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CHARACTERIZING NON-MATRIX PROPERTIES OF VARIETIES OF ALGEBRAS IN THE LANGUAGE OF FORBIDDEN OBJECTS

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Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. We discuss characterizations of some non-matrix properties of varieties of associative algebras in the language of forbidden objects. Properties under consideration include the Engel property, Lie nilpotency, permutativity. We formulate a few open problems.

1. Introduction. In this survey all rings and algebras are assumed associative. The word "algebra" means an "algebra over a field" or a "$\mathbb{Z}$-algebra" (i.e., a ring).

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Many important properties studied within the theory of PI-rings are expressed in terms of a sequence of identities rather than individual identities. (For instance, Lie nilpotency of a variety $\mathcal{V}$ means that $\mathcal{V}$ satisfies the identity $[x_1, \ldots, x_n] = 0$ for some $n = 2, 3, \ldots$.) Therefore in order to recognize if a variety defined by a system of identities enjoys such a property, one has to solve a potentially infinite sequence of inference problems. It is well known that even individual inference problems may be very hard, and, although A. Belov [2, Theorem 3] claims that this problem is decidable in the realm of associative algebras, no feasible algorithm for deciding inference has been developed so far. The situation changes drastically if the property under consideration admits an indicator characterization.

An *indicator characterization* of a variety property θ is a statement of the following kind. *A variety $\mathcal{V}$ satisfies θ if and only if $\mathcal{V}$ does not contain the algebras $A_1, A_2, \ldots$.*

The algebras $A_1, A_2, \ldots$ are called "forbidden algebras" for θ. Suppose, we know this forbidden list for θ and want to prove that a variety $\mathcal{V}$ defined by a system of identities Σ satisfies θ, i.e., $\mathcal{V}$ is a *θ-variety.* We verify whether the identities from Σ hold in the forbidden algebras. If none of the algebras satisfies Σ, then $\mathcal{V}$ is a θ-variety. If the list of forbidden algebras is not very extensive and the algebras are well-constructed then the proof is almost routine. In our paper all considered properties are given in terms of identities of special kind. In this case to form the forbidden list one can find *almost θ-varieties*[1], i.e., elements minimal with respect to inclusion in the set of all non-θ-varieties. Algebras generating the almost θ-varieties form a complete list of forbidden algebras. Indeed, every θ-variety $\mathcal{V}$ satisfies an identity of special kind. Hence, the variety does not contain any almost θ-variety as a subvariety. Therefore, none of the forbidden algebras belongs to $\mathcal{V}$. Conversely, every non θ-variety $\mathcal{V}$ does not satisfy any special for θ identity. Then, by Zorn's Lemma, $\mathcal{V}$ contains some almost θ-variety as a subvariety. Therefore, $\mathcal{V}$ contains the forbidden algebra generating this almost θ-variety.

Indicator characterizations have been found for numerous variety properties. In this paper we consider *non-matrix properties* of varieties. Let us recall that θ is called non-matrix if any variety satisfying θ contains no algebra of all matrices 2×2 over a field. Here we survey indicator characterizations for some natural non-matrix variety properties.

We adopt the following notation.

Let F be a field or $\mathbb{Z}$. We denote by $F\langle X \rangle$ the free F-algebra generated

[1]In some papers an alternative term "just non-θ- varieties" is used.

by a countable set X. As usual, the elements of $F\langle X\rangle$ are called *polynomials.* An ideal I of $F\langle X\rangle$ is called *a T-ideal* if it is closed under endomorphisms. Let A be an algebra, Σ a set of polynomials, $\mathcal{V}$ a variety. We denote by $\operatorname{var} A$ the variety generated by A, and by $\operatorname{var}\Sigma$ the variety defined by Σ. The ideal of the identities of a variety $\mathcal{V}$ (or an algebra A) is the set of all polynomials $f(x)$ such that $f(x)=0$ is an identity of $\mathcal{V}$ (or A). We denote the ideal of the identities of $\mathcal{V}$ by $T(\mathcal{V})$ (or $T(A)$, respectively), and we write $T(\Sigma)$ in the place of $T(\operatorname{var}\Sigma)$.

We denote by $\mathcal{V}^\star$ the variety dual to $\mathcal{V}$. In the case of algebras $A^\star$ stands for the algebra anti-isomorphic to A.

By $\overline{x}$ we denote a tuple of variables $x_1, x_2, \dots$.

As usual, $GF(q)$ is a finite field of q elements. Let U be an arbitrary algebra. As usual, we denote by $M_n(U)$ the algebra of all $n\times n$ matrices with entries in U, and by $UT_n(U)$ the algebra of all $n\times n$ upper triangular matrices with entries in U. We define

$$A(U)=\begin{pmatrix} U & U \\ 0 & 0 \end{pmatrix},$$

$$C(U,m)=\left\{\begin{pmatrix} a & b \\ 0 & a^m \end{pmatrix} \mid a,b\in U\right\}.$$

For a finite field F we also introduce:

$$B(F,G,\sigma)=\left\{\begin{pmatrix} b & c \\ 0 & \sigma(b) \end{pmatrix} \mid b,c\in G\right\},$$

where G is a finite extension of F, σ is an F-automorphism of G such that the invariant field G^σ is a unique maximal subfield of G.

Let F be a field. For a positive integer n denote by $K_{F,n}$ the F-algebra generated by $k_1, k_2, \dots$ subject to the relations

$$k_ik_j = k_jk_i,\ k_i^n = 0,\ i,j = 1,2\dots.$$

By $K_{F,0}$ we denote the F-algebra generated by $k_1, k_2, \dots$ subject to the relations

$$k_ik_j = k_jk_i,\ i,j = 1,2\dots.$$

We will often use various iterated Lie commutators. We define

$$W_2(x_1,x_2) = [x_1,x_2] = V_1(x_1,x_2),$$

where $[x_1,x_2] = x_1x_2 - x_2x_1$ and then continue inductively:

$$W_n(x_1,\dots,x_n) = [W_{n-1}(x_1,\dots x_{n-1}),x_n],$$

$$V_n(x_1, \ldots, x_{2^n}) = [V_{n-1}(x_1, \ldots x_{2^{n-1}}), V_{n-1}(x_{2^{n-1}+1}, \ldots x_{2^n})].$$

Let us denote by $E_n(x, y)$ the Engel polynomial $W_n(x, y, \ldots, y)$.

A variety or an algebra is called *Lie nilpotent* if it satisfies the identity $W_n(x_1, \ldots, x_n) = 0$ for some n.

A variety or an algebra is called *Engel* if it satisfies the identity $E_n(x, y) = 0$ for some n.

A variety or an algebra is called *Lie solvable* if it satisfies the identity $V_n(x_1, \ldots, x_{2^n}) = 0$ for some n.

A polynomial identity $u = v$ where u, v are distinct semigroup words is called a *semigroup identity.* The semigroup identity $u = v$ is called *reduced* if the first letters of the words u, v are different and the last letters of the words u, v are different as well.

In the following proposition we collect several easy statements establishing that the considered properties are non-matrix. Some proofs are well-known but we give them here for convenience. The algebras from the proposition will be used as forbidden algebras for these properties.

