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SPECIAL COMPOSITIONS IN AFFINELY CONNECTED SPACES WITHOUT A TORSION

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Abstract. Let A_N be an affinely connected space without a torsion. With the help of N independent vector fields and their reciprocal covectors is built an affinor which defines a composition $X_n \times X_m$ $(n+m = N)$. The structure is integrable. New characteristics by the coefficients of the derivative equations are found for special compositions, studied in [1], [3] . Two-dimensional manifolds, named as bridges, which cut the both base manifolds of the composition are introduced. Conditions for the affine deformation tensor of two connections where the composition is simultaneously of the kind (g-g) are found.

1. Preliminary. Let A_N be a space with a symmetric affine connection, define by $\Gamma^{\gamma}_{\alpha\beta}$. Consider in the space A_N a composition $X_n \times X_m$ $(n + m = N)$ of two base differential manifolds X_n and X_m $(n + m = N)$. Two positions $P(X_n)$ and $P(X_m)$ of the base manifolds pass thought any point of the space $A_N(X_n \times X_m)$.

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Let in $A_N(X_n \times X_m)$ be introduced as an arbitrary coordinate system x^α $(\alpha = 1,2,\dots,N)$ as well as a coordinate system $(u^i, u^{\overline{i}})$ $(i = 1,2,\dots,n;$ $\overline{i} = n+1, n+2, \dots, N)$ adapted to the composition [3].

According to [1], [3] any composition is completely defined by the field of the affinor a_α^β, satisfying the condition

$$a_\alpha^\sigma a_\sigma^\beta = \delta_\alpha^\beta. \tag{1}$$

The affinor a_α^β is called an affinor of the composition [1]. According to [3], [5] the condition for integrability of the structure is characterized by the equality

$$a_\beta^\sigma \nabla_{[\alpha}\ a_{\sigma]}^\nu - a_\alpha^\sigma \nabla_{[\beta}\ a_{\sigma]}^\nu = 0. \tag{2}$$

The projecting affinors $\overset{n}{a}{}_\alpha^\beta$, $\overset{m}{a}{}_\alpha^\beta$ are defined by the equalities $\overset{n}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta)$, $\overset{m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta)$ [3], [4].

According to [4] for any vector $v^\alpha \in A_N(X_n \times X_m)$ is fullfiled $v^\alpha = \overset{n}{a}{}_\sigma^\alpha v^\sigma + \overset{m}{a}{}_\sigma^\alpha v^\sigma = \overset{n}{V}{}^\alpha + \overset{m}{V}{}^\alpha$, where $\overset{n}{V}{}^\alpha = \overset{n}{a}{}_\sigma^\alpha v^\sigma \in P(X_n)$, $\overset{m}{V}{}^\alpha = \overset{m}{a}{}_\sigma^\alpha v^\sigma \in P(X_m)$.

The following characteristics for some special compositions are defined in [3]:

The composition of the type $(c{-}c) \in A_N(X_n \times X_m)$ for which the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line in the space A_N is characterized by the condition $\nabla_\alpha\ a_\beta^\sigma = 0$.

The composition of the type $(ch - ch) \in A_N(X_n \times X_m)$ for which the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of X_m and X_n, respectively is characterized by the condition $\nabla_{[\alpha}\ a_{\beta]}^\sigma = 0$.

The composition of the type $(g - g) \in A_N(X_n \times X_m)$ for which the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of X_n and X_m, respectively is characterized by the condition $a_\beta^\sigma\ \nabla_\alpha\ a_\sigma^\nu + a_\alpha^\sigma\ \nabla_\sigma\ a_\beta^\nu = 0$.

