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On Sandwich Theorem of Analytic Functions Involving Integral Operator

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Making use of the integral operator I_p^α, we give some applications of the first order differential subordination for normalized p-valent functions defined on the open unit disc $U = \{z : |z| < 1\}$.

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1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in C : |z| < 1\}$. Let $H(U)$ be the class of analytic functions in U and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} ... \ (a \in \mathbb{C}).$$

For simplicity, let $H[a] = H[a, 1]$. Also, let $A_1 = A(1)$ be the subclass of $H(\mathrm{U})$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + \dots . \tag{1.2}$$

If $f,\ g \in H(\mathrm{U})$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z)),\ z \in \mathrm{U}$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (cf., e.g., [5], [8]; see also [9]):

$$f(z) \prec g(z)(z \in \mathrm{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U}).$$

For $p, h \in H(\mathrm{U})$, let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathrm{U} \to C$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [10] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [10], Bulboača [3] considered certain classes of first order differential superordination as well as superordination-preserving integral operators [4]. Ali et al. [1], have used the results of Bulboača [3] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U. Also, Tuneski [17] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [14] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [14] also obtained results for functions defined by using Carlson-Shaffer operator.

Motivated essentially by Jung et al. [7], Shams et al. [13] introduced the operator $\mathrm{I}_p^\alpha : A(p) \to A(p)$ as follows:

$$(i)\ \mathrm{I}_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt\ (z \in \mathrm{U};\ \alpha > 0;\ p \in \mathbb{N})$$

and

(1.5) $$(ii)\mathrm{I}_p^0 f(z) = f(z) \quad (\alpha = 0;\ p \in \mathbb{N}).$$

Note that the one-parameter family of integral operator $\mathrm{I}^\alpha \equiv \mathrm{I}_1^\alpha$ was defined by Jung et al. [7].

For $f \in A(p)$ given by (1.1), it was shown that (see [13])

(1.6) $$\mathrm{I}_p^\alpha f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+1}{k+p+1}\right)^\alpha a_{k+p} z^{k+p} \quad (\alpha \geq 0;\ p \in \mathbb{N}).$$

Using the definition (1.6), it is easy to verify the identity (see [13])

(1.7) $$z(\mathrm{I}_p^\alpha f(z))' = (p+1)\mathrm{I}_p^{\alpha-1} f(z) - \mathrm{I}_p^\alpha f(z).$$

In the present paper, we apply a method based on the differential subordination in order to obtain subordination results for a normalized analytic function$f$ defined by using I_p^α operator and satisfy:

$$q_1(z) \prec \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \prec q_2(z)\ (z \in \mathrm{U}^* = \mathrm{U} \backslash \{0\}),$$

where q_1 and q_2 are given univalent functions in U.

2. Definitions and preliminaries

In order to prove our subordination and superordination results, we need to the following definition and lemmas.

Deinition 1. Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$,

$$E(f)=\{\xi\in\partial U:\lim_{z\to\xi}f(z)=\infty\},$$

and are such that $f'(\xi)\neq 0$ for $\xi\in\partial U\setminus E(f)$.

Lemma 1 *[9]. Let $q(z)$ be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\varphi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\mathrm{Re}\frac{zh'(z)}{Q(z)}>0,\ z\in U$.

If p is analytic in U with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p\prec q$, and q is the best dominant of (2.1).

Lemma 2 *[6]. Let h be convex (univalent) function in U with $h(0)=1$. Also let*

$$p(z)=1+a_m\ z^m+a_{m+1}\ z^{m+1}+\dots\ ,$$

be analytic in U. If

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z)\ (\gamma\in\mathbb{C}^*;\ \mathrm{Re}(\gamma)\geq 0;\ z\in U), \tag{2.2}$$

then

$$p(z)\prec q(z)=\frac{\gamma}{mz^{\frac{\gamma}{m}}}\int_0^z t^{\frac{\gamma}{m}-1}h(t)dt. \tag{2.3}$$

Lemma 3 *[3]. Let q be a univalent function in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}>0$ for $z\in U$,

(ii) $h(z)=zq'(z)\varphi(q(z))$ is starlike in U.

If $p \in H[q(0),1] \cap Q$ with $p(U) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.4}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (2.5).