Proposition 1. *Let F be a field. Then the following statements hold:*

1) *$A(F)$ and $A(F)^\star$ satisfy neither a reduced semigroup identity nor $E_n(x, y) = 0$ for any n.*
2) *$A(K_{F,2})$ and $A(K_{F,2})^\star$ do not satisfy an identity $W_n(\overline{x}) = 0$ for any n.*
3) *If F is finite, then $B(F, G, \sigma)$ does not satisfy $E_n(x, y) = 0$ for any n.*
4) *If $|F| = q$, then $C(K_{F,0}, q^m)$ and $C(K_{F,0}, q^m)^\star$ do not satisfy a semigroup identity.*
5) *If F is infinite, then $UT_2(F)$ does not satisfy a semigroup identity.*
6) *If* $\mathrm{char} F \neq 2$, *then $M_2(F)$ does not satisfy $V_n(\overline{x}) = 0$ for any n.*
7) *If* $\mathrm{char} F = 2$, *then $M_3(F)$ does not satisfy $V_n(\overline{x}) = 0$ for any n.*

Proof. 1) It is easy to see that $A(F)$ is non-Engel. Indeed, we have

$$E_n(e_{12}, e_{11}) = (-1)^n e_{12} \neq 0.$$

By the dual argument, $A(F)^\star$ is non-Engel. Further assume that $u(x, y, \overline{z})x = v(x, y, \overline{z})y$ is a reduced semigroup identity, where u, v are words and x, y are different variables. Let us substitute $x = z_1 = z_2 = \cdots = e_{11}$, $y = e_{11} + e_{12}$. Obviously, $u(x, y, z_1, \ldots)x = e_{11}$ and $u(x, y, z_1, \ldots)y = e_{11} + e_{12}$. This proves that $A(F)$ does not satisfy any reduced semigroup identity. Similarly, by dual arguments $A(F)^\star$ does not satisfy any reduced semigroup identity of the form $xu(x, y, \overline{z}) = yv(x, y, \overline{z})$, $x \neq y$.

2) Let us consider $A(K_{F,2})$ and substitute

$$x_1 = \begin{pmatrix} 0 & k_1 \\ 0 & 0 \end{pmatrix}, \quad x_i = \begin{pmatrix} k_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 2, \ldots, n.$$

Then we obtain

$$W_n(x_1, \ldots, x_n) = \begin{pmatrix} 0 & (-1)^{n-1} k_1 \cdots k_n \\ 0 & 0 \end{pmatrix} \neq 0.$$

The dual argument shows that $A(K_{F,2})^\star$ also is not Lie nilpotent.

3) Let a belong to G and $\sigma(a) \neq a$. Then we have $[e_{12}, ae_{11} + \sigma(a)e_{22}] = (\sigma(a) - a)e_{12}$ and $E_n(e_{12}, ae_{11} + \sigma(a)e_{22}) = (\sigma(a) - a)^n e_{12} \neq 0$ for every n. This proves that $B(F, G, \sigma)$ is non-Engel.

4) and 5) Let $u(x, y, \overline{z})$, $v(x, y, \overline{z})$, $w(x, y, \overline{z})$ be semigroup words and let

$$f(x, y, \overline{z}) = uxw - vyw.$$

We verify that $C(K_{F,0}, q^m)$ and $UT_2(F)$ do not satisfy the semigroup identity $f(x, y, \overline{z}) = 0$. If the lengths of u and v are different then $f(x, x, \ldots, x) = x^k - x^l$ for $k \neq l$. It is evident that $f(x, x, \ldots, x) = 0$ is not an identity of $C(K_{F,0}, q^m)$ and of $UT_2(F)$ for an infinite F. Now suppose, that the length of u is equal to the length of v. Denote by N the length of u and v and by S the length of w. Let b and c be elements of $K_{F,0}$ in the case of 4) or elements of F in the case of 5). If F is infinite then we choose $q = 2$ (or another integer greater than 1). Consider the matrices $B = \begin{pmatrix} b & 0 \\ 0 & b^{q^m} \end{pmatrix}$ and $C = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$. It is easy to see that $CBC = C^2 = 0$ and $CB = B^{q^m} C$. Let us substitute $y = z_i = B$ $(i = 1, 2, \ldots)$, $x = B + C$. Collecting the similar terms we obtain

$$f(B + C, B, B, \ldots) = (B^N C - \sum_{i>0} \alpha_i B^{N-i} C B^i) B^S$$

$$= (B^N - \sum_{i>0} \alpha_i B^{N-i+iq^m}) B^{Sq^m} C = \begin{pmatrix} 0 & (b^N - \sum_{i>0} \alpha_i b^{N-i+iq^m}) b^{Sq^m} c \\ 0 & 0 \end{pmatrix}.$$

Now, suppose that $f(x,y,\overline{z})=0$ is an identity of $C(K_{F,0},q^m)$. Then for some α_i the algebra $K_{F,2}$ satisfies the identity

$$(b^N-\sum_{i>0}\alpha_i b^{N-i+iq^m})b^{Sq^m}c=0$$

which is not true. The same argument shows that $f(x,y,\overline{z})=0$ is not an identity of $UT_2(F)$ if F is infinite.

6) It is easy to see that the following equalities hold in $M_2(F)$:

$$[e_{12},[e_{12},e_{21}]]=-2e_{12},$$

$$[e_{21},[e_{12},e_{21}]]=2e_{21}.$$

Hence, if $\mathrm{char}F\neq 2$ then $M_2(F)$ is not Lie solvable. Let us remark that if $\mathrm{char}F=2$ then $M_2(F)$ satisfies the identity $[[[x,y],[z,t]],u]=0$ and, hence, $V_3(\overline{x})=0$.

7) Let us fix $x_1=e_{12}+e_{31}$ and $x_2=e_{23}+e_{31}$. By assumption, $\mathrm{char}F=2$. Therefore we have

$$[[[x_2,x_1],x_1],x_1]=x_1,$$

$$[[[x_1,x_2],x_2],x_2]=x_2,$$

Hence, $M_3(F)$ is not Lie solvable. □

2. Forbidden algebras for non-matrix properties. In this section we discuss indicator characterizations for the properties from Proposition 1 as well as some others. For all considered properties θ the lists of forbidden algebras coincide with the lists of algebras generating almost-θ-varieties. Therefore for a property θ we give either an indicator characterization or a list of almost-θ-varieties.

2.1. Nilpotency. The list of almost nilpotent varieties of algebras over a commutative ring was found by I. L'vov [11]. Here we give a specialization of his result in the cases of algebras over a field and of rings.

Theorem 1. *Let $\mathcal{V}$ be a variety of F-algebras. Then the following statements hold:*

1) *Let F be an infinite field,* $\mathrm{char}F=p\geq 0$*. Then $\mathcal{V}$ is almost nilpotent if and only if it is generated by $K_{F,p}$.*

2) *Let F be a finite field,* $\mathrm{char} F = p$*. Then $\mathcal{V}$ is almost nilpotent if and only if it is generated either by $K_{F,p}$ or by F.*

3) *Suppose, $F = \mathbb{Z}$. Then $\mathcal{V}$ is almost nilpotent if and only if it is generated either by $GF(p)$ or by $K_{GF(p),p}$ where p is a prime.*

Let us remark that in the ring case the variety $\mathrm{var}\, K_{GF(p),p}$ (p is a prime) is not generated by a finite algebra but every of its proper subvarieties is generated by a finite algebra. In other words, $\mathrm{var}\, K_{GF(p),p}$ is an almost Cross variety. L'vov [11] found the complete list of such varieties. The list consists of the varieties $\mathrm{var}\, K_{GF(p),p}$ for prime p and the variety $\mathrm{var}\,\{xy = 0\}$.

