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External Characterization of I-Favorable Spaces

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We provide both a spectral and an internal characterizations of arbitrary I-favorable spaces with respect to co-zero sets. As a corollary we establish that any product of compact I-favorable spaces with respect to co-zero sets is also I-favorable with respect to co-zero sets. We also prove that every C^*-embedded I-favorable with respect to co-zero sets subspace of an extremally disconnected space is extremally disconnected.

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1. Introduction

In this paper we assume that the topological spaces are Tychonoff and the single-valued maps are continuous. Moreover, all inverse systems are supposed to have surjective bonding maps.

P. Daniels, K. Kunen and H. Zhou [2] introduced the so called open-open game between two players, and the spaces with a winning strategy for the first player were called I-favorable. Recently A. Kucharski and S. Plewik (see [3], [4] and [5]) investigated the connection of I-favorable spaces and skeletal maps. In particular, they proved in [4] that the class of compact I-favorable spaces and the skeletal maps are adequate in the sense of E. Shchepin [8].

On the other hand, the author announced [13, Theorem 3.1(iii)] a characterization of the class of spaces admitting a lattice [8] of skeletal maps (the skeletal maps in [13] were called ad-open maps) as dense subset of the limit spaces of σ-complete almost continuous inverse systems with skeletal projections. Moreover, an internal characterization of the above class was also announced [13, Theorem 3.1(ii)]. In this paper we are going to show that the later class coincides with that one of I-*favorable spaces with respect to co-zero sets*, and to provide the proof of these characterizations. Therefore, we obtain both a spectral and an internal characterizations of I-favorable spaces with respect to co-zero sets.

The following theorem is our main result:

Theorem 1.1. *For a space X the following conditions are equivalent:*

(i) X is I*-favorable with respect to co-zero sets;*

(ii) Every C^*-embedding of X in another space is π-regular;

(iii) X is skeletally generated.

We say that a subspace $X \subset Y$ is *π-regularly embedded* in Y [13] if there exists a π-base $\mathcal{B}$ for X and a function $\mathrm{e}\colon \mathcal{B} \to \mathcal{T}_Y$, where $\mathcal{T}_Y$ is the topology of Y, such that:

(1) $\mathrm{e}(U) \cap X$ is a dense subset of U;

(2) $\mathrm{e}(U) \cap \mathrm{e}(V) = \varnothing$ provided $U \cap V = \varnothing$.

It is easily seen that the above definition doesn't change if $\mathcal{B}$ is either a base for X or $\mathcal{B} = \mathcal{T}_X$.

A space X is *skeletally generated* if there exists an inverse system $S = \{X_\alpha, p^\beta_\alpha, A\}$ of separable metric spaces X_α such that:

(3) All bonding maps p^β_α are surjective and skeletal;

(4) The index set A is σ-complete (every countable chain in A has a supremum in A);

(5) For every countable chain $\{\alpha_n : n \geq 1\} \subset A$ with $\beta = \sup\{\alpha_n : n \geq 1\}$ the space X_β is a (dense) subset of $\varprojlim\{X_{\alpha_n}, p^{\alpha_{n+1}}_{\alpha_n}\}$;

(6) X is embedded in $\varprojlim S$ such that $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the α-th limit projection;

(7) For every bounded continuous function $f\colon X \to \mathbb{R}$ there exists $\alpha \in A$ and a continuous function $g\colon X_\alpha \to \mathbb{R}$ with $f = g \circ (p_\alpha | X)$.

We say that an inverse system S satisfying conditions (3) – (6) is *almost σ-continuous.* Let us note that condition (6) implies that X is a dense subset of $\varprojlim S$.

There exists a similarity between I-favorable spaces with respect to co-zero sets and κ-metrizable compacta [9]. Item (*ii*) is analogical to Shirokov's

[12] external characterization of κ-metrizable compacta, while the definition of skeletally generated spaces resembles that one of openly generated compacta [10]. Moreover, according to Shapiro's result [12], every continuous image of a κ-metrizable compactum is skeletally generated, so it is I-favorable with respect to co-zero sets. So, next question seems reasonable.

Question. Is there any characterization of κ-metrizable compacta in terms of a game between two players?

It is shown in [2, Corollary 1.7] that the product of I-favorable spaces is also I-favorable. Next corollary shows that a similar result is true for I-favorable spaces with respect to co-zero sets.

Corollary 1.2. *Any product of compact* I*-favorable spaces with respect to co-zero sets is also* I*-favorable with respect to co-zero sets.*

Corollary 1.3 below is similar to a result of Bereznickiĭ [1] about specially embedded subset of extremally disconnected spaces.

Corollary 1.3. *Let X be a C^*-embedded subset of an extremally disconnected space. If X is* I*-favorable with respect to co-zero sets, then it is also extremally disconnected.*

2. I-favorable spaces with respect to co-zero sets

In this section we consider a modification of the open-open game when the players are choosing co-zero sets only. Let us describe this game. Players are playing in a topological space X. Player I choose a non-empty co-zero set $A_0 \subset X$, then Player II choose a non-empty co-zero set $B_0 \subset A_0$. At the n-th round Player I choose a non-empty co-zero set $A_n \subset X$ and the Player II is replying by choosing a non-empty co-zero set $B_n \subset A_n$. Player I wins if the union $B_0 \cup B_1 \cup \ldots$ is dense in X, otherwise Player II wins. The space X is called I-*favorable with respect to co-zero sets* if Player I has a winning strategy. Denote by Σ_X the family of all non-empty co-zero sets in X. A winning strategy, see [3], is a function $\sigma : \bigcup\{\Sigma_X^n : n \geq 0\} \to \Sigma_X$ such that for each game

$$(\sigma(\varnothing), B_0, \sigma(B_0), B_1, \sigma(B_0, B_1), B_2, \ldots, B_n, \sigma(B_0, B_1, .., B_n), B_{n+1},,,),$$

where B_k and $\sigma(\varnothing)$ belong to Σ_X and $B_{k+1} \subset \sigma(B_0, B_1, .., B_k)$ for every $k \geq 0$, the union $\bigcup_{n\geq 0} B_n$ is dense in X. For example, every space with a countable π-base $\mathcal{B}$ of co-zero sets is I-favorable with respect to co-zero sets (the strategy for Player I is to keep choosing every member of $\mathcal{B}$, see [2, Theorem 1.1]). Let us mention that if in the above game the players are choosing arbitrary open subsets of X and Player I has a winning strategy, then X is called I-favorable, see [2].

Proposition 2.1 *If X is* I*-favorable with respect to co-zero sets, so is βX.*

Proof. Let $\sigma : \bigcup\{\Sigma_X^n : n \geq 0\} \to \Sigma_X$ be a winning strategy for Player I. Observe that for every co-zero set U in X there exists a co-zero set $c(U)$ in βX with $c(U) \cap X = U$. Now define a function $\overline{\sigma} : \bigcup\{\Sigma_{\beta X}^n : n \geq 0\} \to \Sigma_{\beta X}$ by

$$\overline{\sigma}(U_1, .., U_n) = c\big(\sigma(U_1 \cap X, .., U_n \cap X)\big).$$

Suppose

$$(\overline{\sigma}(\varnothing), U_0, \overline{\sigma}(U_0), U_1, \overline{\sigma}(U_0, U_1), ..., U_n, \overline{\sigma}(U_0, U_1, .., U_n), U_{n+1},,,)$$

is a sequence such that $\overline{\sigma}(\varnothing)$ and all U_k belong to $\Sigma_{\beta X}$ with $U_{k+1} \subset \overline{\sigma}(U_0, U_1, .., U_k)$ for each $k \geq 0$. Consequently, $U_{k+1} \cap X \subset \sigma(U_0 \cap X, .., U_k \cap X)$, $k \geq 0$. So, the set $X \cap \bigcup_{k\geq 0} U_k$ is dense in X which implies that $\bigcup_{k\geq 0} U_k$ is dense βX. Therefore, βX is I-favorable with respect to co-zero sets. ■

A map $f\colon X \to Y$ is said to be skeletal if the closure $\overline{f(U)}$ of $f(U)$ in Y has a non-empty interior in Y for every open set $U \subset X$. The proof of next lemma is standard.

