containing it as a kernel-function, and applied the obtained results to prove the existence and uniqueness of the solution of a corresponding integral equation. Further, some properties of $E^{\gamma}_{\alpha,\beta}(z)$, including classical and fractional order differentiations and integrations, are proved by Kilbas, Saigo and Saxena [6]. An integral operator with such a kernel-function is also studied in the space $L(a,b)$. The functions (1.2) and series in them have been used recently to express solutions of the generalized Langevin equation, by Sandev, Tomovski and Dubbeldam [21]. Another type of 3-parameter Mittag-Leffler function (q-analogue of the M-L function) is also considered, see for example in Rajkovic, Marinkovic and Stankovic [20].

In our previous papers ([16], [17]), we considered series in systems of Mittag-Leffler functions and, resp. in [12] - [15], series in the multi-index ($2m$-indices) analogues (in the sense of [8],[9],[10]) of the M-L functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus ([10]). Properties of these functions are studied by many authors, among them see for example, [1], [7], [8],[9], [18], [20], etc. We studied the convergence of such series in the complex plane $\mathbb{C}$, and proved Cauchy-Hadamard, Abel and Tauberian type theorems.

To be able to prove similar convergence theorems for series in the 3-parametric Mittag-Leffler functions (1.2), we need first some inequalities in the complex plane, on its compact subsets and asymptotic formulae for "large" values of indices of these functions, obtained as results of this paper.

2. Auxiliary statements

Consider now the generalized (3-parametric) Mittag-Leffler functions (1.2) for indices of the kind $\beta = n$; $n = 0, 1, 2, \ldots$, namely:

$$E^{\gamma}_{\alpha,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!}, \quad \alpha,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0,\ n \in \mathbb{N}_0. \tag{2.1}$$

Note 2.1. Given a number γ, suppose that some coefficients in (2.1) equal zero, that is, there exists a number $p \in \mathbb{N}_0$, such that the representation (2.1) can be written as follows:

$$E^{\gamma}_{\alpha,n}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}. \tag{2.2}$$

Note 2.2. In what follows, we will use the notations $\mathbb{Z}^-$ (resp. $\mathbb{N}$) for the set of negative (resp. positive) integers and $\mathbb{Z}^-_0 = \mathbb{Z}^- \cup \{0\}$. Further, set

$$a_k = \frac{1}{\Gamma(\alpha k + n)}, \quad b_k = (\gamma)_k, \quad c_k = a_k b_k / k!, \quad k = 0, 1, 2, \ldots.$$

We consider three main cases.

Lemma 2.1. *If $\gamma \in \mathbb{C}$, but $\gamma \notin \mathbb{Z}^-_0$, then*
1. $p = 0$ for $n \in \mathbb{N}$,
2. $p = 1$ for $n = 0$.

P r o o f. Obviously, in the first case, $b_k \neq 0$ and $\alpha k + n$ are not non-positive integers. Because of that $a_k \neq 0$ and therefore $c_k \neq 0$ for all the values of k. In the second case, since only a_0 equals zero, then $c_0 = 0$ but $c_k \neq 0$ for all the natural values of k. ∎

Note 2.3. Actually, in the other cases, the functions (2.1) reduce to polynomials of power $-\gamma$, and denoting $m = -\gamma$, we can rewrite representation (2.2) in the form:

$$E^{\gamma}_{\alpha,n}(z) = z^p \sum_{k=p}^{m} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}. \tag{2.3}$$

Lemma 2.2. *If $\gamma \in \mathbb{Z}^-$, $m = -\gamma$, then (2.1) can be expressed by the formula (2.3) with the following values of p:*
1. $p = 0$ for $n \in \mathbb{N}$,
2. $p = 1$ for $n = 0$.

Proof. The values of the numbers a_k are the same as in the proof of Lemma 1. Moreover,

$$b_k = (-m)_k = -m(-m+1)\dots(-m+k-1) = (-1)^k m(m-1)\dots(m-k+1),$$

then $b_k \neq 0$ only for $k \leq m$ and because of that $c_k = 0$ for all $k > m$ and therefore the values of p are the same as required. ■

Corollary. *If $\gamma \in \mathbb{Z}^-$, $m = -\gamma$, then (2.3) can be written in the form:*

$$E_{\alpha,n}^{-m}(z) = z^p \sum_{k=p}^{m} (-1)^k \binom{m}{k} \frac{z^{k-p}}{\Gamma(\alpha k + n)}. \tag{2.4}$$

with the corresponding values of p.