2.2. Commutativity. The question whether $[x, y]$ follows from the identities or some other conditions on the algebra is well known and studied in the theory of varieties. One of the pioneer results is Herstein's theorem establishing the commutativity of a ring with the condition that every element a of the ring satisfies $a^{n(a)} = a$. Similar conditions (the so called "commutativity conditions") were intensively studied in the 1980s (see [9]). Almost commutative varieties were investigated by Yu. Maltsev. In the case of an infinite field the complete list of such varieties was obtained.

Theorem 2. *Let F be an infinite field. There exists a unique almost commutative variety of F-algebras*

$$\mathrm{var}\{xyz = 0,\ x^2 = 0\}, \quad \text{if } \ \mathrm{char} F \neq 2,$$

$$\mathrm{var}\{xyz = 0\}, \quad \text{if } \ \mathrm{char} F = 2.$$

In the case of algebras over a finite field or of rings Yu. Maltsev proved that every almost commutative variety is generated by a finite algebra. He found a complete list of the non-nilpotent algebras and described the properties of the nilpotent algebras (cf. [13], [16], [14]).

Theorem 3. *In the case of a finite base field or $\mathbb{Z}$-algebras (i.e., rings) the following statements hold:*

1) *A non-nilpotent variety is almost commutative if and only if it is generated by one of the algebras $A(F)$, $A(F)^\star$ or $B(F, G, \sigma)$, where F is the base field or in the case of rings F is a prime field.*

2) *Every nilpotent almost commutative variety satisfies the identities*

$$[x,y]z = 0, \quad z[x,y] = 0, \quad x_1 \cdots x_n = 0,$$

and an identity of the kind

$$[x,y] = \sum_{i=1}^{n-1} \alpha_i x^i y^{n-1-i} \quad \textit{for some } n > 2.$$

The finite algebras generating the varieties from part 2) of Theorem 3 are described completely for $n = 3, 4$ (see [17]). Additional results concerning the algebras are in the paper of E. Zakharova [26]. But the full description is still unknown.

Problem 1. *Find a complete list of nilpotent algebras generating an almost commutative variety.*

Remark. Every proper subvariety of a variety generated by an algebra from part 1) of Theorem 3 is commutative, and, so, both Lie nilpotent and Engel. According to parts 1) and 2) of Proposition 1, the algebras generate almost Engel and almost Lie nilpotent varieties.

2.2.1. Engel property. Almost Engel varieties over a field of characteristic 0 were found by Yu. Maltsev [12]. The case of algebras over an infinite field of positive characteristic was considered by the author [5]. These two descriptions are equivalent to the following statement.

Theorem 4. *A variety of algebras over an infinite field F is Engel if and only if it does not contain the algebras $A(F)$ and $A(F)^\star$.*

The case of rings was considered by Yu. Maltsev [15]. He proved that the locally finite almost Engel varieties are exactly the non-nilpotent almost commutative varieties. The question whether there exist other almost Engel varieties had been open for a long time (cf. [4], Question 3.53). The results of the author [5] answer it in the negative.

Theorem 5. *In the case of a finite base field or rings a variety is Engel if and only if it does not contain $A(F)$, $A(F)^\star$, and $B(F,G,\sigma)$, where F is the base field or in the case of rings F is a prime field.*

The following corollaries of Theorems 4, 5, and 3 are evident.

Corollary 1. *If all finite rings (finite-dimensional algebras) of a variety are Engel then the variety itself is Engel.*

Corollary 2. *Let all finite nilpotent rings (finite-dimensional nilpotent algebras) of a variety be commutative. Then the variety is commutative if it is Engel.*

2.2.2. Lie nilpotency. In the case of zero characteristic the list of forbidden algebras for Lie nilpotency coincide with the list of such algebras for the Engel property, because every Engel variety is Lie nilpotent and vice versa (cf. [8]). Therefore Theorem 4 gives an indicator characterization for Lie nilpotency in this case.

In a positive characteristic these properties are not equivalent. In other words, there exist Engel algebras which are not Lie nilpotent. A simple example is a nil algebra which is not Lie nilpotent. Indeed, it is easy to show that

$$E_n(x,y) = \sum_{k=0}^{n} C_n^k (-1)^k y^k x y^{n-k}.$$

Thus, if the characteristic of the base field F is equal to p then $E_{p^t}(x,y) = xy^{p^t} - y^{p^t}x$ for any positive integer t. Hence, every nil algebra satisfies the identity $E_{p^t}(x,y) = 0$ for sufficiently large t. Let us consider the algebra $A(K_{F,2})$. The algebra $K_{F,2}$ satisfies the identity $x^p = 0$. Therefore, $A(K_{F,2})$ satisfies the identity $x^p y = 0$. Hence, this algebra is nil but it is not Lie nilpotent by Proposition 1. Thus, $A(K_{F,2})$ is an Engel algebra which is not Lie nilpotent. This algebra generates the variety $\mathrm{var}\langle x^p y = 0,\ [x,y]z = 0\rangle$. It is not hard to show that the variety is one of the almost Lie nilpotent varieties.

To formulate results concerning such varieties we need to recall the following definition due to A. Kemer.

A variety of F-algebras is called *prime* (or *verbally prime*) if for its T-ideal T every inclusion $I_1 \cdot I_2 \subseteq T$ holding for two T-ideals I_1 and I_2 of $F\langle X\rangle$ implies $I_1 \subseteq T$ or $I_2 \subseteq T$.

All almost Lie nilpotent non-prime varieties are found by the author [6].

Theorem 6. *Let $\mathcal{V}$ be a non-prime variety of algebras over an infinite field F of positive characteristic. Then $\mathcal{V}$ is almost Lie nilpotent if and only if $\mathcal{V}$ is generated either by $A(K_{F,2})$ or by $A(K_{F,2})^\star$.*

A proof of the following theorem in the case of a finite base field can be found in [6]. In Section 3 we show how to reduce the case of rings to this case.

Theorem 7. *Let $\mathcal{V}$ be a non-prime variety of algebras over a finite field or of rings. Then $\mathcal{V}$ is almost Lie nilpotent if and only if $\mathcal{V}$ is generated by one of the following algebras $A(F)$, $A(F)^\star$, $B(F,G,\sigma)$, $A(K_{F,2})$, $A(K_{F,2})^\star$, where F is the base field or in the case of rings F is a prime field.*

Corollary 3. *Let a variety of F-algebras $\mathcal{V}$ satisfy an identity*

$$W_{n_1}(\overline{x}_1)W_{n_2}(\overline{x}_2)\cdots W_{n_k}(\overline{x}_k) = 0,$$

where $n_1, \dots, n_k$ are positive integers, and $\overline{x}_1, \dots, \overline{x}_k$ are disjoint sets of variables. Then the following statements hold:

1) *If F is an infinite field of positive characteristic, then $\mathcal{V}$ is Lie nilpotent if and only if $\mathcal{V}$ does not contain the algebras $A(K_{F,2})$ and $A(K_{F,2})^\star$.*

2) *If F is a finite field, then $\mathcal{V}$ is Lie nilpotent if and only if $\mathcal{V}$ does not contain the algebras $A(F)$, $A(F)^\star$, $B(F, G, \sigma)$, $A(K_{F,2})$, $A(K_{F,2})^\star$.*

3) *If $F = \mathbb{Z}$, then $\mathcal{V}$ is Lie nilpotent if and only if $\mathcal{V}$ does not contain the algebras $A(H)$, $A(H)^\star$, $B(H, G, \sigma)$ and $A(K_{H,2})$, $A(K_{H,2})^\star$ for any prime field H.*

P r o o f. Because of the identity $W_{n_1}(\overline{x}_1)\cdots W_{n_k}(\overline{x}_k) = 0$, all prime subvarieties of $\mathcal{V}$ are Lie nilpotent. So, $\mathcal{V}$ is not Lie nilpotent if and only if it contains a non-prime almost Lie nilpotent variety, and, hence, the algebra generating such variety. □

Examples of almost Lie nilpotent prime varieties for algebras over an infinite field of positive characteristic were found by Yu.Razmyslov [20, section 39]. In the case of a finite base field or of rings such examples are not known.