Lemma 2.2. *For a map $f\colon X \to Y$ the following are equivalent:*

(i) f is skeletal;

(ii) $\overline{f(U)}$ is regularly closed in Y, i.e., its interior $\mathrm{Int}\overline{f(U)}$ in Y is dense in $\overline{f(U)}$ for every open $U \subset X$;

(iii) Every open $U \subset X$ contains an open set V_U such that $f(V_U)$ is dense in some open subset of Y.

If in addition f is closed, the above three conditions are equivalent to $f(U)$ has a non-empty interior in Y for every open $U \subset X$.

A space X is said to be an *almost limit* of the inverse system $S = \{X_\alpha, p_\alpha^\beta, A\}$ if X can be embedded in $\varprojlim S$ such that $p_\alpha(X) = X_\alpha$ for each α. We denote this by $X = \mathrm{a}-\varprojlim S$, and it implies that X is a dense subset of $\varprojlim S$. Let $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ be a well ordered inverse system with (surjective) bonding maps p_α^β, where τ is a given cardinal. We say that S is *almost continuous* if for every limit cardinal $\gamma < \tau$ the space X_γ is naturally embedded in the limit space $\varprojlim\{X_\alpha, p_\alpha^\beta, \alpha < \beta < \gamma\}$. If always $X_\gamma = \varprojlim\{X_\alpha, p_\alpha^\beta, \alpha < \beta < \gamma\}$, S is called *continuous*.

Lemma 2.3. *Let $X = \mathrm{a}-\varprojlim\{X_\alpha, p_\alpha^\beta, A\}$ such that all bonding maps p_α^β are skeletal. Then all p_α and the restrictions $p_\alpha|X: X \to X_\alpha$ are also skeletal.*

Proof. Since X is dense in $\varprojlim\{X_\alpha, p_\alpha^\beta, A\}$, p_α is skeletal iff so is $p_\alpha|X$, $\alpha \in A$. To prove that a given p_α is skeletal, let $U \subset \varprojlim\{X_\alpha, p_\alpha^\beta, A\}$ be an open set. We are going to show that $\mathrm{Int}\overline{p_\alpha(U)} \neq \varnothing$ (both, the interior and the closure are in X_α). We can suppose that $U = p_\beta^{-1}(V)$ for some β with $V \subset X_\beta$ being open. Moreover, since A is directed, there exists $\gamma \in A$ with $\beta < \gamma$ and $\alpha < \gamma$. Then, $p_\alpha(U) = p_\alpha^\gamma(W)$, where $W = (p_\beta^\gamma)^{-1}(V)$. Finally, because p_α^γ is skeletal, $\mathrm{Int}\overline{p_\alpha(U)} \neq \varnothing$. ■

Lemma 2.4. *Every skeletally generated space is* I*-favorable with respect to co-zero sets.*

Proof. Let $X = \mathrm{a} - \varprojlim S$, where $S = \{X_\alpha, p_\alpha^\beta, A\}$ satisfies conditions (3)-(7). Condition (7) implies that for every co-zero set $U \subset X$ there exists $\alpha \in A$ and a co-zero set $V \subset X_\alpha$ with $U = p_\alpha^{-1}(V)$. So, Σ_X is the family of all $p_\alpha^{-1}(V)$, where $\alpha \in A$ and V is open in X_α. Using this observation, we can apply the arguments from the proof of [5, Theorem 2] to define a winning strategy $\sigma : \bigcup\{\Sigma_X^n : n \geq 0\} \to \Sigma_X$. ■

We are going to show that every compactum X which is I-favorable with respect to co-zero sets can be represented as a limit of a continuous system with skeletal bonding maps and I-favorable spaces with respect to co-zero sets of weight less than the weight $w(X)$ of X.

Let us introduced few notations. Suppose $X \subset \mathbb{I}^A$ is a compact space and $B \subset A$. Let $\pi_B: \mathbb{I}^A \to \mathbb{I}^B$ be the natural projection and p_B be restriction map $\pi_B|X$. Let also $X_B = p_B(X)$. If $U \subset X$ we write $B \in k(U)$ to denote that $p_B^{-1}(p_B(U)) = U$. For every co-zero set $U \subset X$ there exist a countable $B \subset A$ such that $B \in k(U)$ with $p(U)$ being a co-zero set in X_B. A base $\mathcal{B}$ for the topology of $X \subset \mathbb{I}^A$ consisting of co-zero sets is called *special* if for every finite $B \subset A$ the family $\{p_B(U) : U \in \mathcal{B}, B \in k(U)\}$ is a base for $p_B(X)$.

Proposition 2.5. *Let $X \subset \mathbb{I}^A$ be a compactum and $\mathcal{B}$ a special base for X. If $\sigma : \bigcup\{\mathcal{B}^n : n \geq 0\} \to \mathcal{B}$ is a function such that for each game*

$$(\sigma(\varnothing), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, ..., U_n, \sigma(U_0, U_1, .., U_n), U_{n+1}, , ,),$$

where $\sigma(\varnothing) \in \mathcal{B}$, $U_i \in \mathcal{B}$ and $U_{i+1} \subset \sigma(U_0, U_1), U_2, ..., U_i)$ for all $i \geq 0$, the union $\bigcup_{n\geq 0} U_n$ is dense in X, then X is skeletally generated.

Proof. For any finite set $B \subset A$ fix a countable family $\lambda_B \subset \mathcal{B}$ such that $\{p_B(U) : U \in \lambda_B\}$ is a base for X_B and $B \in k(U)$ for every $U \in \lambda_B$. Let

$\gamma_B = \bigcup\{\lambda_H : H \subset B\}$ and Γ be the family of all countable sets $B \subset A$ satisfying the following condition:

- If $C \subset B$ is finite and $U_0, U_1, ..., U_n \in \gamma_C$, $n \geq 0$, then $B \in k(\sigma(U_0, U_1, .., U_n))$.

Obviously, if $B_1 \subset B_2 \subset ..$ is a chain in Γ, then $\bigcup_{i\geq 1} B_i \in \Gamma$. We claim that $X = \varprojlim\{X_B, p^C_B, B \subset C, \Gamma\}$. It suffices to show that every countable subset of A is contained in an element of Γ. To this end, let $B_0 \subset A$ be countable. Construct by induction countable sets $B(m) \subset A$ such that for all $m \geq 0$ we have:

- $B_0 \subset B(m) \subset B(m+1)$;

- $B(m+1) \in k(\sigma(U_0, U_1, .., U_n))$, where $U_0, U_1, .., U_n \in \gamma_C$ with $n \geq 0$ and $C \subset B(m)$ finite.

Suppose $B(j)$, $j \leq m$, are already constructed for some $m \geq 1$. For every finite $C \subset B(m)$ and $U_0, U_1, .., U_n \in \gamma_C$ there exist a countable set $B(U_0, U_1, .., U_n) \subset A$ with $B(U_0, U_1, .., U_n) \in k(\sigma(U_0, U_1, .., U_n))$. Let $B(m+1)$ be the union of $B(m)$ and all $B(U_0, U_1, .., U_n)$, where $U_0, U_1, .., U_n \in \gamma_C$ with C being a finite subset of $B(m)$ and $n \geq 0$. Obviously $B(m+1)$ is countable and satisfies the required conditions. This completes the inductive step. Finally, $B_\infty = \cup_{m=0}^{\infty} B(m)$ belongs to Γ. Hence, $X = \varprojlim\{X_B, p^C_B, B \subset C, \Gamma\}$.

Next two claims complete the proof of Proposition 2.5.