Lemma 4 *[12]. The function $q(z) = (1-z)^{-2ab}$ is univalent in U if and only if $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$.*

3. Subordination results for analytic functions

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\alpha \geq 0$, $p \in \mathbb{N}$, $\eta, \varrho, \delta \in \mathbb{C}$ and $\upsilon, \mu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and the powers are understood as principle values.

Theorem 1. *Let $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$\operatorname{Re}\left\{1 + \frac{\delta}{\upsilon}q(z) + \frac{2\varrho}{\upsilon}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{3.1}$$

and

$$M(f,\alpha,p,\mu,\varrho,\delta,\upsilon) = \eta + \delta\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu + \varrho\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^{2\mu}$$

$$+\upsilon\mu(p+1)\left(\frac{\mathrm{I}_p^{\alpha-1} f(z)}{\mathrm{I}_p^\alpha f(z)} - 1\right) \quad (z \in \mathrm{U}^*). \tag{3.2}$$

If q satisfies the following subordination:

$$M(f,\alpha,p,\mu,\varrho,\delta,\upsilon) \prec \eta + \delta q(z) + \varrho(q(z))^2 + \upsilon\frac{zq'(z)}{q(z)}, \tag{3.3}$$

then

$$\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \prec q(z) \quad (z \in \mathrm{U}^*)$$

and q is the best dominant of (3.3).

P r o o f. Let the function $p(z)$ be defined by

$$p(z) = \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \quad (z \in \mathrm{U}^*), \tag{3.4}$$

then, differentiating (3.4) logarithmically with respect to z, we deduce that

$$\frac{zp'(z)}{p(z)} = \mu\left(\frac{z(I_p^\alpha f(z))'}{I_p^\alpha f(z)} - p\right). \tag{3.5}$$

Using the identity (1.7) in (3.5), a simple computation shows that

$$\frac{zp'(z)}{p(z)} = \mu(p+1)\left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} - 1\right).$$

In order to prove our result we will use Lemma 1 with

$$\theta(w) = \eta + \delta\omega + \rho\omega^2 \quad \text{and} \quad \varphi(w) = \tfrac{v}{\omega}.$$

Then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = v\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = \eta + \delta q(z) + \rho(q(z))^2 + v\frac{zq'(z)}{q(z)},$$

we find that Q is a starlike univalent in U and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\delta}{v}q(z) + \frac{2\rho}{v}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q(z)}\right\} > 0,$$

then, by using Lemma 1, we deduce that the subordination (3.3) implies $\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \prec q(z)$ $(z \in \mathrm{U}^*)$, and the function q is the best dominant of (3.3). ■xvc

Putting $\alpha = 0$ in Theorem 1, then we obtain the following corollary:

Corollary 1. *Let $f \in A(p)$ and*

$$(3.6)\quad S(f,p,\mu,\varrho,\delta,\upsilon)=\eta+\delta\left(\frac{f(z)}{z^p}\right)^{\mu}+\rho\left(\frac{f(z)}{z^p}\right)^{2\mu}+\upsilon\mu\left(\frac{zf'(z)}{f(z)}-p\right).$$

If q satisfies the following subordination:

$$(3.7)\qquad S(f,p,\mu,\rho,\delta,\upsilon)\prec\eta+\delta q(z)+\rho(q(z))^2+\upsilon\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\mu}\prec q(z)\ (z\in \mathrm{U}^*)$$

and q is the best dominant of (3.7).

Putting $\alpha=0$ and $p=1$ in Theorem 1 (or $p=1$ in Corollary 1), we obtain the following corollary which corrects the result obtained by Shanmugam et al. [15, Theorem 1, for $a=c=1$].

Corollary 2. *Let $f\in A$, (3.1) holds true and*

$$N(f,\mu,\varrho,\delta,\upsilon)=\eta+\delta\left(\frac{f(z)}{z}\right)^{\mu}+\varrho\left(\frac{f(z)}{z}\right)^{2\mu}+\upsilon\mu\left(\frac{zf'(z)}{f(z)}-1\right).$$

If q satisfies the following subordination:

$$(3.8)\qquad N(f,\mu,\rho,\delta,\upsilon)\prec\eta+\delta q(z)+\rho(q(z))^2+\upsilon\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu}\prec q(z)\ (z\in \mathrm{U}^*)$$

and q is the best dominant of (3.8).

