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On Some Generalizations of Classical Integral Transforms

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Using the generalized confluent hypergeometric function [6] some new integral transforms are introduced. They are generalizations of some classical integral transforms, such as the Laplace, Stieltjes, Widder-potential, Glasser etc. integral transforms. The basic properties of these generalized integral transforms and their inversion formulas are obtained. Some examples are also given.

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1. Introduction

The important role of the method of integral transforms is well known in applied mathematics [4, 5, 1, 3], in solving some boundary value problems of mathematical physics, astronomy, in the theory of the differential and integral equations, etc.

Let us recall the definitions if some classical integral transforms [3]:

the Laplace transform:

$$L\{f(x);y\} = \int_0^\infty e^{-xy} f(x)\,dx, \tag{1}$$

the Widder-potential transform:

$$P\{f(x);y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}\,dx, \tag{2}$$

the generalized Stieltjes' transform:

$$S_p\{f(x);y\} = \int_0^\infty \frac{f(x)}{(x+y)^p}\,dx. \tag{3}$$

We define the new generalized transforms by means of the (τ, β) – *generalized confluent hypergeometric function* ${}_1\Phi_1^{\tau,\beta}(a;c;z)$, see [6]:

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{1}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} {}_1\Psi_1 \left[\begin{array}{c} (c;\tau) \\ (c;\beta) \end{array} \middle| zt^{\tau} \right] dt, \qquad (4) $$

where $\operatorname{Re} c > \operatorname{Re} a > 0$, $\{\tau, \beta\} \subset \mathbf{R}$; $\tau > 0$; $\tau - \beta < 1$; $B(\ldots)$ is the classic beta-function, and ${}_1\Psi_1[\ldots]$ is a special case of the *generalized hypergeometric Wright function* ([4]):

$$ {}_p\Psi_q \left[\begin{array}{c} (a_i;\alpha_i)_{1,p} \\ (b_j;\beta_j)_{1,q} \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + n\alpha_i)}{\prod_{j=1}^{q} \Gamma(b_j + n\beta_j)} \frac{z^n}{n!}, \qquad (5) $$

with $z \in \mathbf{C}$, $a_i, b_j \in \mathbf{C}$, $\{\alpha_i, \beta_j\} \subset \mathbf{R}\{-\infty, +\infty\}$;

$$ (a_i, b_j \neq 0; i = 1,2,\ldots,p; j = 1,2,\ldots,q),\ 1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0. $$

2. The generalized integral transforms and their properties

Some definitions:

1) *The generalized Laplace integral transforms*:

$$ L_{\gamma_1,\gamma_2}\{f(x);y\} = \int_0^{\infty} x^{\gamma_2} e^{-(xy)^{\gamma_1}} f(x)\,dx, \qquad (6) $$

$$ \widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(x);y\} = \int_0^{\infty} x^{\gamma_2} e^{-(xy)^{\gamma_1}} {}_1\Phi_1^{\tau,\beta}(\alpha; c; -b(x,y)^{\gamma\gamma_1}) f(x)\,dx = g(y), \qquad (7) $$

where $x > 0$, $\gamma \in \mathbf{C}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $b \geq 0$; $f(x) \equiv 0$ as $x < 0$; $x^{\gamma_2} f(x) < Me^{s_0 x^{\gamma_1}}$, $M > 0$ and s_0 are consts as $x > 0$.

Let us note that when $\gamma_2 = 0$, $\gamma_1 = 1$, $b = 0$, the transform (7) coincides with the transform (1).