Claim 1. *If $B \in \Gamma$, then for each open $V \subset X$ there exists a finite set $C \subset B$ and a finite family $U_0, U_1, .., U_n \in \gamma_C$ such that $p_B(U) \cap p_B(V) \neq \varnothing$ for any $U \in \gamma_H$, where $H \subset B$ is finite and $U \subset \sigma(U_0, U_1, .., U_n)$.*

Assume Claim 1 does not hold. Then there exists an open set $V \subset X$ such that for any finite $C \subset B$ and any $U_0, U_1, .., U_n \in \gamma_C$ there exists finite $H \subset B$ and $U \in \gamma_H$ such that $U \subset \sigma(U_0, U_1, .., U_n)$ and $p_B(U) \cap p_B(V) = \varnothing$. This allows us to construct by induction a sequence $\{C(m)\}_{m\geq 0}$ of finite subsets of B and families $\{U_0, U_1, .., U_m\} \subset \gamma_{C(m)}$ such that $U_m \subset \sigma(U_0, U_1, .., U_{m-1})$ and $p_B(U_m) \cap p_B(V) = \varnothing$. Indeed, we take $\sigma(\varnothing) \in \mathcal{B}$ with $B \in k(\sigma(\varnothing))$ and suppose the sets $C(1), ..., C(m)$ and the families $\{U_0, U_1, .., U_m\} \subset \gamma_{C(m)}$ satisfying the above conditions are already constructed. Consequently, there exists $U_{m+1} \in \gamma_D$, where $D \subset B$ is finite, such that $U_{m+1} \subset \sigma(U_0, U_1, .., U_m)$ and $p_B(U_{m+1}) \cap p_B(V) = \varnothing$. Observe that both $\{U_0, U_1, .., U_m\} \subset \gamma_{C(m)}$ and

$U_{m+1} \in \gamma_D$ implies the inclusion $\{U_0, U_1, .., U_m, U_{m+1}\} \subset \gamma_{C(m+1)}$, where $C(m+1) = C(m) \cup D$. This completes the inductive step. So, we obtained a sequence

$$\sigma(\varnothing), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, ..., U_n, \sigma(U_0, U_1, .., U_n), U_{n+1}, ..$$

from $\mathcal{B}$ such that $U_{i+1} \subset \sigma(U_0, U_1, U_2, ..., U_i)$, $B \in k(U_i)$ and $p_B(U_i) \cap p_B(V) = \varnothing$ for all i. The last two conditions yields $U_i \cap V = \varnothing$ for all $i \geq 0$ which contradicts the density of the set $\bigcup_{i\geq 0} U_i$ in X.

Claim 2. p_B is a skeletal map for each $B \in \Gamma$.

Suppose $V \subset X$ is open. Then there a finite set $C \subset B$ and a family $U_0, U_1, .., U_n \in \gamma_C$ satisfying the conditions from Claim 1. Since $B \in k(\sigma(U_0, U_1, .., U_m))$, $p_B(\sigma(U_0, U_1, .., U_m))$ is open in X_B. Hence, it suffices to show the inclusion $p_B(\sigma(U_0, U_1, .., U_m)) \subset \overline{p_B(V)}$. Assuming the contrary, we obtain that $p_B(\sigma(U_0, U_1, .., U_m)) \backslash \overline{p_B(V)}$ is a non-empty open subset of X_B. Moreover, $\bigcup\{p_B(\gamma_C) : C \subset B \text{ is finite}\}$ is a base for X_B. Therefore, there is $U \in \gamma_C$ with $C \subset B$ finite such that $p_B(U)$ is contained in $p_B(\sigma(U_0, U_1, .., U_m)) \backslash \overline{p_B(V)}$. Consequently, $U \subset \sigma(U_0, U_1, .., U_m)$ and $p_B(U) \cap p_B(V) = \varnothing$, a contradiction. ■

Theorem 2.6. *Let X be a compact I-favorable space with respect to co-zero sets and $w(X) = \tau$ is uncountable. Then there exists a continuous inverse system $S = \{X_\alpha, p_\alpha^\beta, \tau\}$ of compact I-favorable spaces X_α with respect to co-zero sets and skeletal bonding maps p_α^β such that $w(X_\alpha) < \tau$ for each $\alpha < \tau$ and $X = \varprojlim S$.*

Proof. Let $\sigma : \bigcup\{\Sigma_X^n : n \geq 0\} \to \Sigma_X$, where Σ_X is the family of all co-zero sets in X, be a winning strategy for Player I. We embed X in a Tychonoff cube $\mathbb{I}^A$ with $|A| = \tau$ and fix a base $\{U_\alpha : \alpha < \tau\}$ for X of cardinality τ which consists of co-zero sets such that for each α there exists a finite set H_α with $H_\alpha \in k(U_\alpha)$. For any finite set $C \subset A$ let γ_C be a fixed countable base for X_C. Observe that for every $U \in \Sigma_X$ there exists a countable set $B(U) \subset A$ such that $B(U) \in k(U)$ and $p_{B(U)}(U)$ is a co-zero set in $X_{B(U)}$. This follows from the fact that each continuous function f on X can be represented in the form $f = g \circ p_B$ with $B \subset A$ countable and g being a continuous function on X_B. We identify A with all infinite cardinals $\alpha < \tau$ and construct by transfinite induction subsets $A(\alpha) \subset A$ and families $\mathcal{U}(\alpha) \subset \Sigma_X$ satisfying the following conditions:

(8) $|A(\alpha)| \leq \alpha$ and $|\mathcal{U}(\alpha)| \leq \alpha$;

(9) $A(\alpha) \in k(U)$ for all $U \in \mathcal{U}(\alpha)$;

(10) $p_C^{-1}(\gamma_C) \subset \mathcal{U}(\alpha)$ for each finite $C \subset A(\alpha)$;

(11) $\{U_\beta : \beta < \alpha\} \subset \mathcal{U}(\alpha)$ and $\{\beta : \beta < \alpha\} \subset A(\alpha)$;

(12) $\sigma(U_1, .., U_n) \in \mathcal{U}(\alpha)$ for every finite family $\{U_1, .., U_n\} \subset \mathcal{U}(\alpha)$;

(13) $A(\alpha) = \bigcup\{A(\beta) : \beta < \alpha\}$ and $\mathcal{U}(\alpha) = \bigcup\{\mathcal{U}(\beta) : \beta < \alpha\}$ for all limit cardinals α.

Suppose all $A(\beta)$ and $\mathcal{U}(\beta)$, $\beta < \alpha$, have already been constructed for some $\alpha < \tau$. If α is a limit cardinal, we put $A(\alpha) = \bigcup\{A(\beta) : \beta < \alpha\}$ and $\mathcal{U}(\alpha) = \bigcup\{\mathcal{U}(\beta) : \beta < \alpha\}$. If $\alpha = \beta+1$, we construct by induction a sequence $\{C(m)\}_{m\geq 0}$ of subsets of A, and a sequence $\{\mathcal{V}_m\}_{m\geq 0}$ of co-zero families in X such that:

- $C_0 = A(\beta) \cup \{\beta\}$ and $\mathcal{V}_0 = \mathcal{U}(\beta) \cup \{U_\beta\}$;
- $C(m+1) = C(m) \bigcup\{B(U) : U \in \mathcal{V}_m\}$;
- $\mathcal{V}_{2m+1} = \mathcal{V}_{2m} \bigcup\{\sigma(U_1, .., U_s) : U_1, .., U_s \in \mathcal{V}_{2m}, s \geq 1\}$;
- $\mathcal{V}_{2m+2} = \mathcal{V}_{2m+1} \bigcup\{p_C^{-1}(\gamma_C) : C \subset C(2m+1) \text{ is finite}\}$.

Now, we define $A(\alpha) = \bigcup_{m\geq 0} C(m)$ and $\mathcal{U}(\alpha) = \bigcup_{m\geq 0} \mathcal{V}_m$. It is easily seen that $A(\alpha)$ and $\mathcal{U}(\alpha)$ satisfy conditions (8)-(13).

For every $\alpha < \tau$ let $X_\alpha = X_{A(\alpha)}$ and $p_\alpha = p_{A(\alpha)}$. Moreover, if $\alpha < \beta$, we have $A(\alpha) \subset A(\beta)$. In such a situation let $p_\alpha^\beta = p_{A(\alpha)}^{A(\beta)}$. Since $A = \bigcup_{\alpha<\tau} A(\alpha)$, we obtain a continuous inverse system $S = \{X_\alpha, p_\alpha^\beta, \tau\}$ whose limit is X. Observe also that each X_α is of weight $< \tau$ because $p_\alpha(\mathcal{U}(\alpha))$ is a base for X_α (see condition (10)).

Claim 3. Each X_α is I-favorable with respect to co-zero sets.