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le A<B\le 1)$ in Theorem 1, the condition (3.1) reduces to

(3.9)
$$\mathrm{Re}\left\{1+\frac{\delta}{\upsilon}\frac{1+Az}{1+Bz}+\frac{2\varrho}{\upsilon}\left(\frac{1+Az}{1+Bz}\right)^2-\frac{(A-B)z}{(1+Az)(1+Bz)}-\frac{2Bz}{1+Bz}\right\}>0,$$

hence, we obtain the next result:

Corollary 3 . *Assume that (3.9) holds true, $f\in A(p)$, $-1\le A<B\le 1$ and*

$$M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)\prec\eta+\delta\frac{1+Az}{1+Bz}$$

$$+\varrho\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\upsilon z(A-B)}{(1+Az)(1+Bz)}, \tag{3.10}$$

where $M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)$ is defined in (3.2), then

$$\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu\prec\frac{1+Az}{1+Bz}\quad(z\in\mathrm{U}^*)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.10).

Taking $q(z)=\left(\frac{1+z}{1-z}\right)^\sigma$, $0<\sigma\le 1$ in Theorem 1, the condition (3.1) reduces to

$$\mathrm{Re}\left\{1+\frac{\delta}{\upsilon}\left(\frac{1+z}{1-z}\right)^\sigma+\frac{2\varrho}{\upsilon}\left(\frac{1+z}{1-z}\right)^{2\sigma}-\frac{2z^2}{1-z^2}\right\}>0, \tag{3.11}$$

hence, we have the following corollary:

Corollary 4. *Assume that (3.11) holds true, $f\in A(p)$, $0<\sigma\le 1$ and*

$$M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)\prec\eta+\delta\left(\frac{1+z}{1-z}\right)^\sigma+\varrho\left(\frac{1+z}{1-z}\right)^{2\sigma}+\frac{2\upsilon\sigma z}{1-z^2}, \tag{3.12}$$

where $M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)$ is defined in (3.2), then

$$\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu\prec\left(\frac{1+z}{1-z}\right)^\sigma\quad(z\in\mathrm{U}^*)$$

and $\left(\frac{1+z}{1-z}\right)^{\sigma}$ *is the best dominant of (3.12).*

Taking $q(z) = e^{\mu A z}$, $|\mu A| < \pi$ in Theorem 1, it is easy to check that the assumption (3.1) holds, hence we obtain the next result.

Corollary 5. *Let* $f \in A(p)$, $|\mu A| < \pi$ *and*

$$M(f, \alpha, p, \mu, \varrho, \delta, \upsilon) \prec \eta + \delta e^{\mu A z} + \varrho e^{2\mu A z} + \upsilon \mu A z, \tag{3.13}$$

where $M(f, \alpha, p, \mu, \varrho, \delta, \upsilon)$ *is defined in (3.2), then*

$$\left(\frac{I_p^{\alpha} f(z)}{z^p}\right)^{\mu} \prec e^{\mu A z} \quad (z \in U^*)$$

and $e^{\mu A z}$ *is the best dominant of (3.13).*

Putting $\eta = 1$, $\alpha = \delta = \varrho = 0$, $\upsilon = \frac{1}{ab}$ $(a, b \in \mathbb{C}^*)$, $\mu = a$, and $q(z) = (1-z)^{-2ab}$ in Theorem 1, it is easy to check that the assumption (3.1) holds, hence combining this together with Lemma 4 we obtain the next result.

Corollary 6. *Let* $f \in A(p), a, b \in C^*$ *such that* $|2ab - 1| \leq 1$ *or* $|2ab + 1| \leq 1$. *If*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right) \prec \frac{1+z}{1-z}, \tag{3.14}$$

then

$$\left(\frac{f(z)}{z^p}\right)^{a} \prec (1-z)^{-2ab} \quad (z \in U^*)$$

and $(1-z)^{-2ab}$ *is the best dominant of (3.14).*

Remark 1. (i) For $p = 1$, Corollary 6 reduces to the result obtained by Obradović et al. [11, Theorem 1];
(ii) For $p = a = 1$, Corollary 6 reduces to the recent result of Srivastava and Lashin [16, Theorem 3] and the recent result of Shanmugam et al. [15, Corollary 3.6].