2. Integrability of the structure of the space of the compositions $\boldsymbol{A_N(X_n \times X_m)}$. Let M_N be an arbitrary differential manifold and let us assume:

$$\begin{aligned} &\alpha,\beta,\gamma,\sigma,\nu \in \{1,2,\dots,N\}; \qquad i,j,k,p,q,r,s \in \{1,2,\dots,n\}; \\ &\overline{i},\overline{j},\overline{k},\overline{p},\overline{q},\overline{r},\overline{s} \in \{n+1, n+2, \dots, n+m = N\}; \end{aligned} \tag{3}$$

Choose N independent vector fields of directions $\underset{1}{v}{}^\alpha, \underset{2}{v}{}^\alpha, \dots, \underset{N}{v}{}^\alpha$. The reciprocal covector fields $\overset{\alpha}{v}_\sigma$ are defined by the equalities

$$\underset{\alpha}{v}{}^\sigma\ \overset{\beta}{v}_\sigma = \delta_\alpha^\beta \quad \text{iff} \quad \underset{\alpha}{v}{}^\sigma\ \overset{\alpha}{v}_\nu = \delta_\nu^\sigma. \tag{4}$$

Following [6], [7] let us consider the affinor

$$a_\alpha^\beta = \underset{i}{v}^\beta \overset{i}{v}_\alpha - \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha, \tag{5}$$

which according to (4) satisfy (1). Now the projective affinors have the following representation $\overset{n}{a}{}_\alpha^\beta = \underset{i}{v}^\beta \overset{i}{v}_\alpha$, $\overset{m}{a}{}_\alpha^\beta = \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha$.

Let an affine connection $\Gamma_{\alpha\beta}^\sigma$ without a torsion be given in M_N and let us denote the space by A_N in this case. The affinor (5) defines a composition $X_n \times X_m$ in A_N.

We chose the independent vector fields $\underset{\beta}{v}^\alpha$ as coordinate one and the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$ will be called coordinate. Then we have

$$\begin{aligned} &\underset{1}{v}^\alpha(1,0,\ldots,0), \quad \underset{2}{v}^\alpha(0,1,\ldots,0), \quad \ldots, \quad \underset{N}{v}^\alpha(0,0,\ldots,1); \\ &\overset{1}{v}_\alpha(1,0,\ldots,0), \quad \overset{2}{v}_\alpha(0,1,\ldots,0), \quad \ldots, \quad \overset{N}{v}_\alpha(0,0,\ldots,1). \end{aligned} \tag{6}$$

The chosen coordinate net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$ defines adapted with the composition coordinates [1], [3]. That is why as in [3] and here in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$ the matrix (a_α^β) has the following form

$$(a_\alpha^\beta) = \begin{pmatrix} \delta_i^j & 0 \\ 0 & -\delta_{\bar{i}}^{\bar{j}} \end{pmatrix} \tag{7}$$

Theorem 1. *The structure defined by the affinor* (5) *is integrable.*

P r o o f. Let us consider the tensor $N_{\beta\alpha}^\nu = a_\beta^\sigma \nabla_{[\alpha} a_{\sigma]}^\nu - a_\alpha^\sigma \nabla_{[\beta} a_{\sigma]}^\nu$. Taking into account (7) and $\Gamma_{\alpha\beta}^\sigma = \Gamma_{\beta\alpha}^\sigma$, we find for $N_{\beta\alpha}^\nu$ in the chosen coordinate net

$$N_{\beta\alpha}^\nu = \frac{1}{2}\left[\Gamma_{\sigma\delta}^\nu \left(a_\beta^\delta a_\alpha^\sigma - a_\alpha^\delta a_\beta^\sigma\right)\right]. \tag{8}$$