The proof of (2.4) is evident according to Lemma 2.2 because $c_k = a_k b_k / k!$. ■

Note 2.4. Let us mention that $(-1)^p \binom{m}{p} = (-m)_p$ when $p = 0$ or $p = 1$ and $m \in \mathbb{N}$.

Lemma 2.3. *If $\gamma = 0$, then:*
1. $E_{\alpha,n}^0(z) = \dfrac{1}{\Gamma(n)}$ for $n \in \mathbb{N}$,
2. $E_{\alpha,n}^0(z) = 0$ for $n = 0$.

The proof follows directly, taking in view that $b_k = 0$ for all $k \in \mathbb{N}$. ■

Note 2.5. Let us mention that if γ is non-positive integer, as it is seen above, the functions (1.2) reduce to polynomials, but when $\gamma \notin \mathbb{Z}_0^-$, they are entire functions of z of order $\rho = 1/Re(\alpha)$ and type $\sigma = 1$ and this is not difficult to verify.

The above lemmas and Note 2.4 show that the functions $E_{\alpha,n}^{\gamma}(z)$ can be written in the following form

$$E_{\alpha,n}^{\gamma}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p + n)} z^p \left(1 + \theta_{\alpha,n}^{\gamma}(z)\right) \tag{2.5}$$

with

$$\theta_{\alpha,n}^{\gamma}(z) = \sum_{k=p+1}^{\infty} \frac{(\gamma)_k}{(\gamma)_p} \frac{\Gamma(\alpha p + n)}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!} \quad \text{for} \quad \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{2.6}$$

and respectively, for $\gamma = -m$, $m \in \mathbb{N}$,

$$\theta_{\alpha,n}^{-m}(z) = \sum_{k=p+1}^{m} \frac{(-m)_k}{(-m)_p} \frac{\Gamma(\alpha p+n)}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!}$$

$$= \sum_{k=p+1}^{m} \frac{(-1)^{k-p} \binom{m}{k}}{\binom{m}{p}} \frac{\Gamma(\alpha p+n)}{\Gamma(\alpha k+n)} z^{k-p}. \tag{2.7}$$

Note 2.6. In the above representations (2.5)-(2.7), γ is different from zero, and the parameter p is determined by the previous lemmas. More precisely, $p = 0$ for all the natural values of n and $p = 1$ for $n = 0$.

3. Inequalities and asymptotic formulae

Our aim is to estimate the entire functions $\theta_{\alpha,n}^{\gamma}(z)$. To this end we transform the expressions in the equalities (2.6) and (2.7), to the following forms:

$$\theta_{\alpha,n}^{\gamma}(z) = \frac{\Gamma(\alpha p+n)}{\Gamma(\alpha(p+1)+n)} \sum_{k=p+1}^{\infty} \gamma_{n,k} \frac{(\gamma)_k}{(\gamma)_p} \frac{z^{k-p}}{k!}, \tag{3.1}$$

and respectively

$$\theta_{\alpha,n}^{-m}(z) = \frac{\Gamma(\alpha p+n)}{\Gamma(\alpha(p+1)+n)} \sum_{k=p+1}^{m} (-1)^{k-p} \gamma_{n,k} \frac{\binom{m}{k}}{\binom{m}{p}} z^{k-p}, \tag{3.2}$$

with

$$\gamma_{n,k} = \frac{\Gamma(\alpha(p+1)+n)}{\Gamma(\alpha k+n)}. \tag{3.3}$$

Theorem 3.1. *Let $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then there exists an entire function τ such that*

$$\left|\theta_{\alpha,n}^{\gamma}(z)\right| \le \frac{|\Gamma(\alpha p+n)|}{|\Gamma(\alpha(p+1)+n)|} \tau(|z|;\alpha,\gamma) \tag{3.4}$$

for all the values of $z \in \mathbb{C}$.