Problem 2. *Find all almost Lie nilpotent prime varieties both in the case of algebras over a field of positive characteristic and in the case of rings.*

2.2.3. Lie solvability. In the case of zero characteristic the class of all Lie solvable varieties is the class of all non-matrix varieties by the result of Kemer (see [8], Corollary 1).

Theorem 8. *A variety of algebras over a field F of characteristic zero is Lie solvable if and only if it does not contain the algebra $M_2(F)$.*

The algebra $M_2(F)$ generates an almost Lie solvable variety in the case of a finite base field F ($\mathrm{char}F \neq 2$) as well. Moreover, the following statement holds.

Proposition 2. *Let F be a finite field. Suppose that a variety of F-algebras $\mathcal{V}$ is generated by a finite algebra. Then $\mathcal{V}$ is almost Lie solvable if and only if it is generated by $M_2(F)$ if $\mathrm{char}F > 2$ and by $M_3(F)$ if $\mathrm{char}F = 2$.*

P r o o f. By parts 6) and 7) of Proposition 1 the algebras $M_2(F)$ (if $\mathrm{char}F > 2$) and $M_3(F)$ are non Lie solvable. Let A be a finite algebra such that $\mathrm{var}A$ does not contain $M_2(F)$ if $\mathrm{char}F > 2$ or $M_3(F)$ if $\mathrm{char}F = 2$. Let J be the Jacobson radical of A. Consider the semisimple algebra A/J. If $\mathrm{char}F > 2$ then A/J is commutative because it is a direct sum of fields. If $\mathrm{char}F = 2$ then A/J satisfies the identity $V_3(\overline{x}) = 0$ because it is a direct sum of fields and algebras of all 2×2 matrices over a field of characteristic 2 (the latter algebras satisfy $V_3(\overline{x}) = 0$). It is well known that the radical J is nilpotent. Therefore, in both cases A satisfies an identity $V_3(\overline{x}_1)V_3(\overline{x}_2)\cdots V_3(\overline{x}_n) = 0$. Hence, the algebra A is Lie solvable. Thus, every variety which is not Lie solvable has to contain the algebra $M_2(F)$ if $\mathrm{char}F > 2$ or the algebra $M_3(F)$ if $\mathrm{char}F = 2$. This concludes the proof. □

In the other cases almost Lie solvable varieties remain much less understood. In fact, we can only formulate the following easy statement. (It will be proved in Section 3.)

Proposition 3. *Every almost Lie solvable variety is prime.*

Problem 3. *Find a complete list of all almost Lie solvable varieties both in the case of a base field of positive characteristic and in the case of rings.*

2.2.4. Semigroup identities. Let F be a field of characteristic zero. The varieties of F-algebras satisfying semigroup identities both reduced and non-reduced were studied by I. Golubchik and A. Mikhalev [7]. For the non-reduced case they found a unique forbidden algebra $UT_2(F)$. Besides that, they proved that if a variety satisfies a reduced semigroup identity then it is Engel and vice versa. If F is an infinite field of positive characteristic then these results remain true. Again, $UT_2(F)$ is the only forbidden algebra for this property [24]. The fact that in positive characteristic a reduced semigroup identity is equivalent to the Engel property was established by D. M. Riley and M. C. Wilson [22].

Theorem 9. *A variety of algebras over an infinite field F satisfies a semigroup identity if and only if it does not contain the algebra $UT_2(F)$.*

The following theorem is a combination of known results [7, 22] and Theorem 4.

Theorem 10. *A variety of algebras over an infinite field F satisfies a reduced semigroup identity if and only if it does not contain the algebras $A(F)$ and $A(F)^\star$.*

The situation with algebras over a finite field is quite different. Every Engel algebra satisfies an identity of the kind $xy^{p^t} - y^{p^t}x = 0$ were p is the characteristic of the base field F (see the text in the beginning of section 2.2.2). It is a reduced semigroup identity. But the Engel property does not follow from a reduced semigroup identity. Indeed, let us consider the algebra $B(F, G, \sigma)$. It is finite. Hence, there exists an integer n such that x^n is an idempotent for any element x from $B(F, G, \sigma)$. It is easy to see that $B(F, G, \sigma)$ contains only two idempotents: 0 and 1. Therefore $B(F, G, \sigma)$ satisfies a reduced semigroup identity $x^n y = yx^n$. But $B(F, G, \sigma)$ is non-Engel by Proposition 1.

Nevertheless, for the reduced case a list of forbidden algebras is obtained both for the case of a finite field and for the case of rings.

Theorem 11 (Finogenova, unpublished). *Let F be a finite field, $|F| = q$, and let $\mathcal{V}$ be a variety of F-algebras. Then the following statements are equivalent:*

1) *$\mathcal{V}$ satisfies a reduced semigroup identity.*

2) *$\mathcal{V}$ satisfies the identity $x^n y = yx^n$ for some positive integer n.*

3) *$\mathcal{V}$ does not contain the algebras $A(F)$, $A(F)^\star$, $C(K_{F,0}, q^m)$, and $C(K_{F,0}, q^m)^\star$ for all integers $m > 0$.*

Theorem 12 (Finogenova, unpublished). *Let $\mathcal{V}$ be a variety of rings. Then the following statements are equivalent:*

1) *$\mathcal{V}$ satisfies a reduced semigroup identity.*

2) *$\mathcal{V}$ satisfies the identity $x^n y = yx^n$ for some positive integer n.*

3) *$\mathcal{V}$ does not contain the rings $A(F)$, $A(F)^\star$, $C(K_{F,0}, p^m)$, and $C(K_{F,0}, p^m)^\star$ for all prime fields F of order p and all integers $m > 0$.*

Theorems 12 and 5 show that every Engel variety of rings also satisfies a reduced semigroup identity. Indeed, comparing the lists from the theorems we see that the every forbidden algebra from Theorem 5 belongs to a variety generated by a suitable forbidden algebras from Theorem 12. Therefore an Engel variety cannot contain any forbidden algebra from Theorem 12. Hence, the variety satisfies a reduced identity. This result is proved by Riley and Wilson [23] (Corollary 2) in a different way.

The case of non-reduced identities is more complicated: no complete list of forbidden algebras is known so far. We can prove only the following partial result. In the proposition an "SI-variety" means "a variety satisfying a semigroup identity".

Proposition 4. *Let F be a finite field, $|F| = q$. Then the algebra $C(K_{F,0}, q^m)$ and the algebra $C(K_{F,0}, q^m)^\star$ generate almost SI-varieties.*

Proof. By Proposition 1, $C(K_{F,0}, q^m)$ and $C(K_{F,0}, q^m)^\star$ do not satisfy any semigroup identity. But every proper subvariety of $\operatorname{var} C(K_{F,0}, q^m)$ or of $\operatorname{var} C(K_{F,0}, q^m)^\star$ satisfies a reduced semigroup identity by Theorem 11. □

Problem 4. *Find a complete list of forbidden algebras for the property "to satisfy a semigroup identity" in the case of varieties of rings and algebras over a finite field.*

2.2.5. Adjoint semigroup identities. In the case of an infinite base field Riley and Wilson found another condition equivalent to the Engel property [22]. To explain it we recall some definitions and notation. Given an algebra R, the *circle composition* $\circ$ on R is defined by letting $a \circ b = a + b - ab$ for all a, b in the algebra[2]. It is easy to see that the circle composition is associative, and thus we obtain the *adjoint semigroup* $\langle R, \circ\rangle$ of the algebra. If the algebra has an identity element 1, then its multiplicative semigroup and its adjoint semigroup are isomorphic via the map $a \mapsto 1 - a$. However, for algebras without identity element the adjoint semigroup can be very different from the multiplicative semigroup. For instance, if $\langle R, +, \cdot\rangle$ is such that the semigroup $\langle R, \cdot\rangle$ is nilpotent then the semigroup $\langle R, \circ\rangle$ turns out to be a group. An identity of the adjoint semigroup is called *an adjoint semigroup identity* of the algebra.