Then from (7) and (8) we obtain

$$\begin{aligned} N_{ij}^s &= \frac{1}{2}\left(\Gamma_{ij}^s - \Gamma_{ji}^s\right) = 0, & N_{\bar{i}j}^s &= \frac{1}{2}\left(-\Gamma_{\bar{i}j}^s + \Gamma_{j\bar{i}}^s\right) = 0, \\ N_{i\bar{j}}^s &= \frac{1}{2}\left(-\Gamma_{i\bar{j}}^s + \Gamma_{\bar{j}i}^s\right) = 0, & N_{\bar{i}\bar{j}}^s &= \frac{1}{2}\left(\Gamma_{\bar{i}\bar{j}}^s - \Gamma_{\bar{j}\bar{i}}^s\right) = 0, \\ N_{ij}^{\bar{s}} &= \frac{1}{2}\left(\Gamma_{ij}^{\bar{s}} - \Gamma_{ji}^{\bar{s}}\right) = 0, & N_{\bar{i}j}^{\bar{s}} &= \frac{1}{2}\left(-\Gamma_{\bar{i}j}^{\bar{s}} + \Gamma_{j\bar{i}}^{\bar{s}}\right) = 0, \\ N_{i\bar{j}}^{\bar{s}} &= \frac{1}{2}\left(-\Gamma_{i\bar{j}}^{\bar{s}} + \Gamma_{\bar{j}i}^{\bar{s}}\right) = 0, & N_{\bar{i}\bar{j}}^{\bar{s}} &= \frac{1}{2}\left(\Gamma_{\bar{i}\bar{j}}^{\bar{s}} - \Gamma_{\bar{j}\bar{i}}^{\bar{s}}\right) = 0. \end{aligned} \tag{9}$$

The equalities (9) means $N^{\sigma}_{\alpha\beta} = 0$, i.e. the condition (2) is fulfilled. □

3. Other characteristics of the special compositions in A_N. Let the affinor (5) define a composition $X_n \times X_m$ in A_N. Denote by $(\underset{\alpha}{v})$ the lines which are determined by the vector fields $\underset{\alpha}{v}^{\sigma}$. According to [6] and [8] the following derivative equations are fulfilled

$$\nabla_\sigma \underset{\alpha}{v}^{\beta} = \overset{\nu}{\underset{\alpha}{T}}_{\sigma} \underset{\nu}{v}^{\beta} \quad , \quad \nabla_\sigma \overset{\alpha}{v}_{\beta} = -\overset{\alpha}{\underset{\nu}{T}}_{\sigma} \overset{\nu}{v}_{\beta} \quad . \tag{10}$$

From [8] it is known that the vector fields $\underset{\alpha}{v}^{\sigma}$ are parallelly translated along the lines $(\underset{\tau}{v})$ if and only if the coefficients of the derivative equations (10) satisfy the conditions

$$\overset{\beta}{\underset{\alpha}{T}}_{\sigma} \underset{\tau}{v}^{\sigma} = 0, \;\; \alpha \neq \beta. \tag{11}$$

Proposition 1. *The composition $X_n \times X_m$ is of the kind $(ch - ch)$ if and only if the coefficients of the derivative equations* (10) *satisfy the conditions*

$$\overset{\bar{k}}{\underset{i}{T}}_{\sigma} \underset{\bar{j}}{v}^{\sigma} = 0, \;\; \overset{k}{\underset{\bar{i}}{T}}_{\sigma} \underset{j}{v}^{\sigma} = 0 \; . \tag{12}$$

P r o o f. Let the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of X_m and X_n, respectively. Then the equalities (12) follow from (11).

According to (6), in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$, (12) accepts the form

$$\overset{\bar{k}}{\underset{i}{T}}_{\bar{j}} = 0, \;\; \overset{k}{\underset{\bar{i}}{T}}_{j} = 0 \; . \tag{13}$$

Then from (13) and $\nabla_\sigma \underset{\alpha}{v}^{\beta} = \partial_\sigma \underset{\alpha}{v}^{\beta} + \Gamma^{\beta}_{\sigma\nu} \underset{\alpha}{v}^{\nu} = \overset{\nu}{\underset{\alpha}{T}}_{\sigma} \underset{\nu}{v}^{\beta}$ we establish

$$\overset{\beta}{\underset{\alpha}{T}}_{\sigma} = \Gamma^{\beta}_{\sigma\alpha} \; . \tag{14}$$

Now from (13) and (14) the equalities $\Gamma^{\bar{k}}_{i\bar{j}} = 0$, $\Gamma^{k}_{\bar{i}j} = 0$ follow [3]. □

Proposition 2. *The composition $X_n \times X_m$ is of the kind $(g-g)$ if and only if the coefficients of the derivative equations* (10) *satisfy the conditions*

$$\overset{\overline{k}}{\underset{i}{T}}{}_\sigma \underset{j}{v}{}^\sigma = 0, \quad \overset{k}{\underset{\overline{i}}{T}}{}_\sigma \underset{\overline{j}}{v}{}^\sigma = 0. \tag{15}$$

P r o o f. Let the positions $P(X_n)$ and $P(X_m)$ be parallelly translated along any line of X_n and X_m, respectively. Then the equalities (15) follow from (11).

