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WARPED PRODUCT CR-SUBMANIFOLDS IN LORENTZIAN PARA SASAKIAN MANIFOLDS*

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Abstract. Many research articles have recently appeared exploring existence or non existence of warped product submanifolds in known spaces (cf. [2, 5, 8]). The objective of the present paper is to study the existence or non-existence of contact CR-warped products in the setting of LP-Sasakian manifolds.

1. Introduction. In 1989, Matsumoto [6] introduced the idea of LP-Sasakian manifolds. Then Mihai and Rosca [7] introduced the same notion and obtained several results in this manifold. U.C. De and K. Arslan obtained some curvature conditions on LP-Sasakian manifolds [4].

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In [1] the notion of warped product manifolds was introduced by Bishop and O'Neill in 1969. These manifolds appear in differential geometric studies in a natural way and these are generalizations of Riemannian product manifolds. Recently, B.Y. Chen has introduced the notion of warped product CR-submanifolds in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_\perp \times {}_f N_T$ in a Kaehler manifold. He considered only the warped product of the type $M = N_T \times {}_f N_\perp$ and called it a CR-warped product submanifold [2, 3]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_\perp \times {}_f N_T$ in Sasakian manifolds are trivial, i.e., simply contact CR-product submanifolds, where N_T and $N_\perp$ are $\phi-$invariant and anti-invariant submanifolds of Sasakian manifold, respectively [5].

In the present paper, we prove that the warped product in the form $M = N_1 \times {}_f N_2$ does not exist if the vector field ξ is tangent to N_2, where N_1 and N_2 are any real submanifolds of an LP-Sasakian manifold $\bar{M}$. Also, we have shown that there exist no proper warped product CR-submanifold of the type $M = N_T \times {}_f N_\perp$, when ξ is tangent to N_T and thus, we consider the warped product submanifolds in the form $M = N_\perp \times {}_f N_T$, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of an LP-Sasakian manifold $\bar{M}$, respectively.

2. Preliminaries.

Let $\bar{M}$ be a $(2n+1)$-dimensional Lorentzian almost paracontact manifold [6] with the almost paracontact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a contravariant vector field, η is a $1-$form and g is a Lorentzian metric with signature $(-,+,+,\cdots,+)$ on $\bar{M}$, satisfying:

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \tag{2.2}$$

$$\Phi(X,Y) = g(\phi X, Y) = g(X, \phi Y) = \Phi(Y,X), \tag{2.3}$$

for all $X, Y \in T\bar{M}$, where Φ is the fundamental two form, defined above.

A Lorentzian almost contact metric structure on $\bar{M}$ is called a *Lorentzian para-Sasakian structure* if

$$\begin{cases} (\bar{\nabla}_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \\ \bar{\nabla}_X \xi = \phi X, \end{cases} \tag{2.4}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to g. The manifold $\bar{M}$ in this case is called *Lorentzian para-Sasakian* (in brief, *LP-Sasakian*) manifold.

Let M be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for any $X,\ Y \in TM$ and $N \in T^{\perp}M$, where TM is the Lie algebra of vector field in M and $T^{\perp}M$ is the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection in the normal bundle, h the second fundamental form and A_N is the Weingarten endomorphism associated with N. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.7}$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and fN is the normal component of ϕN.

The covariant derivatives of the tensor fields $\phi,\ P$ and F are defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \qquad \forall\ X, Y \in T\bar{M} \tag{2.10}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \qquad \forall\ X, Y \in TM \tag{2.11}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \qquad \forall\ X, Y \in T\bar{M}. \tag{2.12}$$

Moreover, for a submanifold M of an LP-Sasakian manifold $\bar{M}$, we have

$$(\bar{\nabla}_X P)Y = A_{FY} X + th(X, Y) + g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \tag{2.13}$$

$$(\bar{\nabla}_X F)Y = fh(X,Y) - h(X,PY). \tag{2.14}$$

for all $X, Y \in TM$.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following.

(i) A submanifold M tangent to ξ is called an *invariant* submanifold if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be an *anti-invariant* submanifold if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

(ii) A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits an invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}_x^{\perp}) \subset T_x^{\perp}M$, for every $x \in M$.

3. Warped and doubly warped product submanifolds. Let (N_1, g_1) and (N_2, g_2) be two semi-Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the manifold $N_1 \times {}_f N_2 = (N_1 \times N_2,\ g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

We recall the following general formula on a warped product [1].