2) *The generalized Stieltjes integral tranforms*:

$$ P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(u);x\} = \widetilde{P}_1\{f(u);x\} = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)} $$

$$ \times \int_0^{\infty} \frac{u^{\gamma_2} f(u)}{(x^{\gamma_1} + u^{\gamma_1})^{\gamma_3}} {}_2\Psi_1 \left[\begin{array}{l} (a_1;\tau);(a_2;\gamma) \\ (c;\beta) \end{array} \middle| -b \left(\frac{u^{\gamma_1}}{x^{\gamma_1} + u^{\gamma_1}} \right)^{\gamma_4} \right] du = g_1(x), \qquad (8) $$

$$ P_2^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(u);x\} = \widetilde{P}_2\{f(u);x\} = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)} $$

$$ \int_0^{\infty} \frac{u^{\gamma_2} f(u)}{(x^{\gamma_1} + u^{\gamma_1})^{\gamma_3}} {}_2\Psi_1 \left[\begin{array}{l} (a_1;\tau);(a_2;\gamma) \\ (c;\beta) \end{array} \middle| -b \left(\frac{x^{\gamma_1}}{x^{\gamma_1} + u^{\gamma_1}} \right)^{\gamma_4} \right] du = g_2(x), \qquad (9) $$

where $\operatorname{Re} a_1 > 0$, $\operatorname{Re} a_2 > 0$, $\operatorname{Re} c > 0$, $\gamma_1 > 0$, $i = \overline{1,4}$; $\{\tau, \beta\} \subset \mathbf{R}$; $\tau > 0$; $\tau - \beta < 1$; $b \geq 0$, ${}_2\Psi_1$ is the function of the form (5).

Let us notice that as $b = 0$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = p$, the transforms (8), (9) coincide with the Stieltjes integral transform (3).

For the transform (7) the following properties are valid:

i) *Linearity*:

$$\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\left\{\sum_{i=1}^{n} c_i f_i(x); y\right\} = \sum_{i=1}^{n} c_i g_i(y), \tag{10}$$
$$(c_i = const, i = \overline{1,4}).$$

ii) *Similarity*:

$$\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(ax); y\} = \frac{1}{a^{\gamma_2+1}}\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\left\{f(x); \frac{y}{a}\right\}, \tag{11}$$
$$(a = const > 0).$$

iii) If the functions $f(x) \in L(0; +\infty)$, $g(x) \in L(0; +\infty)$, then the following equality is valid:

$$\int_0^{\infty} u^{\gamma_2} L_{\gamma_1,\gamma_2,\gamma}\{f(t); u\} g(u)\, du = \int_0^{\infty} t^{\gamma_2} L_{\gamma_1,\gamma_2,\gamma}\{g(u); t\} f(t)\, dt, \tag{12}$$

under the absolute convergence of integrals.

iv) Under conditions of existing and convergence of integrals (6)-(9), the relations are valid:

$$L_{\gamma_1,\gamma_2}\left\{\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{g(u); x\}; y\right\} = \frac{1}{\gamma_1}\Gamma\left(\frac{\gamma_2+1}{\gamma_1}\right) P_1^{\gamma_1,\gamma_2,\frac{\gamma_2+1}{\gamma_1},\gamma}\{g(u); y\}, \tag{13}$$

$$\widetilde{L}_{\gamma_1,\gamma_2}\left\{L_{\gamma_1,\gamma_2}\{g(u); x\}; y\right\} = \frac{1}{\gamma_1}\Gamma\left(\frac{\gamma_2+1}{\gamma_1}\right) P_2^{\gamma_1,\gamma_2,\frac{\gamma_2+1}{\gamma_1},\gamma}\{g(u); y\}, \tag{14}$$

$$\int_0^{\infty} x^{\gamma_2} \widetilde{P}_1\{f(t); x\} g(x)\, dx = \int_0^{\infty} x^{\gamma_2} \widetilde{P}_2\{g(t); x\} f(x)\, dx, \tag{15}$$

$$\int_0^{\infty} x^{\gamma_2} L_{\gamma_1,\gamma_2}\{h(y); x\}\, \widetilde{L}\{g(u); x\}\, dx$$
$$= \frac{1}{\gamma_1}\Gamma\left(\frac{\gamma_2+1}{\gamma_1}\right)\int_0^{\infty} y^{\gamma_2} h(y)\, \widetilde{P}_1^{\gamma_1,\gamma_2,\frac{\gamma_2+1}{\gamma_1},\gamma}\{g(u); y\}\, dy. \tag{16}$$

(15), (16) are *equalities of Parseval type.*

Let us give some examples of the transform (7).