According to (6), in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$, (15) accepts the form

$$\overset{\overline{k}}{\underset{i}{T}}{}_j = 0, \quad \overset{k}{\underset{\overline{i}}{T}}{}_{\overline{j}} = 0. \tag{16}$$

From (14) and (16) the equalities $\Gamma^{\overline{k}}_{ij} = 0$, $\Gamma^{k}_{\overline{i}\,\overline{j}} = 0$ follow [3]. $\square$

Proposition 3. *The composition $X_n \times X_m$ is of the kind $(c-c)$ if and only if the coefficients of the derivative equations* (10) *satisfy the conditions*

$$\overset{\overline{k}}{\underset{i}{T}}{}_\sigma \underset{\alpha}{v}{}^\sigma = 0, \quad \overset{k}{\underset{\overline{i}}{T}}{}_\sigma \underset{\alpha}{v}{}^\sigma = 0. \tag{17}$$

P r o o f. Let the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of A_N. Then the equalities (17) follow from (11).

According to (6), in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$, (15) accepts the form

$$\overset{\overline{k}}{\underset{i}{T}}{}_\alpha = 0, \quad \overset{k}{\underset{\overline{i}}{T}}{}_\alpha = 0. \tag{18}$$

From (14) and (18) the equalities $\Gamma^{\overline{k}}_{i\alpha} = 0$, $\Gamma^{k}_{\overline{i}\alpha} = 0$ follow [3].

Let us denote by $\left(\sum \underset{s}{v}\right)$ and $\left(\sum \underset{\overline{s}}{v}\right)$ the lines defined by vector fields $\underset{1}{v}{}^\alpha + \underset{2}{v}{}^\alpha + \dots + \underset{n}{v}{}^\alpha$ and $\underset{n+1}{v}{}^\alpha + \underset{n+2}{v}{}^\alpha + \dots + \underset{N}{v}{}^\alpha$, respectively. From (11) easily follow

Proposition 4. 4.1 *The positions $P(X_n)$ and $P(X_m)$ are parallelly translated along the lines $\left(\sum \underset{\overline{s}}{v}\right)$ and $\left(\sum \underset{s}{v}\right)$, respectively if and only if the coefficients* *of the derivative equations* (10) *satisfy the equalities*

$$\overset{\overline{k}}{\underset{s}{T}}{}_\alpha(\underset{n+1}{v}{}^\alpha+\underset{n+2}{v}{}^\alpha+\cdots+\underset{N}{v}{}^\alpha)=0,\quad \overset{k}{\underset{\overline{s}}{T}}{}_\alpha(\underset{1}{v}{}^\alpha+\underset{2}{v}{}^\alpha+\cdots+\underset{n}{v}{}^\alpha)=0. \tag{19}$$

4.2 *The positions* $P(X_n)$ *and* $P(X_m)$ *are parallelly translated along the lines* $\left(\sum \underset{s}{v}\right)$ *and* $\left(\sum \underset{\overline{s}}{v}\right)$ *, respectively if and only if the coefficients of the derivative equations* (10) *satisfy the equalities*

$$\overset{\overline{k}}{\underset{s}{T}}{}_\alpha(\underset{1}{v}{}^\alpha+\underset{2}{v}{}^\alpha+\cdots+\underset{n}{v}{}^\alpha)=0,\quad \overset{k}{\underset{\overline{s}}{T}}{}_\alpha(\underset{n+1}{v}{}^\alpha+\underset{n+2}{v}{}^\alpha+\cdots+\underset{N}{v}{}^\alpha)=0. \tag{20}$$