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Unal [10]. A *doubly warped product manifold* of N_1 and N_2, denoted as ${}_{f_2}N_1 \times {}_{f_1}N_2$ is endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \tag{3.3}$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2 respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X \tag{3.4}$$

for each X in TN_1 and Z in TN_2 [8].

If neither f_1 nor f_2 is constant we have a non trivial doubly warped product $M = {}_{f_2}N_1 \times {}_{f_1}N_2$. Obviously in this case both N_1 and N_2 are totally umbilical submanifolds of M.

We now consider a doubly warped product of two semi-Riemannian manifolds N_1 and N_2 embedded into an LP-Sasakian manifold $\bar{M}$ such that the structure vector field ξ is tangential to the submanifold $M = {}_{f_2}N_1 \times {}_{f_1}N_2$.

Theorem 3.1. *Let $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ be a doubly warped product submanifold of an LP-Sasakian manifold $\bar{M}$ where N_1 and N_2 are submanifolds of $\bar{M}$. Then f_2 is constant and N_2 is anti-invariant if the structure vector field ξ is tangent to N_1 and f_1 is constant and N_1 is anti-invariant if ξ is tangent to N_2.*

Proof. Consider ξ tangent to N_1, then for $V \in TN_2$ we get

$$\nabla_V \xi = (\xi \ln f_1)V + (V \ln f_2)\xi. \tag{3.5}$$

Thus from equations (2.4), (2.5), (2.8) and (3.5), we get

$$\bar{\nabla}_V \xi = (\xi \ln f_1)V + (V \ln f_2)\xi + h(V, \xi) = PV + FV. \tag{3.6}$$

On comparing tangential and normal parts and using the fact that ξ, V and PV are mutually orthogonal vector fields, (3.6) implies that

$$V \ln f_2 = 0, \quad \xi \ln f_1 = 0$$

$$h(V, \xi) = FV, \quad PV = 0.$$

Showing that f_2 is constant and N_2 is an anti-invariant submanifold of $\bar{M}$.

Similarly, if ξ is tangent to N_2 and $U \in TN_1$ we have

$$\bar{\nabla}_U \xi = (\xi \ln f_2)U + (U \ln f_1)\xi + h(U, \xi) = PU + FU, \tag{3.7}$$

which gives

$$U \ln f_1 = 0, \quad \xi \ln f_2 = 0$$

$$PU = 0, \quad h(U, \xi) = FU.$$

Which shows that f_1 is constant and N_1 is an anti-invariant submanifold of $\bar{M}$. This completes the proof. □

The following corollaries are immediate consequences of the above theorem.

Corollary 3.1. *There does not exist a proper doubly warped product submanifold in LP-Sasakian manifolds.*

Corollary 3.2. *There does not exist a warped product submanifold* $N_1 \times {}_fN_2$ *of an LP-Sasakian manifold* $\bar{M}$ *such that* ξ *tangential to* N_2*.*

Thus the only remaining case to study is the warped product submanifold $N_1 \times {}_fN_2$ with structure vector field ξ tangential to N_1. In particular, warped products of the type $M = N_T \times {}_fN_\perp$ and $M = N_\perp \times {}_fN_T$, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of an LP-Sasakian manifold $\bar{M}$ are discussed in the following section.

4. CR-warped product submanifolds. Throughout this section the structure vector field ξ is either tangent to the invariant submanifold N_T or tangent to the anti-invariant submanifold $N_\perp$. There are two types of warped product submanifolds in an LP-Sasakian manifold $\bar{M}$, namely $N_T \times {}_fN_\perp$ and $N_\perp \times {}_fN_T$ are called *CR-warped product* submanifolds, with ξ tangential to N_T and $N_\perp$, respectively. In the following theorem we deal with the case ξ is tangent to the submanifold N_T.

Theorem 4.1. *There does not exist a proper warped product submanifold* $N_T \times {}_fN_\perp$ *where* N_T *is an invariant and* $N_\perp$ *is an anti-invariant submanifold of an LP-Sasakian manifold* $\bar{M}$ *such that* ξ *is tangent to* N_T*.*

Proof. Let $M = N_T \times {}_fN_\perp$. For any $X \in TN_T$ and $Z \in TN_\perp$, by (3.2) we deduced that