Riley and Wilson [22] showed for the case of an infinite base field that whenever an algebra is Engel then it satisfies an adjoint semigroup identity. Combining this fact with Theorem 4 we derive the following result.

Theorem 13. *A variety of algebras over an infinite field F satisfies an adjoint semigroup identity if and only if it does not contain the algebras $A(F)$ and $A(F)^\star$.*

In the case of a finite base field and rings the above equivalence does not hold. It is not hard to show that an Engel algebra over a field of positive characteristic p satisfies an adjoint semigroup identity. Indeed, every Engel algebra satisfies the identity $E_{p^t}(x, y) = xy^{p^t} - y^{p^t}x = 0$. By induction on n one can show that $\underbrace{y \circ y \circ \cdots \circ y}_{n} = \sum_{k=1}^{n} C_n^k (-1)^{k+1} y^k$. Hence, $y^p = \underbrace{y \circ \cdots \circ y}_{p}$ and

[2] In some papers one means by the circle composition the operation $\circ'$ defined by $a \circ' b = a + b + ab$. This does not make real difference because $\langle R, \circ'\rangle$ is isomorphic to $\langle R, \circ\rangle$ via the map $a \mapsto -a$.

$y^{p^t} = \underbrace{y \circ \cdots \circ y}_{p^t}$. Thus

$$xy^{p^t} - y^{p^t}x = x \circ y^{p^t} - y^{p^t} \circ x =$$

$$x \circ \underbrace{y \circ \cdots \circ y}_{p^t} - \underbrace{y \circ \cdots \circ y}_{p^t} \circ y = 0.$$

The latter equality is an adjoint semigroup identity of the Engel algebra. But the algebras over a finite field satisfying adjoint semigroup identities are not necessarily Engel. As an example we can take the non-Engel algebra $B(F, G, \sigma)$. Its adjoint and multiplicative semigroups are isomorphic because $B(F, G, \sigma)$ has an identity element. The multiplicative semigroup of the algebra satisfies an identity (see the text after Theorem 10). Hence, the adjoint semigroup satisfies an identity, too.

Remark. As we can see, in the case of an infinite field, a variety satisfies a reduced semigroup identity if and only if it satisfies an adjoint semigroup identity. Moreover, Riley and Wilson [22] showed that the variety also satisfies the corresponding reduced adjoint semigroup identity. The proofs of these equivalences require the base field to be infinite. It is unclear if these conditions stay so closely connected in the case of a finite field.

Problem 5.

1) *Find a complete list of forbidden algebras for the property "to satisfy an adjoint semigroup identity" in the case of rings and algebras over a finite field.*

2) *Is the property "to satisfy an adjoint semigroup identity" equivalent to the property "to satisfy a reduced semigroup identity"?*

3) *Is the property "to satisfy an adjoint semigroup identity" equivalent to the property "to satisfy the corresponding reduced adjoint semigroup identity"?*

2.2.6. Permutativity. An important special class of semigroup identities is the class of *permutative* identities. A variety (an algebra) is called *permutative* if it satisfies an identity of the kind

$$x_1x_2 \cdots x_n = x_{1\sigma}x_{2\sigma} \cdots x_{n\sigma},$$

where σ is a nontrivial permutation of the set $\{1, 2, \ldots, n\}$. To formulate indicator characterizations of the permutativity we introduce the following algebras. Let F be a field. Letting U be an arbitrary algebra, we define

$$TZ(U) \cong \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} \right\}, \quad TD(U) \cong \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\}$$

where a, b, c, d belong to U, and

$$R(U) \cong \begin{pmatrix} F & U \\ 0 & U \end{pmatrix}.$$

Theorem 14 (Finogenova, [19]). *A variety of algebras over an infinite field F of characteristic $p \geq 0$ is permutative if and only if it does not contain the algebras $TZ(K_{F,p})$ or $TD(K_{F,p})$.*

For algebra U we denote by U^1 the smallest unitary algebra containing U. The following result was proved by the author in [18].

Theorem 15. *Suppose that a variety $\mathcal{V}$ of algebras over a finite field F is generated by a finite algebra. Then $\mathcal{V}$ is almost permutative if and only if it is generated by one of the algebras $B(F, G, \sigma)$, $TZ(F)$, $UT_2(F)$, or J^1, where J generates a nilpotent almost commutative variety.*

The following theorem completes the description of almost permutative varieties of algebras over a finite field.

Theorem 16 (Finogenova, [19]). *Let F be a finite field of characteristic p and $|F| = q$. Suppose that a variety of F-algebras $\mathcal{V}$ is not generated by any finite algebra. Then $\mathcal{V}$ is almost permutative if and only if it is generated by one of the algebras $TZ(K_{F,p})$, $TD(K_{F,p})$, $C(K_{F,pq^m}, q^m)$, $C(K_{F,pq^m}, q^m)^\star$, $R(K_{F,p})$, $R(K_{F,p})^\star$.*

Problem 6. *Find a complete list of forbidden algebras for permutativity in the case of varieties of rings.*

3. General points of proofs.

3.1. Where to find identities? Let the property θ be defined by a system of polynomials $\{f_\delta\}$, $\delta \in \Delta$. In other words, the variety $\mathcal{V}$ satisfies θ whenever $f_\delta \in T(\mathcal{V})$ for some $\delta \in \Delta$.

The following easy lemma is the main "provider" of identities.

Lemma 1. *Let $\mathcal{V}$ be an almost θ-variety, $g(\overline{x})$ a polynomial. Then either $g \in T(\mathcal{V})$ or $f_\delta \in T(\{g\}) + T(\mathcal{V})$ for some $\delta \in \Delta$.*

Proof. If $g \notin T(\mathcal{V})$ then $T(\{g\}) + T(\mathcal{V})$ defines a proper subvariety of $\mathcal{V}$. Every proper subvariety of $\mathcal{V}$ satisfies θ, therefore $f_\delta \in T(\{g\}) + T(\mathcal{V})$ for some $\delta \in \Delta$. □

We demonstrate the utility of the lemma proving Proposition 3.