Example 1.

$$f(x) = \eta(x) = \begin{cases} 1 & \text{as} \quad x > 0, \\ 0 & \text{as} \quad x < 0. \end{cases}$$

$$\widetilde{L}_{\gamma_1,\gamma_2,\gamma_3}\{\eta(x); y\} = \frac{1}{\gamma_1}\frac{\Gamma(c)}{\Gamma(a)} y^{-\gamma_2-1} {}_2\Psi_1\left[\begin{array}{l} (a;\tau); \left(\frac{\gamma_2+1}{\gamma_1};\gamma\right) \\ (c;\beta) \end{array}\Bigg| -b\right].$$

Example 2.

$$f(x) = x^k,$$

$$\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{x^k; y\} = \frac{1}{\gamma_1}\frac{\Gamma(c)}{\Gamma(a)} y^{-k-\gamma_2-1} {}_2\Psi_1\left[\begin{array}{l} (a;\tau); \left(\frac{\gamma_2+k+1}{\gamma_1};\gamma\right) \\ (c;\beta) \end{array}\Bigg| -b\right].$$

Example 3.

$$f(x) = e^{-kx^{\gamma_1}},$$

$$\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\left\{e^{-kx^{\gamma_1}}; y\right\} = \frac{1}{\gamma_1}\frac{\Gamma(c)}{\Gamma(a)} (y^{\gamma_1}+k)^{\frac{\gamma_2+1}{\gamma_1}}$$

$$\times {}_2\Psi_1\left[\begin{array}{l} (a;\tau); \left(\frac{\gamma_2+1}{\gamma_1};\gamma\right) \\ (c;\beta) \end{array}\Bigg| -b\left(\frac{y^{\gamma_2}}{y^{\gamma_1+k}}\right)^{\gamma}\right].$$

3. The inversion formulae

Theorem 1. *Under the conditions of existing of the integral transform* $P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(x); y\}$ *the following inversion formula is valid:*

$$f(y) = \widetilde{L}^{-1}_{\gamma_1,\gamma_2,\gamma}\left\{L^{-1}_{\gamma_1,\gamma_2,a_2-1}\left\{\frac{\Gamma(a_2)}{\gamma_1} g_1(z); x\right\}; y\right\}, \tag{17}$$

where $g_1(z) = P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(u); z\}$*;* $\gamma_4 = \gamma$*,* $\gamma_3 = a_2$*.*

P r o o f. Let us consider the equality

$$\int_0^\infty x^{\gamma_1 a_2 - 1} y^{\gamma_2} e^{-(y^{\gamma_1}+z^{\gamma_1})x^{\gamma_1}} {}_1\Phi_1(a; c: -b(xy)^{\gamma\gamma_1})\, dx$$

$$= \frac{1}{\gamma_1}\frac{\Gamma(c)}{\Gamma(a_1)} \frac{y^{\gamma_2}}{(y^{\gamma_1}+z^{\gamma_1})^{a_2}} {}_2\Psi_1\left[\begin{array}{l} (a;\tau); (a_2;\gamma) \\ (c;\beta) \end{array}\Bigg| -b\left(\frac{y^{\gamma_1}}{(y^{\gamma_1}+z^{\gamma_1})}\right)^{\gamma}\right]. \tag{18}$$