Indeed, by conditions (9)-(10), $\mathcal{B}_\alpha = p_\alpha(\mathcal{U}(\alpha))$ is a special base for X_α consisting of co-zero sets. We define a function $\sigma_\alpha : \bigcup\{\mathcal{B}_\alpha^n : n \geq 0\} \to \mathcal{B}_\alpha$ by

$$\sigma_\alpha(p_\alpha(U_0), p_\alpha(U_1), .., p_\alpha(U_n)) = p_\alpha(\sigma(U_0, U_1, .., U_n)).$$

This definition is correct because of conditions (9) and (12). Condition (9) implies that σ_α satisfies the hypotheses of Proposition 2.5. Hence, according to this proposition, X_α is skeletally generated. Finally, by Lemma 2.4, X_α is I-favorable with respect to co-zero sets.

Claim 4. *All bonding maps* p^β_α *are skeletal.*

It suffices to show that all p_α are skeletal. And this is really true because each family $\mathcal{U}(\alpha)$ is stable with respect to σ, see (12). Hence, by [3, Lemma 9], for every open set $V \subset X$ there exists $W \in \mathcal{U}(\alpha)$ such that whenever $U \subset W$ and $U \in \mathcal{U}(\alpha)$ we have $V \cap U \neq \varnothing$. The last statement yields that p_α is skeletal. Indeed, let $V \subset X$ be open, and $W \in \mathcal{U}(\alpha)$ be as above. Then $p_\alpha(W)$ is a co-zero set in X_α because of condition (9). We claim that $p_\alpha(W) \subset \overline{p_\alpha(V)}$. Otherwise, $p_\alpha(W)\backslash\overline{p_\alpha(V)}$ would be a non-empty open subset of X_α. So, $p_\alpha(U) \subset p_\alpha(W)\backslash\overline{p_\alpha(V)}$ for some $U \in \mathcal{U}(\alpha)$ (recall that $p_\alpha(\mathcal{U}(\alpha))$ is a base for X_α). Since, by (9), $p_\alpha^{-1}(p_\alpha(U)) = U$ and $p_\alpha^{-1}(p_\alpha(W)) = W$, we obtain $U \subset W$ and $U \cap V = \varnothing$ which is a contradiction. ∎

3. Proof of Theorem 1.1 and Corollaries 1.2 - 1.3

Suppose $X = \mathrm{a} - \varprojlim S$ with $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ being almost continuous, and $H \subset X$. The set

$$q(H) = \{\alpha : \mathrm{Int}\big(((p^{\alpha+1}_\alpha)^{-1}(\overline{p_\alpha(H)}))\backslash\overline{p_{\alpha+1}(H)}\big) \neq \varnothing\}$$

is called a *rank of* H.

Lemma 3.1. *Let* $X = \mathrm{a} - \varprojlim S$ *and* $U \subset X$ *be open, where* $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ *is almost continuous with skeletal bonding maps. Then we have:*

(i) $\alpha \notin q(U)$ *if and only if* $(p^{\alpha+1}_\alpha)^{-1}(\mathrm{Int}\overline{p_\alpha(U)}) \subset \overline{p_{\alpha+1}(U)}$;

(ii) $q(U) \cap [\alpha, \tau) = \varnothing$ *provided* $U = p_\alpha^{-1}(V)$ *for some open* $V \subset X_\alpha$.

Proof. The first item follows directly from the definition of $q(U)$. For the second one, suppose $\beta \in q(U)$ for some $\beta \geq \alpha$. Then $W = (p^{\beta+1}_\beta)^{-1}(\mathrm{Int}\overline{p_\beta(U)})\backslash\overline{p_{\beta+1}(U)} \neq \varnothing$ is open in $X_{\beta+1}$. Since $p^{\beta+1}_\beta$ is skeletal, $\mathrm{Int}\overline{p^{\beta+1}_\beta(W)}$ is a non-empty open subset of X_β which is contained in $\overline{p_\beta(U)}$. Observe that $p_\beta(U)$ is open in X_β because $p_\beta(U) = (p^\alpha_\beta)^{-1}(V)$. Hence, $p_\beta(U) \cap p^{\beta+1}_\beta(W) \neq \varnothing$. The last relation implies $W \cap p_{\beta+1}(U) \neq \varnothing$ since $p_{\beta+1}(U) = (p^{\beta+1}_\alpha)^{-1}(V) = (p^{\beta+1}_\alpha)^{-1}(p_\beta(U))$. On the other hand, $W \cap p_{\beta+1}(U) = \varnothing$, a contradiction. ∎

Lemma 3.2. *Let* $S = \{X_\alpha, p^\beta_\alpha, 1 \leq \alpha < \beta < \tau\}$ *be an inverse system with skeletal bonding maps and* $X = \varprojlim S$. *Suppose* $U \subset X$ *is open such that* $(p^\alpha_1)^{-1}(\mathrm{Int}\overline{p_1(U)}) \subset \mathrm{Int}\overline{p_\alpha(U)}$ *for all* $\alpha < \tau$. *Then* $p_1^{-1}(\mathrm{Int}\overline{p_1(U)}) \subset \overline{U}$.

Proof. Suppose $W = p_1^{-1}(\mathrm{Int}\overline{p_1(U)})\backslash\overline{U} \neq \varnothing$. Then there exists $\mu < \tau$ and open $V \subset X_\mu$ with $p_\mu^{-1}(V) \subset W$. Hence $p_1^\mu(V) \subset \mathrm{Int}\overline{p_1(U)}$, so $V \subset (p_1^\mu)^{-1}(\mathrm{Int}\overline{p_1(U)}) \subset \mathrm{Int}\overline{p_\mu(U)}$. The last inclusion implies that $p_\mu^{-1}(V)$ meets $\overline{p_\alpha(U)}$, a contradiction. ■

Lemma 3.3. *Let $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ be a continuous inverse system with skeletal bonding maps and $X = \varprojlim S$. Assume $U, V \subset X$ are open with $q(U)$ and $q(V)$ finite and $\overline{U} \cap \overline{V} = \varnothing$. If $q(U) \cap q(V) \cap [\gamma, \tau) = \varnothing$ for some $\gamma < \tau$, then $\mathrm{Int}\overline{p_\gamma(U)}$ and $\mathrm{Int}\overline{p_\gamma(V)}$ are disjoint.*

Proof. Suppose $\mathrm{Int}\overline{p_\gamma(U)} \cap \mathrm{Int}\overline{p_\gamma(V)} \neq \varnothing$. We are going to show by transfinite induction that $\mathrm{Int}\overline{p_\beta(U)} \cap \mathrm{Int}\overline{p_\beta(V)} \neq \varnothing$ for all $\beta \geq \gamma$. Assume this is done for all $\beta \in (\gamma, \alpha)$ with $\alpha < \tau$. If α is not a limit cardinal, then $\alpha - 1$ belongs to at least one of the sets $q(U)$ and $q(V)$. Suppose $\alpha - 1 \notin q(V)$. Hence, $(p_{\alpha-1}^\alpha)^{-1}(\mathrm{Int}\overline{p_{\alpha-1}(V)}) \subset \mathrm{Int}\overline{p_\alpha(V)}$ (see Lemma 3.1(i)). Because of our assumption, $\mathrm{Int}\overline{p_{\alpha-1}(U)} \cap \mathrm{Int}\overline{p_{\alpha-1}(V)} \neq \varnothing$. Moreover, $p_{\alpha-1}^\alpha(\overline{p_\alpha(U)})$ is dense in $\overline{p_{\alpha-1}(U)}$. Hence, $\mathrm{Int}\overline{p_{\alpha-1}(V)}$ meets $p_{\alpha-1}^\alpha(\overline{p_\alpha(U)})$. This yields $\mathrm{Int}\overline{p_\alpha(V)} \cap \overline{p_\alpha(U)} \neq \varnothing$. Finally, since by Lemma 2.2(ii) $\overline{p_\alpha(U)}$ is the closure of its interior, $\mathrm{Int}\overline{p_\alpha(V)} \cap \mathrm{Int}\overline{p_\alpha(U)} \neq \varnothing$.