Putting $\alpha = \delta = \varrho = 0$, $\eta = 1$, $\upsilon = \frac{1}{\mu}$ and $q(z) = (1 + Bz)^{\frac{\mu(A-B)}{B}}$ $(\mu \in \mathbb{C}^*, -1 \leq B < A \leq 1, B \neq 0)$ in Theorem 1, it is easy to check that the assumption (3.1) holds, hence we get the next corollary:

Corollary 7 . *Let $f \in A(p)$, $\mu \in \mathbb{C}^*$, $-1 \le B < A \le 1$, with $B \neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B} - 1\right| \le 1$ or $\left|\frac{\mu(A-B)}{B} + 1\right| \le 1$. If*

$$1 + \frac{zf'(z)}{f(z)} - p \prec \frac{1+Az}{1+Bz}, \tag{3.15}$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}} \quad (\mu \in \mathbb{C}^*;\ z \in \mathrm{U}^*)$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.15).

Remark 2. For $p = 1$, Corollary 7 reduces to the result obtained by Shanmugam et al. [15 , Corollary 3.7].

Remark 3. Putting $\eta = p = 1$, $\alpha = \delta = \varrho = 0$, $\upsilon = \frac{e^{i\lambda}}{ab\cos\lambda}$ $(a, b \in \mathbb{C}^*;\ |\lambda| < \frac{\pi}{2})$, $\mu = a$ and $q(z) = (1-z)^{-2ab\cos\lambda e^{-i\lambda}}$ in Theorem 1, we obtain the result obtained by Aouf et al. [2, Theorem 1].

Theorem 2. *Let $h \in H(U)$, $h(0) = 1, h'(0) \neq 0$ which satisfy the inequality*

$$\mathrm{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \quad (z \in \mathrm{U}).$$

If $f \in A(p)$ satisfies the differential subordination

$$\frac{I_p^{\alpha} f(z)}{z^p} \prec h(z) \quad (z \in \mathrm{U}^*),$$

then

$$\frac{I_p^{\alpha+1} f(z)}{z^p} \prec g(z) \quad (z \in \mathrm{U}^*), \tag{3.16}$$

where

$$g(z) = \frac{p+1}{mz^{\frac{(p+1)}{m}}} \int_0^z h(t) t^{\frac{(p+1)}{m}-1} dt \quad (z \in \mathrm{U}), \tag{3.17}$$

and $g(z)$ is the best dominant of (3.16).

P r o o f. Let the function $p(z)$ be defined by

$$p(z) = \frac{I_p^{\alpha+1} f(z)}{z^p} \quad (z \in \mathrm{U}^*), \tag{3.18}$$

then, differentiating (3.18) logarithmically with respect to z, we deduce that

$$\frac{zp'(z)}{p(z)} = \frac{z\left(I_p^{\alpha+1} f(z)\right)'}{I_p^{\alpha+1} f(z)} - p. \tag{3.19}$$

Using the identity (1.7) in (3.19), a simple computation shows that

$$\frac{zp'(z)}{p(z)} = (p+1)\left(\frac{I_p^{\alpha} f(z)}{I_p^{\alpha+1} f(z)} - 1\right)$$

and hence

$$p(z) + \frac{zp'(z)}{p+1} = \frac{I_p^{\alpha} f(z)}{z^p} \quad (z \in \mathrm{U}^*). \tag{3.20}$$

From (3.20) and using Lemma 2, we get the desired result. ■

4. Superordination results for analytic functions

Next, by using Lemma 3, we obtain to following theorem.

Theorem 3. *Let $q(z)$ be analytic and univalent in* U *such that $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in* U. *Let*

$$\operatorname{Re}\left\{\frac{2\varrho}{\upsilon}(q(z))^2 + \frac{\delta}{\upsilon} q(z)\right\} > 0. \tag{4.1}$$

If $f \in A(p)$, $0 \neq \left(\frac{I_p^{\alpha} f(z)}{z^p}\right)^{\mu} \in H[q(0), 1] \cap Q$, and $M(f, \alpha, p, \mu, \varrho, \delta, \upsilon)$ is univalent in U, then

$$(4.2) \qquad \eta + \delta q(z) + \varrho(q(z))^2 + \upsilon \frac{zq'(z)}{q(z)} \prec M(f, \alpha, p, \mu, \varrho, \delta, \upsilon)$$

where $M(f, \alpha, p, \mu, \varrho, \delta, \upsilon)$ is defined in (3.2), then

$$(4.3) \qquad q(z) \prec \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p} \right)^\mu \quad (z \in \mathrm{U}^*)$$

and q is the best subordinant of (4.2).