Setting in (8) $\gamma_4 = \gamma$, $\gamma_3 = a_2$ and taking into account (18), we obtain:

$$P_1^{\gamma_1,\gamma_2,a_2,\gamma}\{f(y);z\} = g_1(z)$$
$$= \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{y^{\gamma_2} f(y)}{(z^{\gamma_1}+y^{\gamma_1})^{a_2}} {}_2\Psi_1 \left[\begin{array}{c} (a;\tau);(a_2;\gamma) \\ (c;\beta) \end{array} \middle| -b\left(\frac{y^{\gamma_1}}{(y^{\gamma_1}+z^{\gamma_1})}\right)^{\gamma} \right] dy$$
$$= \frac{1}{\Gamma(a_2)} \int_0^\infty f(y) \left(\int_0^\infty x^{\gamma_1 a_2 - 1} y^{\gamma_2} e^{-(y^{\gamma_1}+z^{\gamma_1})x^{\gamma_1}} {}_1\Phi_1(a;c;-b(x,y)^{\gamma\gamma_1})\, dx \right) dy$$
$$= \frac{\gamma_1}{\Gamma(a_2)} L_{\gamma_1,\gamma,a_2-1} \left\{ \widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(y);x\};z \right\},$$

from where we get (17). ∎

Theorem 2. *Under the conditions of existing of the integral transform* $\widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(x);y\}$ *the following inversion formula is valid:*

$$f(u) = \frac{\gamma_1 \Gamma(a)}{\Gamma(c)} u^{-\gamma_2} \int_0^\infty (ux)^{-1} g(x) K(ux)\, dx, \tag{19}$$

where

$$K(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\zeta(s)}\, ds,$$

$$g(y) = \widetilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(x);y\}, \quad \zeta(s) = {}_2\Psi_1 \left[\begin{array}{c} (a;\tau);\left(\frac{s}{\gamma_1};\gamma\right) \\ (c;\beta) \end{array} \middle| -b \right].$$

The proof of the theorem follows by using of the Mellin integral transform.

Corollary. *Under the conditions of existing and absolute convergence of integrals, the following equality is valid:*

$$\int_0^\infty x^{\mu-1} \widetilde{L}_{m,m-1,\gamma}\{e^{-tx} g(t);x\}\, dx$$
$$= \frac{1}{m} \frac{\Gamma(c)}{\Gamma(a)} {}_2\Psi_1 \left[\begin{array}{c} (a;\tau);\left(\frac{\mu}{m};\gamma\right) \\ (c;\beta) \end{array} \middle| -b \right] L\left\{ t^{m-1-\mu} g(t);u \right\}, \tag{20}$$

where L is the transform (1).

4. Some applications of the generalized integral transforms

Let us give some applications of the generalized integral transforms for evaluation of some integrals, in the theory of differential equations.

Example 1. Let $g(t) = e^{-at} t^{\nu+\mu-m}$.

Then taking into account formula [3]:

$$L\left\{t^{m-1-\mu}g(t);u\right\} = L\left\{t^{\nu-1}e^{-\alpha t};u\right\} = \Gamma(\nu)(u+\alpha)^{-\nu},$$

$$(\ \mathrm{Re}\ u > -\ \mathrm{Re}\ \alpha)$$

we have:

$$\int_0^\infty x^{\mu-1}\widetilde{L}_{m,m-1,\gamma}\left\{t^{\nu+\mu-m}e^{-(\alpha+u)t};x\right\}dx$$
$$= \frac{1}{m}\frac{\Gamma(c)\Gamma(\nu)}{\Gamma(a)}\ {}_2\Psi_1\left[\begin{array}{l}(a;\tau);(\frac{\mu}{m};\gamma)\\(c;\beta)\end{array}\middle| -b\right](u+\alpha)^{-\nu}.$$

Example 2. Let $g(t) = t^{\mu-m}\sin\alpha t\cdot\sin\beta t$.
Then taking into account formula [3]:

$$L\left\{t^{-1}\sin\alpha t\cdot\sin\beta t;u\right\} = 2^{-2}\ln\frac{u^2+(\alpha+\beta)^2}{u^2+(\alpha-\beta)^2},$$

$$(\ \mathrm{Re}\ u > |\ \mathrm{Im}\ (\pm\alpha\pm\beta)|)$$

we get

$$\int_0^\infty x^{\mu-1}\widetilde{L}_{m,m-1,\gamma}\left\{t^{\mu-m}e^{-tu}\sin\alpha t\sin\beta t;x\right\}dx$$
$$= 2^{-2}\frac{\Gamma(c)}{m\Gamma(a)}\ {}_2\Psi_1\left[\begin{array}{l}(a;\tau);(\frac{\mu}{m};\gamma)\\(c;\beta)\end{array}\middle| -b\right]\ln\frac{u^2+(\alpha+\beta)^2}{u^2+(\alpha-\beta)^2}.$$