P r o o f. To find such a function τ and to prove the inequality (3.4), we estimate the function (3.1) beginning with the values of (3.3). Since

$$\gamma_{0,k} = \frac{\Gamma(\alpha(p+1))}{\Gamma(\alpha k)},$$

$$\gamma_{n,k} = \frac{\Gamma(\alpha(p+1))}{\Gamma(\alpha k)} \frac{(\alpha(p+1))}{(\alpha k)} \cdots \frac{(\alpha(p+1)+n-1)}{(\alpha k)+n-1} \quad \text{for } n \in \mathbb{N},$$

and due to the following inequality

$$\frac{|\alpha(p+1)|}{|\alpha k|} \cdots \frac{|\alpha(p+1)+n-1|}{|\alpha k+n-1|} \le 1,$$

we obtain that

$$|\gamma_{n,k}| \leq \frac{|\Gamma(\alpha(p+1))|}{|\Gamma(\alpha k)|} \quad \text{for all the possible } n \text{ and } k.$$

Finally, the proof of the theorem ends, by taking

$$\tau(z;\alpha,\gamma) = \sum_{k=p+1}^{\infty} \frac{|\Gamma(\alpha(p+1))|}{|\Gamma(\alpha k)|} \frac{|(\gamma)_k|}{|(\gamma)_p|} \frac{z^{k-p}}{k!}. \qquad \blacksquare$$

Theorem 3.2. *Let $\gamma \in \mathbb{Z}^-$ and $m = -\gamma$, then there exists an entire function t such that for all $z \in \mathbb{C}$:*

$$\left|\theta_{\alpha,n}^{-m}(z)\right| \leq \frac{|\Gamma(\alpha p+n)|}{|\Gamma(\alpha(p+1)+n)|} \; t(|z|;\alpha,\gamma). \tag{3.5}$$

P r o o f. Considering (3.2), following the evaluations in the proof of Theorem 3.1 and denoting

$$t(z;\alpha,\gamma) = \sum_{k=p+1}^{m} \frac{|\Gamma(\alpha(p+1))|}{|\Gamma(\alpha k)|} \frac{\binom{m}{k}}{\binom{m}{p}} z^{k-p},$$

we complete the proof. $\blacksquare$

Further, we prove an asymptotic formula for "large" values of the indices n.

Theorem 3.3. *Let $z,\ \alpha,\ \gamma \in \mathbb{C},\ n \in \mathbb{N}_0,\ \gamma \neq 0,\ Re(\alpha) > 0$ and $\theta_{\alpha,n}^{\gamma}$ be given by the formulae (3.1)–(3.3). Then the generalized Mittag-Leffler functions (2.1) satisfy the following asymptotic formulae*

$$E_{\alpha,n}^{\gamma}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p+n)} z^p \left(1+\theta_{\alpha,n}^{\gamma}(z)\right) \text{ and } \theta_{\alpha,n}^{\gamma}(z) \to 0 \text{ as } n \to \infty, \tag{3.6}$$

with the corresponding p, depending on γ. Moreover, on the compact subsets of the complex plane $\mathbb{C}$, the convergence is uniform and

$$\theta_{\alpha,n}^{\gamma}(z) = O\left(\frac{1}{n^{Re(\alpha)}}\right) \quad (n \in \mathbb{N}). \tag{3.7}$$

T h e p r o o f is evident, using formula (2.5), Theorems 3.1. and 3.2. and Stirling's formula. $\blacksquare$

Note 3.1. According to the asymptotic formula (3.6), it follows there exists a natural number M such that the functions $E_{\alpha,n}^{\gamma}$ have not any zeros at all for $n > M$, possibly except of zero.

4. Conclusion

In conclusion, note that the case $\gamma = 1$ gives analogous results related to the classical Mittag-Leffler functions (1.1). Additionally, if the parameters α and β are positive, we obtain our previous results published in the papers [16] and [17].

Acknowledgements: The paper is partially supported by National Science Fund - Ministry of Education, Youth and Science, Bulgaria, under Project D ID 02/25/2009: "Integral Transform Methods, Special Functions and Applications".