Suppose $\alpha > \gamma$ is a limit cardinal. Since $q(U) \cap q(V)$ is a finite set, there exists $\lambda \in (\gamma, \alpha)$ such that $\beta \notin q(U) \cap q(V)$ for every $\beta \in [\lambda, \alpha)$. Then for all $\beta \in [\lambda, \alpha)$ we have $(p_\beta^{\beta+1})^{-1}(\mathrm{Int}\overline{p_\beta(U)}) \subset \mathrm{Int}\overline{p_{\beta+1}(U)}$ and $(p_\beta^{\beta+1})^{-1}(\mathrm{Int}\overline{p_\beta(V)}) \subset \mathrm{Int}\overline{p_{\beta+1}(V)}$. This allows us to find points $x_\beta \in \mathrm{Int}\overline{p_\beta(U)} \cap \mathrm{Int}\overline{p_\beta(V)}$, $\beta \in [\lambda, \alpha)$, such that $p_\theta^\beta(x_\beta) = x_\theta$ for all $\lambda \leq \theta \leq \beta < \alpha$. Because X_α is the limit space of the inverse system $S_\lambda^\alpha = \{X_\theta, p_\theta^\beta, \lambda \leq \theta \leq \beta < \alpha\}$, we obtain a point $x_\alpha \in X_\alpha$ with $p_\theta^\alpha(x_\alpha) = x_\theta$, $\theta \in [\gamma, \alpha)$. Next claim implies $x_\alpha \in \mathrm{Int}\overline{p_\alpha(U)} \cap \mathrm{Int}\overline{p_\alpha(V)}$ which completes the induction.

Claim 5. *For all $\theta \in [\lambda, \alpha)$ we have $(p_\theta^\alpha)^{-1}(\mathrm{Int}\overline{p_\theta(V)}) \subset \mathrm{Int}\overline{p_\alpha(V)}$ and $(p_\theta^\alpha)^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \mathrm{Int}\overline{p_\alpha(U)}$.*

Fix $\theta \in [\lambda, \alpha)$ and let Λ be the set of all $\beta \in [\theta, \alpha)$ such that $(p_\theta^\beta)^{-1}(\mathrm{Int}\overline{p_\theta(U)})\backslash\overline{p_\beta(U)} \neq \varnothing$. Suppose that $\Lambda \neq \varnothing$ and denote by ν the minimal element of Λ. Therefore $W_\nu = (p_\theta^\nu)^{-1}(\mathrm{Int}\overline{p_\theta(U)})\backslash\overline{p_\nu(U)} \neq \varnothing$. Observe that $\nu > \theta$ because $\theta \notin q(U)$. Moreover, ν is a limit cardinal. Indeed, otherwise $(p_\theta^{\nu-1})^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \mathrm{Int}\overline{p_{\nu-1}(U)}$. On the other hand $\nu - 1 \notin q(U)$ yields $(p_{\nu-1}^\nu)^{-1}(\mathrm{Int}\overline{p_{\nu-1}(U)}) \subset \mathrm{Int}\overline{p_\nu(U)}$. Hence, $(p_\theta^\nu)^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \mathrm{Int}\overline{p_\nu(U)}$, a contradiction. So, X_ν is the limit of the inverse system $S_\theta^\nu = \{X_\beta, p_\beta^\mu, \theta \leq \beta \leq$

$\mu < \nu\}$. Now, we apply Lemma 3.2 to the system S_ν and the set $\mathrm{Int}\overline{p_\nu(U)}$, to conclude that $(p^\nu_\theta)^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \overline{p_\nu(U)}$ which contradicts $W_\nu \neq \varnothing$. Consequently, $\Lambda = \varnothing$ and $(p^\beta_\theta)^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \overline{p_\beta(U)}$ for all $\beta \in [\theta, \alpha)$. We can apply again Lemma 3.2 to the system $S^\alpha_\theta = \{X_\mu, p^\beta_\mu, \theta \leq \mu \leq \beta < \alpha\}$ and the set $\mathrm{Int}\overline{p_\alpha(U)}$ to obtain that $(p^\alpha_\theta)^{-1}(\mathrm{Int}\overline{p_\theta(U)}) \subset \mathrm{Int}\overline{p_\alpha(U)}$. Similarly, we can show that $(p^\alpha_\theta)^{-1}(\mathrm{Int}\overline{p_\theta(V)}) \subset \mathrm{Int}\overline{p_\alpha(V)}$ which completes the proof of Claim 5.

Therefore, $\mathrm{Int}\overline{p_\beta(U)} \cap \mathrm{Int}\overline{p_\beta(V)} \neq \varnothing$ for all $\beta \in [\gamma, \tau)$. To finish the proof of this lemma, take $\lambda(0) \in (\gamma, \tau$ such that $(q(U) \cup q(V)) \cap [\lambda(0), \tau) = \varnothing$. Repeating the arguments from Claim 5, we can show that $(p^\alpha_{\lambda(0)})^{-1}(\mathrm{Int}\overline{p_{\lambda(0)}(U)}) \subset \mathrm{Int}\overline{p_\alpha(U)}$ and $(p^\alpha_{\lambda(0)})^{-1}(\mathrm{Int}\overline{p_{\lambda(0)}(V)}) \subset \mathrm{Int}\overline{p_\alpha(V)}$ for all $\alpha \in [\lambda(0), \tau)$. Then apply Lemma 3.2 to the inverse system $S_{\lambda(0)} = \{X_\mu, p^\beta_\mu, \lambda(0) \leq \mu \leq \beta < \tau\}$ and the set U to obtain that $p^{-1}_{\lambda(0)}(\mathrm{Int}\overline{p_{\lambda(0)}(U)}) \subset \mathrm{Int}\overline{U}$. Similarly, we have $p^{-1}_{\lambda(0)}(\mathrm{Int}\overline{p_{\lambda(0)}(V)}) \subset \mathrm{Int}\overline{V}$. Since $\mathrm{Int}\overline{p_{\lambda(0)}(U)} \cap \mathrm{Int}\overline{p_{\lambda(0)}(V)} \neq \varnothing$, the last two inclusions imply $\overline{U} \cap \overline{V} \neq \varnothing$, a contradiction. Hence, $\mathrm{Int}\overline{p_\gamma(U)} \cap \mathrm{Int}\overline{p_\gamma(V)} = \varnothing$. ■

Next proposition was announced in [13]:

Proposition 3.4. *[13, Proposition 3.2] Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ be an almost continuous inverse system with skeletal bonding maps such that $X = \mathrm{a} - \varprojlim S$. Then the family of all open subsets of X having a finite rank is a π-base for X.*

Proof. First, following the proof of [8, Section 3, Lemma 2], we are going to show by transfinite induction that for every $\alpha < \tau$ the open subsets $U \subset X$ with $q(U) \cap [1, \alpha]$ being finite form a π-base for X. Obviously, this is true for finite α, and it holds for $\alpha + 1$ provided it is true for α. So, it remains to prove this statement for a limit cardinal α if it is true for any $\beta < \alpha$. Suppose $G \subset X$ is open. Let $S_\alpha = \{X_\gamma, p^\beta_\gamma, \gamma < \beta < \alpha\}$, $Y_\alpha = \varprojlim S_\alpha$ and $\tilde{p}^\alpha_\gamma \colon Y_\alpha \to X_\gamma$ are the limit projections of S_α. Obviously, X_α is naturally embedded as a dense subset of Y_α and each $\tilde{p}^\alpha_\gamma$ restricted on X_α is p^α_γ. Then, by Lemma 2.3, $\mathrm{Int}\overline{p_\alpha(G)}$ is non-empty and open in X_α (here both interior and closure are taken in X_α). So, there exists $\gamma < \alpha$ and an open set $U_\gamma \subset X_\gamma$ with $(\tilde{p}^\alpha_\gamma)^{-1}(U_\gamma) \subset \mathrm{Int}_{Y_\alpha}\overline{p_\alpha(G)}^{Y_\alpha}$. Consequently, $(p^\alpha_\gamma)^{-1}(U_\gamma) \subset \mathrm{Int}\overline{p_\alpha(G)}$. We can suppose that $U_\gamma = \mathrm{Int}\overline{U_\gamma}$. Then, according to the inductive assumption, $p^{-1}_\gamma(U_\gamma) \cap G$ contains an open set $W \subset X$ such that $q(W) \cap [1, \gamma]$ is finite. So, $W_\gamma = \mathrm{Int}\overline{p_\gamma(W)} \neq \varnothing$ and it is contained in U_γ. Hence, $p^{-1}_\gamma(W_\gamma) \cap G$ is a non-empty open subset of X contained in G.