P r o o f. By setting

$$\theta(w) = \eta + \delta w + \varrho w^2 \qquad \textit{and} \qquad \varphi(w) = \upsilon \frac{w'}{w},$$

it is easy observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\varphi(w)$ is analytic in $\mathbb{C}^*$ and $\varphi(w) \neq 0$, $w \in \mathbb{C}^*$. Since q is convex and univalent function, it follows that

$$(4.4) \qquad \operatorname{Re}\left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \operatorname{Re}\left\{ \frac{2\varrho}{\upsilon}(q(z))^2 + \frac{\delta}{\upsilon} q(z) \right\} > 0,$$

and then, by using Lemma 3 we deduce that the subordination (4.2), implies $q(z) \prec \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p} \right)^\mu$ $(z \in \mathrm{U}^*)$, and q is the best subordinant of (4.2). ∎

Putting $\alpha = 0$ in Theorem 3, it is easy to check that the assumption (4.1) holds, then we obtain the following corollary:

Corollary 8. *Let $f \in A(p)$ and q satisfies the following subordination*

$$(4.5) \qquad \eta + \delta q(z) + \varrho(q(z))^2 + \upsilon \frac{zq'(z)}{q(z)} \prec S(f, p, \mu, \varrho, \delta, \upsilon),$$

where $S(f, p, \mu, \varrho, \delta, \upsilon)$ defined by (3.6) then

$$q(z) \prec \left(\frac{f(z)}{z^p} \right)^\mu \quad (z \in \mathrm{U}^*)$$

and q is the best subordinant of (4.5).

5. Sandwich results

Combining Theorem 1 with Theorem 3, we get the following sandwich theorem:

Theorem 4. *Let q_1, q_2 be convex univalent in U. Suppose q_1 and q_2 satisfies (4.1) and (3.1), respectively. If $f \in A(p)$, $\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \in H[q(0),1] \cap Q$ and $M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)$ is univalent in U, where $M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)$ is defined in* (3.2), *then*

$$\eta + \delta q_1(z) + \varrho(q_1(z))^2 + \upsilon\frac{zq_1'(z)}{q_1(z)} \prec M(f,\alpha,p,\mu,\varrho,\delta,\upsilon)$$

$$\prec \eta + \delta q_2(z) + \varrho(q_2(z))^2 + \upsilon\frac{zq_2'(z)}{q_2(z)}, \tag{5.1}$$

implies

$$q_1(z) \prec \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\mu \prec q_2(z)\ (z \in \mathrm{U}^*) \tag{5.2}$$

and q_1, q_2 are respectively the best subordinant and best dominant of (5.1).

Putting $\alpha = 0$ in Theorem 4, then we obtain the following corollary:

Corollary 9. *Let q_1, q_2 be convex univalent in U. Suppose q_1 and q_2 satisfies (4.1) and (3.1), respectively. If $f \in A(p)$, $\left(\frac{f(z)}{z^p}\right)^\mu \in H[q(0),1] \cap Q$ and $S(f,p,\mu,\varrho,\delta,\upsilon)$ is univalent in U, where $S(f,p,\mu,\varrho,\delta,\upsilon)$ is defined in (3.6), then*

$$\eta + \delta q_1(z) + \varrho(q_1(z))^2 + \upsilon\frac{zq_1'(z)}{q_1(z)} \prec S(f,p,\mu,\varrho,\delta,\upsilon)$$

$$\prec \eta + \delta q_2(z) + \varrho(q_2(z))^2 + \upsilon\frac{zq_2'(z)}{q_2(z)}, \tag{5.3}$$

implies

$$q_1(z) \prec \left(\frac{f(z)}{z^p}\right)^\mu \prec q_2(z)\ (z \in \mathrm{U}^*) \tag{5.4}$$

and q_1, q_2 are respectively the best subordinant and best dominant of (5.3).

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