Proof of Proposition 3. The Lie solvability is defined by the system of identities $\{V_n(\overline{x})\}$, $n = 1, 2, \ldots$. Suppose, $\mathcal{V}$ is almost Lie solvable, and I_1, I_2 are T-ideals such that $I_1 \cdot I_2 \subseteq T(\mathcal{V})$. Also, suppose that there exist polynomials $g_1 \in I_1 \setminus T(\mathcal{V})$ and $g_2 \in I_2 \setminus T(\mathcal{V})$. Then, by Lemma 1, we have $V_m(\overline{x}) \in T(\{g_1\}) + T(\mathcal{V})$ and $V_n(\overline{x}) \in T(\{g_2\}) + T(\mathcal{V})$ for some positive integers m and n. Clearly, we can assume that $m = n$. Then, we obtain

$$V_n(\overline{x})V_n(\overline{y}) \in T(\{g_1 \cdot g_2\}) + T(\mathcal{V}) \subseteq I_1 \cdot I_2 + T(\mathcal{V}) \subseteq T(\mathcal{V}).$$

Hence, $V_{n+1}(\overline{x}) \in T(\mathcal{V})$, i.e., $\mathcal{V}$ is Lie solvable. The contradiction shows that $I_1 \subseteq T(\mathcal{V})$ or $I_2 \subseteq T(\mathcal{V})$. In other words, $\mathcal{V}$ is a prime variety. □

Using Lemma 1 one can simplify the identities. For example, the almost permutative variety $\mathcal{V}$ satisfies the identity $g(\overline{x}) = 0$ whenever $\mathcal{V}$ satisfies $y_1 \cdots y_m g(\overline{x}) = 0$. Indeed, by Lemma 1, if $g \notin T(\mathcal{V})$ then

$$x_1 x_2 \cdots x_n - x_{1\sigma} x_{2\sigma} \cdots x_{n\sigma} \in T(\{g\}) + T(\mathcal{V}).$$

Multiplying the inclusion by $y_1 \cdots y_m$ we obtain

$$y_1 \cdots y_m x_1 x_2 \cdots x_n - y_1 \cdots y_m x_{1\sigma} x_{2\sigma} \cdots x_{n\sigma} \in T(\{y_1 \cdots y_n g\}) + T(\mathcal{V}) \subseteq T(\mathcal{V}).$$

Therefore $\mathcal{V}$ is permutative. The contradiction proves that $g \in T(\mathcal{V})$.

For every considered property θ one can find a special nontrivial simplification class for identities of an almost θ-variety.

3.2. How to reduce from rings to algebras. In this section we assume that a property θ is defined by a system of polynomials $\{f_\delta\}$, $\delta \in \Delta$. All of the polynomials have integer coefficients. The system is the same both for $\mathbb{Q}$-algebras and for algebras over a prime field $GF(p)$. In the latter case the coefficients are naturally interpreted as elements of the field.

For a T-ideal I in the free ring $\mathbb{Z}\langle X\rangle$ we denote by $I_{\mathbb{Q}}$ the T-ideal of the free algebra $\mathbb{Q}\langle X\rangle$ generated by the polynomials from I. The T-ideal I of $\mathbb{Z}\langle X\rangle$

is called *additive torsion-free* if for any polynomial f and an integer $m \neq 0$ we have $f \in T(\mathcal{V})$ whenever $mf \in T(\mathcal{V})$.

The following result was proved by M. Volkov [25, Lemma 8] but we give a self-contained proof for convenience of the reader.

Lemma 2. *Let I be an additive torsion-free T-ideal. Then I coincides with the set of all polynomials with integer coefficients from $I_{\mathbb{Q}}$.*

Proof. The inclusion $I \subseteq I_{\mathbb{Q}} \cap \mathbb{Z}\langle X\rangle$ is evident. Let us prove the converse. By the usual Vandermonde determinant argument, every homogeneous component of a polynomial from I also belongs to I. Therefore every polynomial from $I_{\mathbb{Q}}$ is a sum of polynomials of the kind $\alpha h(u_1, \ldots, u_n)$ where $\alpha \in \mathbb{Q}$, $u_1, \ldots, u_n \in \mathbb{Z}\langle X\rangle$, and $h(\overline{x})$ is a homogeneous polynomial from I. In other words, $I_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space spanned by the elements of I. Thus for every $g \in I_{\mathbb{Q}}$ there exists an integer $m > 0$ such that $mg \in I$. By the hypothesis, if g has only integer coefficients, then $g \in I$. This proves the required inclusion $I_{\mathbb{Q}} \cap \mathbb{Z}\langle X\rangle \subseteq I$. $\square$

Lemma 3. *Let I be a T-ideal of an almost θ-variety of rings. Then either* $\mathrm{var} I_{\mathbb{Q}}$ *satisfies θ or* $\mathrm{var} I_{\mathbb{Q}}$ *is almost θ-variety of $\mathbb{Q}$-algebras.*

Proof. Suppose, g is a polynomial with rational coefficients and $g \notin I_{\mathbb{Q}}$. We shall show that $T(\{g\}) + I_{\mathbb{Q}}$ defines a θ-variety of $\mathbb{Q}$-algebras[3]. Let $m \neq 0$ be an integer such that mg has only integer coefficients. Clearly, $mg \notin I$. Hence by Lemma 1 we have $f_\delta \in T(\{mg\}) + I$ and therefore $f_\delta \in T(\{g\}) + I_{\mathbb{Q}}$. Thus every proper subvariety of $\mathrm{var} I_{\mathbb{Q}}$ satisfies θ. If $\mathrm{var} I_{\mathbb{Q}}$ does not satisfy θ, then $\mathrm{var} I_{\mathbb{Q}}$ is an almost θ-variety of $\mathbb{Q}$-algebras. $\square$

Now we describe a way to obtain forbidden rings having the lists of algebras generating almost θ-vatieties of algebras over a field (both finite and infinite). We shall demonstrate the method only for θ satisfying the following *substitution* condition: For every α, $\beta \in \Delta$ there exists $\gamma \in \Delta$ such that $f_\gamma \in T(\{f_\alpha(f_\beta, \overline{y})\})$. Let us remark that most of the considered properties satisfy the substitution condition. The exeptions are commutativity, permutativity and the property "to satisfy a (reduced, adjoint) semigroup identity". For all of them except the permutativity one can implement a slightly different approach related to the property "to be locally Noetherian". But we shall not discuss it in this paper.

Let $\mathcal{V}$ be an almost θ-variety of rings. Denote by I the ideal $T(\mathcal{V})$. First, suppose that I is an additive torsion-free T-ideal. Then, by Lemma 2, none of

[3]In this case $T(\{g\})$ is a T-ideal of a variety of $\mathbb{Q}$-algebras. We do not state this explicitly because it is clear from the context whether we consider $\mathbb{Q}$-algebras or rings.

the polynomials $\{f_\delta\}$ belongs to $I_{\mathbb{Q}}$. This means that var$I_{\mathbb{Q}}$ does not satisfy θ. Hence, by Lemma 3 the variety var$I_{\mathbb{Q}}$ is an almost θ-variety of $\mathbb{Q}$-algebras.

Now suppose that I is not additive torsion-free. Then there exist a prime integer p and a polynomial g with integer coefficients, $g \notin I$, such that $pg \in I$. If $px \notin I$ then, by Lemma 1, the following conditions hold for α and β from Δ and a polynomial h:

$$f_\alpha(x, \overline{y}) = ph(x, \overline{y})$$

and

$$f_\beta(\overline{x}) \in T(\{g\}).$$

By assumption, there exists $\gamma \in \Delta$ such that $f_\gamma \in T(\{f_\alpha(f_\beta, \overline{y})\}) \subseteq T(\{pg\}) \subseteq I$. This contradiction shows that $px \in I$. Thus the variety $\mathcal{V}$ is an almost θ-variety of $GF(p)$-algebras[4].

Now it is easy to form a list of forbidden rings from two parts. The first part consists of all forbidden $GF(p)$-algebras generating almost θ-varieties. The second part is formed from the rings generating additive torsion-free varieties. We include such a ring R to the second part only if $T(R)_{\mathbb{Q}}$ defines an almost θ-variety of $\mathbb{Q}$-algebras and none of the rings from the first part belongs to var R.