Claim 6. $q(p^{-1}_\gamma(W_\gamma) \cap G) \cap [1, \alpha] = q(W) \cap [1, \gamma)$.

Indeed, for every $\beta \le \gamma$ we have $\overline{p_\beta(p_\gamma^{-1}(W_\gamma)\cap G)} = \overline{p_\beta(W)}$. This implies

$$q(W)\cap[1,\gamma) = q(p_\gamma^{-1}(W_\gamma)\cap G)\cap[1,\gamma). \tag{14}$$

Moreover, if $\beta\in[\gamma,\alpha)$, then

$$\overline{p_\beta(p_\gamma^{-1}(W_\gamma)\cap G)} = \overline{p_\beta(p_\gamma^{-1}(W_\gamma))}$$

because $W_\gamma \subset U_\gamma$ and $(p_\gamma^\alpha)^{-1}(U_\gamma)\subset \overline{p_\alpha(G)}$. Hence,

$$q(p_\gamma^{-1}(W_\gamma)\cap G)\cap[\gamma,\alpha) = q(p_\gamma^{-1}(W_\gamma))\cap[\gamma,\alpha). \tag{15}$$

Obviously, by Lemma 3.1(ii), $q(p_\gamma^{-1}(W_\gamma))\cap[\gamma,\alpha)=\varnothing$. Then the combination of (14) and (15) provides the proof of the claim.

Therefore, for every $\alpha<\tau$ the open sets $W\subset X$ with $q(W)\cap[1,\alpha]$ finite form a π-base for X. Now, we can finish the proof of the proposition. If $V\subset X$ is open we find a set $G\subset V$ with $G=p_\beta^{-1}(G_\beta)$, where G_β is open in X_β. Then there exists an open set $W\subset G$ such that $q(W)\cap[1,\beta]$ is finite. Let $W_\beta = \mathrm{Int}\overline{p_\beta(W)}$ and $U=p_\beta^{-1}(W_\beta\cap G_\beta)$. It is easily seen that $\overline{p_\nu(U)}=\overline{p_\nu(W)}$ for all $\nu\le\beta$. This yields that $q(U)\cap[1,\beta)=q(W)\cap[1,\beta)$. On the other hand, by Lemma 3.1(ii), $q(U)\cap[\beta,\tau)=\varnothing$. Hence $q(U)$ is finite. ∎

Proposition 3.5. *Let X be a compact* I*-favorable space with respect to co-zero sets. Then every embedding of X in another space is π-regular.*

Proof. We are going to prove this proposition by transfinite induction with respect to the weight $w(X)$. This is true if X is metrizable, see for example [6, §21, XI, Theorem 2]. Assume the proposition is true for any compact space Y of weight $<\tau$ such that Y is I-favorable with respect to co-zero sets, where τ is an uncountable cardinal. Suppose X is compact I-favorable with respect to co-zero sets and $w(X)=\tau$. Then, by Theorem 2.6, X is the limit space of a continuous inverse system $S=\{X_\alpha, p_\alpha^\beta, \alpha<\beta<\tau\}$ such that all X_α are compact I-favorable with respect to co-zero sets spaces of weight $<\tau$ and all bonding maps are surjective and skeletal. If suffices to show that there exists a π-regular embedding of X in a Tychonoff cube $\mathbb{I}^A$ for some card(A).

By Proposition 3.4, X has a π-base $\mathcal{B}$ consisting of open sets $U\subset X$ with finite rank. For every $U\in\mathcal{B}$ let $\Omega(U)=\{\alpha_0,\alpha,\alpha+1:\alpha\in q(U)\}$, where $\alpha_0<\tau$ is fixed. Obviously, X is a subset of $\prod\{X_\alpha:\alpha<\tau\}$. For every $U\in\mathcal{B}$ we consider the open set $\Gamma(U)\subset\prod\{X_\alpha:\alpha<\tau\}$ defined by

$$\Gamma(U)=\prod\{\mathrm{Int}\overline{p_\alpha(U)}:\alpha\in\Omega(U)\}\times\prod\{X_\alpha:\alpha\notin\Omega(U)\}.$$

Claim 7. *$\Gamma(U_1) \cap \Gamma(U_2) = \varnothing$ whenever $\overline{U_1} \cap \overline{U_2} = \varnothing$. Moreover, there exists $\beta \in \Omega(U_1) \cap \Omega(U_2)$ with $\overline{p_\beta(U_1)} \cap \overline{p_\beta(U_2)} = \varnothing$.*

Let $\beta = \max\{\Omega(U_1) \cap \Omega(U_2)\}$. Then β is either α_0 or $\max\{q(U_1) \cap q(U_2)\} + 1$. In both cases $q(U_1) \cap q(U_2) \cap [\beta, \tau) = \varnothing$. According to Lemma 3.3, $\mathrm{Int}\overline{p_\beta(U_1)} \cap \mathrm{Int}\overline{p_\beta(U_2)} = \varnothing$. Since $\beta \in \Omega(U_1) \cap \Omega(U_2)$, $\Gamma(U_1) \cap \Gamma(U_2) = \varnothing$.

Suppose $U \subset X$ is open. Since all p_α and p_α^β are closed skeletal maps (see Lemma 2.2 and Lemma 2.3), $U_\alpha = \mathrm{Int}p_\alpha(U)$ is a non-empty subset of X_α for every α.

Claim 8. *$\bigcap\{p_\alpha^{-1}(U_\alpha) \cap U : \alpha \in \Delta\} \neq \varnothing$ for every finite set $\Delta \subset \{\alpha : \alpha < \tau\}$.*

Obviously, this is true if $|\Delta| = 1$. Suppose it is true for all Δ with $|\Delta| \leq n$ for some n, and let $\{\alpha_1, .., \alpha_n, \alpha_{n+1}\}$ be a finite set of $n+1$ cardinals $< \tau$. Then $V = \bigcap_{i \leq n} p_{\alpha_i}^{-1}(U_{\alpha_i}) \cap U \neq \varnothing$. Since $p_{\alpha_{n+1}}$ is skeletal, $W = \mathrm{Int}p_{\alpha_{n+1}}(V)$ is a non-empty subset of $X_{\alpha_{n+1}}$, so $W \subset U_{\alpha_{n+1}}$. Consequently $\bigcap_{i \leq n+1} p_{\alpha_i}^{-1}(U_{\alpha_i}) \cap U \neq \varnothing$.

Claim 9. *$\Gamma(U) \cap X$ is a non-empty subset of $\overline{U}$ for all $U \in \mathcal{B}$.*

We are going to show first that $\Gamma(U) \cap X \neq \varnothing$ for all $U \in \mathcal{B}$. Indeed, we fix such U and let $\Omega(U) = \{\alpha_i : i \leq k\}$ with $\alpha_i \leq \alpha_j$ for $i \leq j$. By Claim 8, there exists $x \in \bigcap_{i \leq k} p_{\alpha_i}^{-1}(U_{\alpha_i}) \cap U$. So, $p_{\alpha_i}(x) \in U_{\alpha_i}$ for all $i \leq k$. This implies $x \in \Gamma(U) \cap X$.

To show that $\Gamma(U) \cap X \subset \overline{U}$, let $x \in \Gamma(U) \cap X$. Define $\beta(U) = \max q(U) + 1$. Then $p_{\beta(U)}(x) \in \mathrm{Int}\overline{p_{\beta(U)}(U)}$. Since $\alpha \notin q(U)$ for all $\alpha \geq \beta(U)$, the arguments from Claim 5 show that $\left(p_{\beta(U)}^{\alpha}\right)^{-1}\left(\mathrm{Int}\overline{p_{\beta(U)}(U)}\right) \subset \mathrm{Int}\overline{p_\alpha(U)}$ for $\alpha \geq \beta(U)$. Hence, applying Lemma 3.2 to the inverse system $S_U = \{X_\alpha, p_\alpha^\beta, \beta(U) \leq \alpha \leq \beta < \tau\}$ and the set U, we obtain $x \in p_{\beta(U)}^{-1}\left(\mathrm{Int}\overline{p_{\beta(U)}(U)}\right) \subset \overline{U}$. This completes the proof of Claim 9.