4.3 *The positions* $P(X_n)$ *and* $P(X_m)$ *are parallelly translated along the lines* $\left(\sum \underset{s}{v}\right)$ *if and only if the coefficients of the derivative equations* (10) *satisfy the equalities*

$$\overset{\overline{k}}{\underset{s}{T}}{}_\alpha(\underset{1}{v}{}^\alpha+\underset{2}{v}{}^\alpha+\cdots+\underset{n}{v}{}^\alpha)=0,\quad \overset{k}{\underset{\overline{s}}{T}}{}_\alpha(\underset{1}{v}{}^\alpha+\underset{2}{v}{}^\alpha+\cdots+\underset{n}{v}{}^\alpha)=0. \tag{21}$$

4.4 *The positions* $P(X_n)$ *and* $P(X_m)$ *are parallelly translated along the lines* $\left(\sum \underset{\overline{s}}{v}\right)$ *if and only if the coefficients of the derivative equations* (10) *satisfy the equalities*

$$\overset{\overline{k}}{\underset{s}{T}}{}_\alpha(\underset{n+1}{v}{}^\alpha+\underset{n+2}{v}{}^\alpha+\cdots+\underset{N}{v}{}^\alpha)=0,\quad \overset{k}{\underset{\overline{s}}{T}}{}_\alpha(\underset{n+1}{v}{}^\alpha+\underset{n+2}{v}{}^\alpha+\cdots+\underset{N}{v}{}^\alpha)=0. \tag{22}$$

Because of (6) in the parameters of the coordinate net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ we can write $\overset{\gamma}{\underset{\beta}{T}}{}_\alpha\,\underset{\sigma}{v}{}^\alpha=\overset{\gamma}{\underset{\beta}{T}}{}_\sigma$. Then from (14) it follows

Corollary 1. *In the parameters of the coordinate net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ (19), (20), (21), (22) *are equivalent to the equalities*

$$\begin{aligned}
&\Gamma^{\overline{k}}_{s\;n+1}+\Gamma^{\overline{k}}_{s\;n+2}+\cdots+\Gamma^{\overline{k}}_{sN}=0, && \Gamma^{k}_{\overline{s}\;1}+\Gamma^{k}_{\overline{s}\;2}+\cdots+\Gamma^{k}_{\overline{s}\;n}=0,\\
&\Gamma^{\overline{k}}_{s1}+\Gamma^{\overline{k}}_{s2}+\cdots+\Gamma^{\overline{k}}_{sn}=0, && \Gamma^{k}_{\overline{s}\;n+1}+\Gamma^{k}_{\overline{s}\;n+2}+\cdots+\Gamma^{k}_{\overline{s}\;N}=0,\\
&\Gamma^{\overline{k}}_{s1}+\Gamma^{\overline{k}}_{s2}+\cdots+\Gamma^{\overline{k}}_{sn}=0, && \Gamma^{k}_{\overline{s}\;1}+\Gamma^{k}_{\overline{s}\;2}+\cdots+\Gamma^{k}_{\overline{s}\;n}=0,\\
&\Gamma^{\overline{k}}_{s\;n+1}+\Gamma^{\overline{k}}_{s\;n+2}+\cdots+\Gamma^{\overline{k}}_{sN}=0, && \Gamma^{k}_{\overline{s}\;n+1}+\Gamma^{k}_{\overline{s}\;n+2}+\cdots+\Gamma^{k}_{\overline{s}\;N}=0,
\end{aligned}$$

respectively.

4. Bridges of compositions. Let the composition $X_n \times X_m$ be defined by the affinor (5). According to accepted notations $\underset{s}{v}^\alpha \in P(X_n)$ and $\underset{\overline{s}}{v}^\alpha \in P(X_m)$.

Consider all pairs from the vector fields $\underset{p}{v}^\alpha$ and $\underset{\overline{k}}{v}^\alpha$. These vector fields determine the lines $\underset{p}{v}$ and $\underset{\overline{k}}{v}$, which form a two-dimensional net $(\underset{p}{v}, \underset{\overline{k}}{v})$.