4. Non-matrix pseudovarieties of algebras. In this section we consider algebras over a finite field and rings. Let us recall some definitions.

A *pseudovariety* is a non-empty class of finite algebras closed under homomorphic images, subalgebras, and finitary direct products[5].

Every pseudovariety can be defined by *pseudoidentities*. Let $\mathcal{A}$ be the class of all finite algebras. An *n-ary implicit operation* π is a family $\{\pi_A \mid A \in \mathcal{A}\}$ where $\pi_A : A^n \mapsto A$ such that for every finite algebras A and B and homomorphism $\phi : A \to B$ we have

$$\phi(\pi_A(a_1, \dots, a_n)) = \pi_B(\phi(a_1), \dots, \phi(a_n)).$$

A pair of n-ary implicit operations is called *pseudoidentity*. It is said that a class $\mathcal{K}$ satisfies a pseudoidentity $\pi = \rho$ if $\pi_A = \rho_A$ for all $A \in \mathcal{K}$. Reiterman [21] proved an analogue of Birkhoff's theorem for pseudovarieties: Pseudovarieties are exactly classes defined by pseudoidentities.

Let $F\langle X\rangle$ be the free F-algebra generated by the countable set X. Every polynomial $f(x_1, \dots, x_n)$ from $F\langle X\rangle$ gives rise to an n-ary implicit operation f

[4] Let us remark that every such variety is an almost θ-variety of rings.

[5] The reader can find a detailed exposition of the theory of pseudovarieties in the book [1].

in the following natural way: For an algebra A and $(a_1, \dots, a_n) \in A^n$, we define $f_A : (a_1, \dots, a_n) \mapsto f(a_1, \dots, a_n)$.

Let $g(\overline{x})$, $h(\overline{x})$ be polynomials from $F\langle X\rangle$. If $g_A = h_A$ for each finite F-algebra A, then we define $d(g,h) = 0$. Otherwise we define $d(g,h) = 2^{-r(g,h)}$, where $r(g,h)$ is the smallest size of finite F-algebras A such that $g_A \neq h_A$. One can prove that $d(g,h)$ is a metric on $F\langle X\rangle$. Let $\{f^{(n)}(\overline{x}) \mid n = 1, 2, \dots\} \subseteq F\langle X\rangle$ be a Cauchy sequence in this metric space. For any finite algebra A there exists N such that $f_A^{(m)} = f_A^{(N)}$ for all $m \geq N$. We define $\pi_A = f_A^{(N)}$. It is well-known and easily verified that the family π is an implicit operation. It is denoted by $\lim_{n\to\infty} f^{(n)}$.

We shall use several such implicit operations. Let $g(\overline{x})$ be a polynomial. We define $g(\overline{x})^\omega = \lim_{n\to\infty} g(\overline{x})^{n!}$. Let us remark that $g(\overline{a})_A^\omega$ is an idempotent of the kind $g(\overline{a})^m$ whatever $g(\overline{x})$, a finite algebra A, and a tuple $\overline{a}$ of elements from A. Also, let $E_\omega(x,y) = \lim_{n\to\infty} E_{n!}(x,y)$.

In this section we shall find bases of the pseudoidentities for pseudovarieties consisting of algebras with properties considered in Section 2.

The following two statements are evident and they are called "theorems" only for uniformity.

Theorem 17. *The pseudovariety of all finite nilpotent algebras (rings) is defined by the pseudoidentity* $x^\omega = 0$.

Theorem 18. *The pseudovariety of all finite Engel algebras (rings) is defined by the pseudoidentity* $E_\omega(x,y) = 0$.

Remark. The Engel property in classes of finite algebras is equivalent to the Lie nilpotency. Thus, the latter theorem describes a basis of the pseudoidentities for the pseudovariety of all finite Lie nilpotent algebras (rings) as well.

Theorem 19. *Let* $\mathcal{LS}_{\text{fin}}$ *be the pseudovariety of all Lie solvable* F*-algebras. Then the following statements hold:*

1) *If* F *is a field,* $\operatorname{char} F > 2$*, then* $\mathcal{LS}_{\text{fin}}$ *is defined by the pseudoidentity* $[x,y]^\omega = 0$.

2) *If* F *is a field of characteristic* 2 *or* $F = \mathbb{Z}$*, then* $\mathcal{LS}_{\text{fin}}$ *is defined by the pseudoidentity* $[[[x,y],[x,xy]],y]^\omega = 0$.

Proof. 1) by Proposition 2, every semisimple algebra from $\mathcal{LS}_{\text{fin}}$ is commutative as a direct sum of fields. Hence, the commutator $[x,y]$ of any elements x, y from every algebra of $\mathcal{LS}_{\text{fin}}$ is nilpotent. Therefore the pseudoidentity

$[x,y]^\omega = 0$ holds in every algebra of $\mathcal{LS}_{\rm fin}$. Remark that $[[[x,y],[x,xy]],y]^\omega = 0$ in $\mathcal{LS}_{\rm fin}$ as well. Conversely, let $\mathcal{P}$ be the pseudovariety defined by $[x,y]^\omega = 0$. Then $M_2(F)$ does not belong to $\mathcal{P}$ because we have $[e_{12}, e_{21}]^\omega = (e_{11} - e_{22})^\omega = e_{11} + e_{22} \neq 0$. By Proposition 2, all algebras from $\mathcal{P}$ are Lie solvable that is $\mathcal{P} = \mathcal{LS}_{\rm fin}$.

2) Let F be a field, char$F = 2$. The pseudovariety $\mathcal{LS}_{\rm fin}$ does not contain $M_3(F)$. Hence, every semisimple algebra from $\mathcal{LS}_{\rm fin}$ satisfies

$$[[[x,y],[x,xy]],y] = 0$$

because both fields and algebras of 2×2 matrices satisfy it. Hence,

$$[[[x,y],[x,xy]],y]$$

is nilpotent in every algebra of $\mathcal{LS}_{fin}$. Therefore, $\mathcal{LS}_{\rm fin}$ satisfies the pseudoidentity $[[[x,y],[x,xy]],y]^\omega = 0$. To prove the sufficiency it remains to see that $[[[x,y],[x,xy]],y]^\omega \neq 0$ in $M_3(F)$. Indeed, for $x = e_{23} + e_{32}$ and $y = e_{12} + e_{21} + e_{23}$ we obtain $[[[x,y],[x,xy]],y] = e_{11} + e_{22}$. Hence, $[[[x,y],[x,xy]],y]^\omega = e_{11} + e_{22} \neq 0$.

The proof can be repeated with slight changes in the case of rings. □

R. A. Dean and T. Evans [3] proved that the direct sum of two semigroups satisfying nontrivial (reduced) identities satisfies some nontrivial (reduced) identity as well. Therefore all finite algebras satisfying (adjoint) semigroup identities form a pseudovariety. This is true also in the case of reduced semigroup identities.

Theorem 20. *The pseudovariety $\mathcal{RSI}_{\rm fin}$ of all finite algebras (rings) satisfying reduced semigroup identities is defined by the pseudoidentity $[x, y^\omega] = 0$.*

Proof. By Theorems 11 and 12, if an algebra satisfies a reduced semigroup identity then it satisfies the identity $[x, y^n] = 0$ for some n. Hence, the algebra satisfies the pseudoidentity $[x, y^\omega] = 0$. Moreover, for a finite algebra there exists a positive integer n such that $x^\omega = x^n$ for every element x of the algebra. Then, this algebra satisfies a reduced semigroup identity $[x, y^n] = 0$ whenever it satisfies $[x, y^\omega] = 0$. □

It was showed by V. Latyshev [10] that every permutative semigroup satisfies an permutative identity of the special kind

$$x_1 \cdots x_n x y x_{n+1} \cdots x_{2n} = x_1 \cdots x_n y x x_{n+1} \cdots x_{2n}.$$

Therefore all finite permutative algebras form a pseudovariety.