Remark. Noticing that

$$L_m\{f(x);y\} = \frac{1}{m}L\left\{f\left(\sqrt[m]{z};y^m\right)\right\},$$

we obtain the inversion formula for $L_m\{f(x);y\}$:

$$f(x) = L_m^{-1}\{F(s);x\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2F\left(\sqrt[m]{s}\right)e^{sx^m}\,ds, \tag{21}$$

where

$$F(s) = L_m\{f(t);s\} = \int_0^\infty t^{m-1}e^{-x^m t^m}f(t)\,dt.$$

Example 3. Let us consider the following problem:

$$t\frac{\partial^2 u}{\partial t^2} + (1-m)\frac{\partial u}{\partial t} - t^{2m-1}\frac{\partial^2 u}{\partial x^2} = t^m x, \quad (x,t>0), \tag{22}$$

$$u(x,0)=0;\quad u(0,t)=1,\quad u'_x(x,0)=0. \tag{23}$$

Rewrite the differential equation (22):

$$\delta_m^2(u(x,t)) - \frac{\partial^2 u}{\partial x^2} = \frac{x}{t^{m-1}}, \tag{24}$$

where

$$\delta_m^2 = (1-m)\frac{1}{t^{2m-1}}\frac{d}{dt} + \frac{1}{t^{2m-2}}\frac{d^2}{dt^2}. \tag{25}$$

Let us apply to (24) the generalized integral Laplace transform L_m. We obtain

$$U''_{xx}(x,s) - m^2 s^{2m} U(x,s) = \frac{\Gamma\left(\frac{1}{m}\right)}{ms} x, \tag{26}$$

where $U(x,s) = L_m\{u(x,t), s\}$.

The solution of the differential equation is:

$$U(x,s) = C_1 e^{ms^m} + C_2 e^{-ms^m} - \frac{\Gamma\left(\frac{1}{m}\right)}{m^3 s^{2m+1}} x.$$

Taking into account (23) we have:

$$C_1 = 0, \quad C_2 = \frac{1}{ms^m},$$

from where

$$U(x,s) = \frac{1}{ms^m} e^{-ms^m} = \frac{\Gamma\left(\frac{1}{m}\right)}{m^3 s^{2m+1}} x.$$

Applying the inversion formula for this integral transform (21) we get

$$u(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left[\frac{1}{s}e^{-ms}\frac{\Gamma\left(\frac{1}{m}\right)}{m^2 s^{2+\frac{1}{m}}} x\right] e^{st^m}\, ds. \tag{27}$$

References

[1] M. Al-Hajri, S.L. Kalla, On a integral transform involving Bessel function, *Bull. Soc. Math. Maced.* **28** (2004), 5-18.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Fricomi, *Higher Transcendental Functions*, Vol. 1, Mc Graw-Hill, New York (1953).

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Fricomi, *Tables of Integral Transforms*, Mc Graw-Hill, New York (1954).

[4] A.A. Kilbas, M. Saigo, *The H-Transforms*, Chapman and Hall / CRC (2004).

[5] H.M. Srivastava, O. Yürekli, A theorem on Widder's potential transfo and its applications, *J. Math. Anal. Appl.* **154** (1991), 585-593.

[6] N. Virchenko, On the generalized confluent hypergeometric function and its applications, *Fract. Calc. Appl. Anal.* **9**, No 2 (2006), 101-108.

[7] N. Virchenko, S. L. Kalla, M. Al.-Zamel, Some results on a generalized hypergeometric function, *Integr. Transf. Spec. Funct.* **12**, No 1 (2011), 89-100.

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