Definition 1. *We will call bridges of the composition $X_n \times X_m$ the two-dimensional manifolds which contain the nets $(\underset{p}{v}, \underset{\overline{k}}{v})$ and let denote them $X_{p\,\overline{k}}$.*

Definition 2. *The bridge $X_{p\,\overline{k}}$ of the composition $X_n \times X_m$ will be called chebyshevian, geodesic, cartesian and i.e. when its corresponding net $(\underset{p}{v}, \underset{\overline{k}}{v})$ is chebyshevian, geodesic, cartesian and i.e., respectively.*

With the help of (11) it is easy to prove: All bridges $X_{p\,\overline{k}}$ of the composition $X_n \times X_m$ are chebyshevian if and only if the composition $X_n \times X_m$ is chebyshevian.

Denote by $P(X_{p\,\overline{k}})$ the positions of the bridges $X_{p\,\overline{k}}$. Then from (11) follows

Proposition 5. 5.1 *The positions $P(X_{p\,\overline{k}})$ of the bridges $X_{p\,\overline{k}}$ of the composition $X_n \times X_m$ are parallelly translated along any line of the X_n if and only if the coefficients of the derivative equations* (10) *satisfy the equalities*

$$\underset{p}{\overset{\overline{i}}{T}}{}_\sigma \underset{k}{v}^\sigma = 0, \quad \underset{\overline{p}}{\overset{i}{T}}{}_\sigma \underset{k}{v}^\sigma = 0. \tag{23}$$

5.2 *The positions $P(X_{p\,\overline{k}})$ of the bridges $X_{p\,\overline{k}}$ of the composition $X_n \times X_m$ are parallelly translated along any line of the X_m if and only if the coefficients of the derivative equations* (10) *satisfy the equalities*

$$\underset{p}{\overset{\overline{i}}{T}}{}_\sigma \underset{\overline{k}}{v}^\sigma = 0, \quad \underset{\overline{p}}{\overset{i}{T}}{}_\sigma \underset{\overline{k}}{v}^\sigma = 0. \tag{24}$$

Since in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ the conditions (23) and (24) accept the form $\underset{p}{\overset{\overline{i}}{T}}{}_k = 0$, $\underset{\overline{p}}{\overset{i}{T}}{}_k = 0$ and $\underset{p}{\overset{\overline{i}}{T}}{}_{\overline{k}} = 0$, $\underset{\overline{p}}{\overset{i}{T}}{}_{\overline{k}} = 0$, respectively then according to (14) we can formulate

Corollary 2. *In the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ the conditions* (23) *and* (24) *accept the form $\Gamma^{\overline{i}}_{pk} = 0$, $\Gamma^{i}_{\overline{p}k} = 0$ and $\Gamma^{\overline{i}}_{p\overline{k}} = 0$, $\Gamma^{i}_{\overline{p}\overline{k}} = 0$, respectively.*

5. Transformation of connections. Let in the space A_N with the connection $\Gamma^{\sigma}_{\alpha\beta}$ be given the composition $X_n \times X_m$ defined by the affinor (5). Let $G^{\sigma}_{\alpha\beta}$ be a new connection in A_N and let the new covariant derivative be denoted by ${}^1\nabla$. According to [2] $T^{\sigma}_{\alpha\beta} = G^{\sigma}_{\alpha\beta} - \Gamma^{\sigma}_{\alpha\beta}$ is the tensor of the affine deformation.