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z. \tag{4.1}$$

In particular, for $X = \xi$

$$\nabla_Z \xi = (\xi \ln f)Z. \tag{4.2}$$

Whereas by formulae (2.4) and (2.5) we have

$$\bar{\nabla}_Z \xi = \phi Z = FZ,$$

or

$$\nabla_Z \xi + h(Z, \xi) = FZ,$$

which on using (4.2), we get

$$\xi \ln f = 0, \quad h(Z, \xi) = FZ. \tag{4.3}$$

Now, for any $X \in TN_T$ and $Z, W \in TN_\perp$ and using (2.2), (2.4), (2.5), (2.6), (2.7), (2.10) and (3.2), we have

$$g(\nabla_X Z, W) = g(\nabla_Z X, W) = g(\bar{\nabla}_Z X, W) = g(\phi \bar{\nabla}_Z X, \phi W) - \eta(\bar{\nabla}_Z X)\eta(W),$$

$$(X \ln f)g(Z, W) = g(\bar{\nabla}_Z \phi X, \phi W) - g((\bar{\nabla}_Z \phi)X, \phi W) = g(\nabla_Z \phi X + h(Z, \phi X), \phi W)$$

or

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + (\phi X \ln f)g(Z, FW) = g(h(Z, \phi X), \phi W).$$

That is

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W). \tag{4.4}$$

Again, we have

$$g(h(Z, \phi X), \phi W) = g(\bar{\nabla}_{\phi X} Z, \phi W). \tag{4.5}$$

Making use of equations (2.3), (2.6), (2.7) and (2.10) and the fact that $\bar{M}$ is LP-Sasakian, we deduce from (4.5) that

$$g(h(Z, \phi X), \phi W) = -g(h(\phi X, W), \phi Z). \tag{4.6}$$

Interchanging Z and W in equation (4.4) and adding the resulting equation with (4.6), we obtain that

$$(X \ln f)g(Z, W) = 0, \tag{4.7}$$

for all $X \in TN_T$. Equations (4.3) and (4.7) imply that f is constant on N_T, proving the result. □

Now, the other case i.e., $N_\perp \times {}_f N_T$ with ξ tangential to $N_\perp$ is dealt with the following theorem.

Theorem 4.2. *Let $M = N_\perp \times {}_f N_T$ be a warped product submanifold of an LP-Sasakian manifold $\bar{M}$, with $\xi \in TN_\perp$, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then*

(i) $\xi \ln f = 0$,

(ii) $th(X, Z) = 0$,

(iii) $g(h(X, Z), FW) = -g(h(X, W), FZ)$

for any $X \in TN_T$ and $Z,\ W \in TN_\perp$.

Proof. The first result is an immediate consequence of the formula $\bar{\nabla}_U \xi = \phi U$, for $U \in TM$, and using formulae (2.4), (3.2) and the fact that U and PU are mutually orthogonal vector fields. Now, for any $U, V \in TM$ we have

$$(\bar{\nabla}_U P)V = A_{FV}U + th(U, V) + g(\phi U, \phi V)\xi + \eta(V)\phi^2 U.$$

Using the above fact for any $X \in TN_T$ and $Z \in TN_\perp$, we get

$$(\bar{\nabla}_Z P)X = th(X, Z). \tag{4.8}$$

Also, for any $X \in TN_T$ and $Z \in TN_\perp$, we have

$$(\bar{\nabla}_Z P)X = \nabla_Z PX - P\nabla_Z X = (Z \ln f)PX - P(Z \ln f)X = 0. \tag{4.9}$$

Part (ii) follows by equations (4.8) and (4.9). For (iii), consider for any $X \in TN_T$ and $Z, W \in TN_\perp$

$$g(A_{\phi Z}X, W) = g(h(X, W), \phi Z).$$

Using (2.5) and (2.2), we get

$$g(A_{\phi Z}X, W) = (\bar{\nabla}_X W, \phi Z) = g(\phi\bar{\nabla}_X W, Z).$$

Then from (2.10), we obtain

$$g(A_{\phi Z}X, W) = g(\bar{\nabla}_X \phi W, Z) - g((\bar{\nabla}_X \phi)W, Z).$$

Thus, on using (2.4) and (2.6), we derive

$$g(A_{\phi Z}X, W) = -g(A_{\phi W}X, Z) - \eta(W)g(X, Z).$$

By orthogonality of two distributions, the second term of right hand side is identically zero. Hence, from (2.7), we obtain

$$g(h(X, W), \phi Z) = -g(h(X, Z), \phi W). \tag{4.10}$$

Part (iii) thus follows by equation (4.10). Hence the theorem is proved. □

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