This work is performed also in the frames of the Research Project "Transform Methods, Special Functions, Operational Calculi and Applications" under the auspices of the Institute of Mathematics and Informatics - Bulgarian Academy of Sciences.

References

[1] L. Gajić, B. Stanković, Some properties of Wright's function. *Publ. l'Institut Math.* Beograd, Nouv. Ser. **20**, 34 (1976), 91–98.

[2] R. Gorenflo, F. Mainardi, On Mittag-Leffler function in fractional evolution processes. *J. Comput. Appl. Math.*, **118** (2000), 283–299.

[3] M. M. Dzrbashjan, *Integral Transforms and Representations in the Complex Domain* (in Russian). Nauka, Moskow (1966).

[4] M. M. Dzrbashjan, *Interpolation and Spectrum Expansions Associated with Fractional Differential Operators* (in Russian). IM, AAS & Univ. of Yerevan, Yerevan (1983).

[5] A. Erdélyi et al. (Ed-s), *Higher Transcendental Functions.* Vols. 1 - 3. McGraw-Hill, New York-Toronto-London (1953).

[6] A. A. Kilbas, M. Saigo and R. K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transforms and Special Functions*, **15**, No. 1 (2004), 31–49.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam, (2006).

[8] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *Journal of Computational and Applied Mathematics*, **118** (2000), 241–259.

[9] V. Kiryakova, The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus. *Computers and Mathematics with Appl.*, **59**, No 5 (2010), 1885–1895, doi:10.1016/j.camwa.2009.08.025

[10] V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. *Computers and Mathematics with Appl.*, **59**, No 3 (2010), 1128–1141, doi:10.1016/j.camwa.2009.05.014.

[11] A. M. Mathai, H. J. Haubold, *Special Functions for Applied Scientists.* Springer (2008).

[12] J. Paneva-Konovska, Cauchy-Hadamard, Abel and Tauber type theorems for series in generalized Bessel-Maitland functions. *Compt. Rend. Acad. Bulg. Sci.*, **61**, No 1 (2008), 9–14.

[13] J. Paneva-Konovska, Theorems on the convergence of series in generalized Lommel-Wright functions. *Fract. Calc. Appl. Anal.*, **10**, No 1 (2007), 59–74.

[14] J. Paneva-Konovska, Tauberian theorem of Littlewood type for series in Bessel functions of first kind. *Compt. Rend. Acad. Bulg. Sci.*, **62**, No 2 (2009), 161–166.

[15] J. Paneva-Konovska, Convergence of series in some multi-index Mittag-Leffler functions. *Proc. of 4th IFAC Workshop "Fractional Differentiation and its Applications' 2010"*, Badajoz, Spain, Oct. 18-20, 2010 (Eds: I. Podlubny, B. M. Vinagre Jara, YQ. Chen, V. Feliu Batlle, I. Tejado Balsera), ISBN 9788055304878, Article No FDA10-147 (2010), 1–4.

[16] J. Paneva-Konovska, Convergence of series in Mittag-Leffler functions. *Compt. Rend. Acad. Bulg. Sci.*, **63**, No 6 (2010), 815–822.

[17] J. Paneva-Konovska, Series in Mittag-Leffler functions: inequalities and convergent theorems. *Fract. Calc. Appl. Anal.*, **13**, No 4 (2010), 403–414.

[18] I. Podlubny, *Fractional Differential Equations.* Acad. Press, San Diego, (1999).

[19] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.*, **19** (1971), 7-15.

[20] P. Rajkovic, S. Marinkovic, M. Stankovic, Diferencijalno-Integralni Racun Baicnih Hipergeometrijskih Funkcija (In Serbian). *Mas. Fak. Univ. u Nisu*, Nis (2008).

[21] T. Sandev, Ž. Tomovski, J. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise. *Physica A*, **390**, Issue 21-22, (2011), 3627–3636, DOI: 10.1016/j.physa.2011.05.039.

Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, bul. Kliment Ohridski, 1000 Sofia, BULGARIA
e-mail: yorry77@mail.bg

Received: November 11, 2011