Theorem 2. *The composition $X_n \times X_m$ defined by the affinor* (5) *is simultaneously of the type $(g-g)$ with respect to the connections $\Gamma^{\sigma}_{\alpha\beta}$ and $G^{\sigma}_{\alpha\beta}$ if and only if the tensor of the affine deformation satisfy the conditions*

$$\underset{s}{v}^{\tau}\ \delta_{\tau}^{[\beta}T_{\sigma\nu}^{\alpha]}\ \underset{s}{v}^{\nu}\ \underset{k}{v}^{\sigma} = 0 \ , \quad \underset{\overline{s}}{v}^{\tau}\ \delta_{\tau}^{[\beta}T_{\sigma\nu}^{\alpha]}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{k}}{v}^{\sigma} = 0 \ . \tag{25}$$

P r o o f. Taking into account the definition of the composition of the type $(g-g)$ and [2] we can conclude that the compositions $X_n \times X_m$ is simultaneously of the type $(g-g)$ in the connections $\Gamma^{\sigma}_{\alpha\beta}$ and $G^{\sigma}_{\alpha\beta}$ if and only if

$$\nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{k}{v}^{\sigma} = \lambda\underset{s}{v}^{\alpha}, \quad {}^1\nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{k}{v}^{\sigma} = \mu\underset{s}{v}^{\alpha}. \tag{26}$$

Because of ${}^1\nabla_{\sigma}\underset{s}{v}^{\alpha} - \nabla_{\sigma}\underset{s}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{s}{v}^{\nu}$, ${}^1\nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha} - \nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}$ [2] we can write

$${}^1\nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{k}{v}^{\sigma} - \nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{k}{v}^{\sigma} = T^{\alpha}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \ \underset{k}{v}^{\sigma}, \quad {}^1\nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha}\ \underset{\overline{k}}{v}^{\sigma} - \nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha}\ \underset{\overline{k}}{v}^{\sigma} = T^{\alpha}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{k}}{v}^{\sigma}. \tag{27}$$

Now from (26) it follows that the equalities (27) are equivalent to the following ones $\lambda\underset{s}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \ \underset{k}{v}^{\sigma}$, $\lambda\underset{\overline{s}}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{k}}{v}^{\sigma}$. From the last equalities we obtain $\lambda\underset{s}{v}^{[\beta}\underset{s}{v}^{\alpha]} = \underset{s}{v}^{[\beta}T^{\alpha]}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \ \underset{k}{v}^{\sigma} = 0$, $\lambda\underset{\overline{s}}{v}^{[\beta}\underset{\overline{s}}{v}^{\alpha]} = \underset{\overline{s}}{v}^{[\beta}T^{\alpha]}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{k}}{v}^{\sigma} = 0$, from where it follows (25). $\square$

Corollary 3. *In the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ the conditions* (25) *accept the form*

$$T^{\bar{i}}_{pk} = T^{i}_{pk} = 0 \ , \quad T^{i}_{\bar{p}\,\bar{s}} = T^{\bar{i}}_{\bar{p}\,\bar{s}} = 0. \tag{28}$$

P r o o f. Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be chosen as a coordinate one. Then (25) accept the form $\delta_s^{[\beta}T_{ps}^{\alpha]} = 0$, $\delta_{\bar{s}}^{[\beta}T_{\bar{p}\,\bar{s}}^{\alpha]} = 0$ or $\delta_s^{\beta}T^{\alpha}_{ps} - \delta_s^{\alpha}T^{\beta}_{ps} = 0$, $\delta_{\bar{s}}^{\beta}T^{\alpha}_{\bar{p}\,\bar{s}} - \delta^{\alpha}_{\bar{s}}\ T^{\beta}_{\bar{p}\,\bar{s}} = 0$. It is easy to verify:

1. When $\alpha = i, \beta = j$ then $T^{i}_{ps} = 0$.
2. When $\alpha = i, \beta = \overline{j}$ then $T^{\overline{j}}_{ps} = 0$, $T^{i}_{\overline{p}\,\overline{s}} = 0$.
3. When $\alpha = \overline{i}, \beta = j$ then $T^{\overline{i}}_{ps} = 0$, $T^{i}_{\overline{p}\,\overline{s}} = 0$.
4. When $\alpha = \overline{i}, \beta = \overline{j}$ then $T^{\overline{i}}_{\overline{p}\,\overline{s}} = 0$. □