According to our assumption, each X_α is π-regularly embedded in $\mathbb{I}^{A(\alpha)}$ for some $A(\alpha)$. So, there exists a π-regular operator $\mathrm{e}_\alpha : \mathcal{T}_{X_\alpha} \to \mathcal{T}_{\mathbb{I}^{A(\alpha)}}$. For every $U \in \mathcal{B}$ consider the open set $\theta_1(U) \subset \prod\{\mathbb{I}^{A(\alpha)} : \alpha < \tau\}$,

$$\theta_1(U) = \prod\{\mathrm{e}_\alpha\left(\mathrm{Int}\overline{p_\alpha(U)}\right) : \alpha \in \Omega(U)\} \times \prod\{\mathbb{I}^{A(\alpha)} : \alpha \notin \Omega(U)\}.$$

Now, we define a function θ from $\mathcal{B}$ to the topology of $\prod\{\mathbb{I}^{A(\alpha)} : \alpha < \tau\}$ by

$$\theta(G) = \bigcup\{\theta_1(U) : U \in \mathcal{B} \text{ and } \overline{U} \subset G\}.$$

Let us show that θ is π-regular. It follows from Claim 7 that $\theta(G_1)\cap\theta(G_2)=\varnothing$ provided $G_1\cap G_2=\varnothing$. It is easily seen that $\theta(G)\cap X=\bigcup\{\Gamma(U)\cap X : U\in\mathcal{B}$ and $\overline{U}\subset G\}$. According to Claim 9, each $\Gamma(U)\cap X$ is a non-empty subset of $\overline{U}$. Hence, $\theta(G)\cap X$ is a non-empty dense subset of G. So, X is π-regularly embedded in $\mathbb{I}^A$, where A is the union of all $A(\alpha)$, $\alpha<\tau$. ■

Lemma 3.6. *Suppose $X=\varprojlim S$, where $S=\{X_\alpha, p^\beta_\alpha, A\}$ is an almost σ-complete inverse system with open bonding maps and second countable spaces X_α. Then X is ccc and for every open $U\subset X$ there exists $\alpha\in A$ such that $p_\beta^{-1}(p_\beta(\overline{U}))=\overline{U}$. Moreover, any continuous function f on X can be represented in the form $f=g\circ p_\alpha$ for some $\alpha\in A$ and a continuous function g on X_α.*

Proof. More general statement was announce in [14], for the sake of completeness we provide a proof. Denote by $\mathcal{B}$ a base of X consisting of all open sets of the form $p_\beta^{-1}(W_\beta)$, $\beta\in A$, where $W_\beta\subset X_\beta$ is open. Let $U\subset X$ be open and $\mathcal{B}(U)=\{V\in\mathcal{B} : V\subset U\}$. We construct by induction an increasing sequence $\{\beta_n\}\subset A$ and countable families $\mathcal{B}_n(U)\subset\mathcal{B}(U)$, $n\geq 1$, satisfying the following conditions:

$(i)_n$ $\mathcal{B}_n(U)\subset\mathcal{B}_{n+1}(U)$ for each n;

$(ii)_n$ The family $\{p_{\beta_n}(W) : W\in\mathcal{B}_n(U)\}$ is dense in $p_{\beta_n}(U)$;

$(iii)_n$ $p_{\beta_{n+1}}^{-1}(p_{\beta_{n+1}}(W))=W$ for all $n\geq 1$ and $W\in\mathcal{B}_n(U)$.

Fix an arbitrary $\beta_1\in A$ and choose a countable family $\mathcal{B}_1(U)\subset\mathcal{B}(U)$ such that $\{p_{\beta_1}(W) : W\in\mathcal{B}_1(U)\}$ is dense in $p_{\beta_1}(U)$ (this can be done because X_{β_1} is second countable). Suppose β_k and $\mathcal{B}_k(U)$ are already constructed for all $k\leq n$. The family $\mathcal{B}_n(U)$ is countable and for each $W\in\mathcal{B}_n(U)$ there exists $\beta_W\in A$ with $p_{\beta_W}^{-1}(p_{\beta_W}(W))=W$. Moreover, A is σ-complete. So, we can find $\beta_{n+1}\geq\beta_n$ satisfying item $(iii)_n$. Next, we choose a countable family $\mathcal{B}_{n+1}\subset\mathcal{B}$ containing $\mathcal{B}_n$ and satisfying condition $(ii)_n$. This completes the induction. Finally, let $\beta=\sup\{\beta_n : n\geq 1\}$ and $\mathcal{B}_0=\bigcup_{n\geq 1}\mathcal{B}_n$. It is easily seen that $\{p_\beta(W) : W\in\mathcal{B}_0\}$ is dense in $p_\beta(U)$ and $p_\beta^{-1}(p_\beta(W))=W$ for all $W\in\mathcal{B}_0$. Since p_β is open, this implies that $\bigcup\mathcal{B}_0$ is dense in U and $p_\beta^{-1}(p_\beta(\overline{U}))=\overline{U}$.

Suppose now $f\colon X\to\mathbb{R}$ is a continuous function. Choose a countable base $\mathcal{U}$ of $\mathbb{R}$. For each $U\in\mathcal{U}$ there exists $\beta(U)\in A$ such that $p_{\beta(U)}^{-1}(p_{\beta(U)}(\overline{U}))=\overline{U}$. Let $\beta=\sup\{\beta(U) : U\in\mathcal{U}\}$. Then $p_\beta^{-1}(p_\beta(\overline{U}))=\overline{U}$ for all $U\in\mathcal{U}$. The last equalities imply that if $p_\beta(x)=p_\beta(y)$ for some $x,y\in X$, then $f(x)=f(y)$. So, the function $g\colon X_\beta\to\mathbb{R}$, $g(z)=f(p_\beta^{-1}(z))$, is well defined and $f=g\circ p_\beta$. Finally, since p_β is open, g is continuous. ■

Proposition 3.7. *Let Y be a limit space of an almost σ-complete inverse system with open bonding maps and second countable spaces. Suppose X is a π-regularly C^*-embedded subspace of Y. Then X is skeletally generated.*

Proof. Suppose $Y = \varprojlim S_Y$ and $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$ is a π-regular operator, where $S_Y = \{Y_\alpha, \pi^\beta_\alpha, A\}$ is an almost σ-complete inverse system with open bonding maps and second countable spaces Y_α. Then the limit projections $\pi_\alpha\colon Y \to Y_\alpha$ are also open.

Let $\mathcal{A}_\beta$ be a countable open base for Y_β. We say that $\beta \in A$ is e-*admissible* if

$$\pi_\beta^{-1}\big(\pi_\beta\big(\overline{\mathrm{e}(\pi_\beta^{-1}(V)\cap X)}\big)\big) = \overline{\mathrm{e}(\pi_\beta^{-1}(V)\cap X)} \tag{16}$$

for every $V \in \mathcal{A}_\beta$. We also denote $X_\beta = \pi_\beta(X)$.