The following result was proved by the author [18].

Theorem 21. *The pseudovariety $\mathcal{P}_{\text{fin}}$ of all finite permutative algebras over a finite field is defined by the pseudoidentity $x^\omega[x,y]y^\omega = 0$.*

Proof. Every permutative semigroup satisfies an identity

$$x_1 \cdots x_n xy x_{n+1} \cdots x_{2n} = x_1 \cdots x_n yx x_{n+1} \cdots x_{2n}.$$

Therefore every finite permutative algebra satisfies the pseudoidentity

$$x^\omega[x,y]y^\omega = 0.$$

Now let us assume that A satisfies this pseudoidentity. Then all finite algebras from the variety var$A$ satisfy it. It is easy to prove that none of the forbidden algebras from Theorem 15 satisfies it. By Theorem 15 the variety var$A$ is permutative. □

From Theorems 19, 20, 21 it follows unexpected but easy corollary.

Corollary 4. *A finite algebra is Lie solvable (is permutative, satisfies a reduced semigroup identity) if and only if each of its 2-generated subalgebras satisfies this property.*

We collect few open questions of this section.

Problem 7.

1) *Find a basis of the pseudoidentities for the pseudovariety of all permutative finite rings.*

2) *Find a basis of the pseudoidentities for the pseudovariety of all finite algebras* (*rings*) *satisfying a semigroup identities.*

3) *Find a basis of the pseudoidentities for the pseudovariety of all finite algebras* (*rings*) *satisfying an adjoint semigroup identities.*

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REFERENCES

[1] J. Almeida. Finite Semigroups and Universal Algebra. Series in Algebra 3. Singapore, World Scientific, 1994.

[2] A. Ya. Belov. Local finite basis property and local representability for varieties of associative rings. *Dokl. Akad. Nauk* **432**, *6* (2010), 727–731 (in Russian); English translation in: *Dokl. Math.* **81**, *3* (2010) 458–461.

[3] R. A. Dean, T. Evans. A remark on varieties of lattices and semigroups. *Proc. Am. Math. Soc.* **21** (1969), 394–396.

[4] The Dniester Notebook. Unsolved Problems in Ring Theory. 4th edn, Institute of Mathematics SO RAN, Novosibirsk, 1993 (in Russian). English translation: Non-associative algebra and its applications. Proceedings of the 5th International Conference, Oaxtepec, July 27–August 2, 2003 (Eds L. Sabinin, L. Sbitneva, I. Shestakov) Lecture Notes in Pure and Applied Mathematics, vol. **246**, 2006.

[5] O. B. Finogenova. Varieties of associative algebras satisfying Engel identities. *Algebra Logika* **43**, *4* (2004), 482–505 (in Russian); English translation in: *Algebra Logic* **43**, *4* (2004), 271–284.

[6] O. B. Finogenova. Almost Lie nilpotent varieties of associative algebras. `arXiv:1108.5670v3 [math.RA]`

[7] I. Z. Golubchik, A. V. Mikhalev. On varieties of algebras with semigroup identity. *Vestn. Mosk. Univ., Ser. I 2* (1982), 8–11 (in Russian); English translation in: *Mosc. Univ. Math. Bull.* **37**, *2* (1982), 5–9.

[8] A. R. Kemer. Nonmatrix varieties. *Algebra Logika* **19** (1980), 255–283 (in Russian); English translation in: *Algebra Logic* 19 (1981), 157–178.

[9] H. Komatsu, H. Tominaga. Chacron's condition and Commutativity theorems. *Math. J. Okayama Univ.* **31** (1989), 101–120.

[10] V. N. Latyshev. On the Specht property of certain varieties of associative algebras. *Algebra Logika* **8** (1969), 660–673 (in Russian); English translation in: *Algebra Logic* **8** (1969), 374–382.

[11] I. V. L'vov. On varieties of associative rings. II. *Algebra Logika* **12** (1973), 667–688 (in Russian); English translation in: *Algebra Logic* **12** (1973), 381–393.

[12] Yu. N. Mal'tsev. Varieties of associative algebras. *Algebra Logika* **15** (1976), 579–584 (in Russian); English translation in: *Algebra Logic* **15** (1976), 361-364.

[13] Yu. N. Mal'tsev. Just non commutative varieties of associative rings. *Sib. Mat. Zh.* **17** (1976), 1086–10964 (in Russian); English translation in: *Sib. Math. J.* **17** (1977), 803–811.

[14] Yu. N. Mal'tsev. Some examples of varieties of associative rings. *Algebra Logika* **19** (1980), 669–675 (in Russian); English translation in: *Algebra Logic* **19** (1981), 433–437.

[15] Yu. N. Mal'tsev. Almost Engel locally finite varieties of associative rings. *Izv. Vyssh. Uchebn. Zaved., Mat. 11(246)* (1982), 41–42 (in Russian); English translation in: *Sov. Math.* **26** (1982), 50–52.

[16] Yu. N. Mal'tsev. Just non commutative varieties of operator algebras and rings with some conditions on nilpotent elements. *Tamkang J. Math.* **27**, *1* (1996), 59-65.

[17] A. N. Oleksenko. Almost commutative nilpotent of index 4 varieties of rings. *Izv. Altaj. Gos. Univ., Ser. Mat. Inform. Fiz. 1(19)* (2001), 30–33 (in Russian).

[18] O. B. Paison. Permutative pseudovarieties of associative algebras over a finite field. *Izv. Vyssh. Uchebn. Zaved., Mat. 1(392)* (1995), 71–79 (in Russian); English translation in: *Russ. Math.* **39**, *1* (1995), 65–73.

[19] O. B. Paison. Indicator characterisations of certain properties of associative ring varieties. Ph.D. Thesis, Ural State University, Ekaterinburg, 1998 (in Russian).

[20] Yu. P. Razmyslov. Identities of Algebras and their Representations. Nauka, Moscow, 1989 (in Russian); English t translation in: Translations of Mathematical Monographs vol. **138** Providence, RI, AMS, 1994.

[21] J. Reiterman. The Birkhoff theorem for finite algebras. *Algebra Univers.* **14** (1982), 1–10.

[22] D. M. Riley, M.C. Wilson. Associative algebras satisfying a semigroup identity. *Glasg. Math. J.* **41**, *3* (1999), 453–462.

[23] D. M. Riley, M.C. Wilson. Associative rings satisfying the Engel condition. *Proc. Amer. Math. Soc.* **127**, *4* (1999), 973–976.

[24] L. M. Samoilov. A note on trinomial identities in associative algebras. *Mat. Zametki* **65**, *2* (1999), 254–260 (in Russian); English translation in: *Math. Notes* **65**, *2* (1999), 208–213.

[25] M. V. Volkov. Lattices of varieties of algebras. *Mat. Sb., N. Ser.* **109(151)**, *1(5)* (1979), 60–79 (in Russian); English translation in: *Math. USSR, Sb.* **37** (1980), 53–69.

[26] E. N. Zakharova. Almost commutative nilpotent varieties of algebras *Izv. Akad. Nauk Mold. SSR, Ser. Fiz.-Tekh. Mat. Nauk 1* (1982), 3-10 (in Russian).

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