Theorem 3. *The bridges $Xs\overline{p}$ of the composition $X_n \times X_m$ defined by the affinor* (5) *is simultaneously geodesic with respect to the connections $\Gamma^{\sigma}_{\alpha\beta}$ and $G^{\sigma}_{\alpha\beta}$ if and only if the tensor of the affine deformation satisfies the conditions*

$$(29) \qquad \underset{s}{v}^{\tau}\ \delta^{[\beta}_{\tau}T^{\alpha]}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \underset{s}{v}^{\sigma} = 0\ , \quad \underset{\overline{s}}{v}^{\tau}\ \delta^{[\beta}_{\tau}T^{\alpha]}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{s}}{v}^{\sigma} = 0\ .$$

P r o o f. According to [2], [8] we can write the equalities ${}^{1}\nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{s}{v}^{\sigma} - \nabla_{\sigma}\underset{s}{v}^{\alpha}\ \underset{s}{v}^{\sigma} = T^{\alpha}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \underset{s}{v}^{\sigma}$, ${}^{1}\nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha}\ \underset{\overline{s}}{v}^{\sigma} - \nabla_{\sigma}\underset{\overline{s}}{v}^{\alpha}\ \underset{\overline{s}}{v}^{\sigma} = T^{\alpha}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{s}}{v}^{\sigma}$, which are equivalent to $\lambda\underset{s}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \underset{s}{v}^{\sigma}$, $\lambda\underset{\overline{s}}{v}^{\alpha} = T^{\alpha}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{s}}{v}^{\sigma}$ since the vector fields $\underset{s}{v}^{\alpha}$ and $\underset{\overline{s}}{v}^{\sigma}$ are simultaneously geodesic with respect to the connections $\Gamma^{\sigma}_{\alpha\beta}$ and $G^{\sigma}_{\alpha\beta}$. But the last equalities are equivalent to $\lambda\underset{s}{v}^{[\beta}\underset{s}{v}^{\alpha]} = \underset{s}{v}^{[\beta}T^{\alpha]}_{\sigma\nu}\ \underset{s}{v}^{\nu}\ \underset{s}{v}^{\sigma} = 0$, $\lambda\underset{\overline{s}}{v}^{[\beta}\underset{\overline{s}}{v}^{\alpha]} = \underset{\overline{s}}{v}^{[\beta}T^{\alpha]}_{\sigma\nu}\ \underset{\overline{s}}{v}^{\nu}\ \underset{\overline{s}}{v}^{\sigma} = 0$ from where it follows (29). □

Corollary 4. *In the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ the conditions* (25) *accept the form*

$$(30) \qquad T^{\overline{i}}_{ss} = T^{i}_{ss} = 0\ , \quad T^{i}_{\overline{s}\,\overline{s}} = T^{\overline{i}}_{\ \overline{s}} = 0.$$

P r o o f. Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be chosen as a coordinate one. Then (25) accept the form $\delta^{[\beta}_{s}T^{\alpha]}_{ss} = 0$, $\delta^{[\beta}_{\overline{s}}T^{\alpha]}_{\overline{s}\,\overline{s}} = 0$ or $\delta^{\beta}_{s}T^{\alpha}_{ss} - \delta^{\alpha}_{s}T^{\beta}_{ss} = 0$, $\delta^{\beta}_{\overline{s}}T^{\alpha}_{\overline{s}\,\overline{s}} - \delta^{\alpha}_{\overline{s}}\ T^{\beta}_{\overline{s}\,\overline{s}} = 0$. It is easy to verify:

1. When $\alpha = i, \beta = j$ then $T^{i}_{ss} = 0$.
2. When $\alpha = i, \beta = \overline{j}$ then $T^{\overline{j}}_{ss} = 0$, $T^{i}_{\overline{s}\,\overline{s}} = 0$.
3. When $\alpha = \overline{i}, \beta = j$ then $T^{\overline{i}}_{ss} = 0$, $T^{j}_{\overline{s}\,\overline{s}} = 0$.
4. When $\alpha = \overline{i}, \beta = \overline{j}$ then $T^{\overline{i}}_{\overline{s}\,\overline{s}} = 0$. □

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