Claim 10. *The map $p_\beta = \pi_\beta | X$ is skeletal for every* e-*admissible $\beta \in A$.*

The proof of this claim is extracted from the proof of [11, Lemma 9]. Let $U \subset X$ be open in X. Because π_β is open, it suffices to show that $\pi_\beta(\mathrm{e}(U)) \cap X_\beta \subset \overline{\pi_\beta(U)}^{X_\beta}$. Suppose there exists a point $z \in \pi_\beta(\mathrm{e}(U)) \cap X_\beta \backslash \overline{\pi_\beta(U)}^{X_\beta}$ and take $V \in \mathcal{A}_\beta$ containing z such that $V \cap \overline{\pi_\beta(U)} = \varnothing$ (here $\overline{\pi_\beta(U)}$ is the closure in Y_β). Since β is e-admissible, $\pi_\beta^{-1}\big(\pi_\beta\big(\overline{\mathrm{e}(U_1)}\big)\big) = \overline{\mathrm{e}(U_1)}$, where $U_1 = \pi_\beta^{-1}(V) \cap X$. Obviously, $U_1 \cap U = \varnothing$ and $\pi_\beta(U_1) = V \cap X_\beta$. Because $\mathrm{e}(U_1) \cap X$ is dense in U_1, we have $\overline{\pi_\beta(\mathrm{e}(U_1) \cap X)} = \overline{\pi_\beta(U_1)} = \overline{V \cap X_\beta}$. Since $\pi_\beta\big(\overline{\mathrm{e}(U_1)}\big)$ is closed in Y_β (recall that π_β being open is a quotient map), $z \in \pi_\beta \overline{\mathrm{e}(U_1)} \cap \pi_\beta(\mathrm{e}(U))$ which implies $\overline{\mathrm{e}(U_1)} \cap \mathrm{e}(U) \neq \varnothing$. So, $\mathrm{e}(U_1) \cap \mathrm{e}(U) \neq \varnothing$, and consequently, $U \cap U_1 \neq \varnothing$. This contradiction completes the proof of Claim 10.

Claim 11. *Let $\{\beta_n\}_{n\geq 1}$ be an increasing sequence of elements of A such that each β_{n+1} satisfies the equality* (16) *with $V \in \mathcal{A}_{\beta_n}$. Then* $\sup\{\beta_n : n \geq 1\}$ *is* e-*admissible. In particular, this is true if all β_n are* e-*admissible.*

The proof of this claim follows from the definition of e-admissible sets.

Claim 12. *For every $\gamma \in A$ there exists an* e-*admissible β with $\gamma < \beta$.*

We construct by induction an increasing sequence $\{\beta_n\}_{n\geq 1}$ such that $\beta_1 = \gamma$ and β_{n+1} satisfies the equality (16) with $V \in \mathcal{A}_{\beta_n}$ for all $n \geq 1$. Suppose β_n is already constructed. By Lemma 3.6, for each $V \in \mathcal{A}_{\beta_n}$ there exists $\beta(V) \in A$ such that $\pi_{\beta(V)}^{-1}\big(\pi_{\beta(V)}\big(\overline{\mathrm{e}(\pi_{\beta(V)}^{-1}(V)\cap X)}\big)\big) = \overline{\mathrm{e}(\pi_{\beta(V)}^{-1}(V)\cap X)}$ and $\beta(V) \geq \beta_n$.

Then $\beta_{n+1} = \sup\{\beta(V) : V \in \mathcal{A}_{\beta_n}\}$ is as desired (to be sure that β_{n+1} exists, we may assume that $\{\beta(V) : V \in \mathcal{A}_{\beta_n}\}$ is an increasing sequence). Finally, by Claim 11, $\beta = \sup\{\beta_n : n \geq 1\}$ is e-admissible.

Now, consider the set $\Lambda \subset A$ consisting of all e-admissible β with the order inherited from A. According to Claim 12, Λ is directed. Claim 11 yields Λ is σ-complete and, by Claim 10, all p_β are skeletal maps. Hence, the bonding maps $p^\alpha_\beta \colon X_\alpha \to X_\beta$, where $\beta, \alpha \in \Lambda$ and $X_\alpha = p_\alpha(X)$, are also skeletal. Moreover, the inverse system $S_X = \{X_\alpha, p^\beta_\alpha, \Lambda\}$ is σ-complete and $X = \mathrm{a}-\varprojlim S_X$. It remains to show that the system S_X satisfies condition (7). So, let $f \colon X \to \mathbb{R}$ be a bounded continuous function. Next, extend f to a continuous function $\overline{f} \colon Y \to \mathbb{R}$ (recall that X is C^*-embedded in Y). Since any inverse σ-complete system with open projections and second countable spaces is factorizable (i.e., its limit space satisfies condition (7)), see Lemma 3.6, there exists $\alpha \in \Lambda$ and a continuous function $g : X_\alpha \to \mathbb{R}$ with $f = g \circ p_\alpha$. Therefore, X is skeletally generated. ∎

Proof of Theorem 1.1. To prove implication $(i) \Rightarrow (ii)$, suppose X is I-favorable with respect to co-zero sets and X is C^*-embedded in a space Y. Then $\overline{X}^{\beta Y}$ is homeomorphic to βX. Since βX is also I-favorable with respect to co-zero sets (see Proposition 2.1), according to Proposition 3.5, βX is π-regularly embedded in βY. This yields that X is π-regularly embedded in Y.

$(ii) \Rightarrow (iii)$ Let X be a C^*-embedded subset of some $\mathbb{I}^A$. Then X is π-regularly embedded in $\mathbb{I}^A$. Since $\mathbb{I}^A$ is openly generated (it is the limit space of the continuous inverse system $\{\mathbb{I}^B, \pi^C_B, B \subset C \subset A\}$ with all B, C being countable subsets of A), we can apply Proposition 3.7 to conclude that X is skeletally generated.

Finally, the implication $(iii) \Rightarrow (i)$ follows from Lemma 2.4. ∎

Proof of Corollary 1.2. Let X_α, $\alpha \in \Lambda$, be a family of compact I-favorable with respect to co-zero sets spaces and $X = \prod_{\alpha\in\Lambda} X_\alpha$. We embed each X_α is a Tychonoff cube $\mathbb{I}^{A(\alpha)}$ and let $K = \prod_{\alpha\in\Lambda} \mathbb{I}^{A(\alpha)}$. By theorem 1.1(ii), there exists a π-regular operator $e_\alpha : \mathcal{T}_{X_\alpha} \to \mathcal{T}_{I^{A(\alpha)}}$ for each $\alpha \in \Lambda$. Let $\mathcal{B}$ be the family of all standard open sets of the form $U = U_{\alpha(1)} \times .. \times U_{\alpha(k)} \times \prod\{X_\alpha : \alpha \neq \alpha_i, i = 1, .., k\}$, where each $U_{\alpha(i)} \subset X_{\alpha(i)}$ is open. For any such $U \in \mathcal{B}$ we define $\gamma(U) = e_{\alpha(1)}(U_{\alpha(1)}) \times ... \times e_{\alpha(k)}(U_{\alpha(k)}) \times \prod\{\mathbb{I}^{A(\alpha)} : \alpha \neq \alpha_i, i = 1, .., k\}$. Finally, we define a function $e \colon \mathcal{T}_X \to \mathcal{T}_K$ by the equality $e(W) = \bigcup\{\gamma(U) : U \in \mathcal{B} \text{ and } U \subset W\}$. It is easily seen that e is π-regular. Since K is the limit space of a continuous σ-complete inverse system consisting of open bounding maps and compact metrizable spaces, by Proposition 3.7, X is skeletally generated. Hence, X is I-favorable with respect to co-zero sets. ∎

Proof of Corollary 1.3. Suppose $X \subset Y$ a C^*-embedded I-favorable space with respect to co-zero sets, where Y is extremally disconnected. Then, by Theorem 1.1(ii), there exists a π-regular operator e: $\mathcal{T}_X \to \mathcal{T}_Y$. We need to show that the closure (in X) of every open subset of X is also open. Since Y is extremally disconnected, $\overline{\mathrm{e}(U)}^Y$ is open in Y. So, the proof will be done if we prove that $\overline{\mathrm{e}(U)}^Y \cap X = \overline{U}^X$ for all $U \in \mathcal{T}_X$. By (1), we have $\overline{U}^X \subset \overline{\mathrm{e}(U)}^Y \cap X$. Assume there exists $x \in \overline{\mathrm{e}(U)}^Y \cap X \setminus \overline{U}^X$ and choose $V \in \mathcal{T}_X$ with $V \subset \overline{\mathrm{e}(U)}^Y \setminus \overline{U}^X$. Then $\mathrm{e}(V) \cap \overline{\mathrm{e}(U)}^Y \neq \varnothing$, so $\mathrm{e}(V) \cap \mathrm{e}(U) \neq \varnothing$. The last one contradicts $U \cap V = \varnothing$. ■

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