TO THE ANALYTICAL THEORY OF ALGEBRAIC EQUATIONS

P. Turán (Budapest)

1. This paper contains a modest contribution to the subject in the title, to which Prof. L. Čakalov added many important contributions. Some results of a lecture* will be detailed in which the necessity of the study of polynomials *in their Hermite-development* was systematically discussed, i. e. in the form

$$f(z)=\sum_{\nu=0}^{n} a_\nu H_\nu(z) \tag{1.1}$$

where $H_\nu(z)$, the νth Hermite-polynomial is, as usual, defined by

$$e^{-z^2} H_\nu(z)=(-1)^\nu (e^{-z^2})^{(\nu)} \tag{1.2}$$

$$\nu=0, 1, \ldots$$

Many signs show that in the questions of reality of zeros or giving strips for the zeros the Hermite-development is a much more suitable tool than the Taylor-development

$$f(z)=\sum_{\nu=0}^{\infty} b_\nu z^\nu. \tag{1.3}$$

The significance of all results in this direction would be greatly enhanced if the Hermite-coefficients of Riemann's Ξ-function could be given in a simple closed form. What is easy to show is that this expansion has the form**

$$\Xi(t)\sim\sum_{\nu=0}^{\infty}(-1)^\nu d_{2\nu} H_{2\nu}(t) \tag{1.4}$$

$$d_{2\nu}>0, \quad \nu=0, 1, \ldots$$

---

* Turan [1]. The results announced in this lecture were found in 1938–39.

** For the $d_{2\nu}$'s the following explicit representation can be given

$$d_{2\nu}=\frac{1}{2^{2\nu-2}(2\nu)!}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4}}\, t^{2\nu} \left\{\sum_{m=1}^{\infty}\left(2m^{4}\pi e^{\frac{9}{2}t} - 3m^{2}e^{\frac{5}{2}t}\right) e^{-m^2\pi e^{2t}}\right\} dt.$$

In lack of such forms we are not aiming best-possible results, rather we confine ourselves to obtain characteristical results with possibly short proofs.

The first result we are going to prove is

Theorem I. If $f(z)=\sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ with arbitrary complex coefficients and

$$\max_{\nu=0,1,\ldots,n-1} |c_{2\nu}| = M,$$

then all zeros of $f(z)$ lie in the strip

$$|\operatorname{Im} z| \leqq \frac{1}{2}\left(1+\frac{5}{\sqrt{2n-1}}\cdot\frac{M}{|c_{2n}|}\right). \tag{1.5}$$

We have to compare this theorem to the other one* according which all zeros of

$$g(z)=\sum_{\nu=0}^{n} a_\nu H_\nu(z)$$

with

$$\max_{\nu=0,1,\ldots,n-1} |a_\nu| = M_1$$

lie in the strip

$$|\operatorname{Im} z| \leqq \frac{1}{2}\left(1+\frac{M}{|a_n|}\right). \tag{1.6}$$

The constant $\frac{1}{2}$ in (1.6) is best-possible.

2. For the proof we need a simple inequality concerning Hermite-polynomials. We consider for $m \geqq 2$ the quotient

$$\frac{H_{m-2}(z)}{H_m(z)}.$$

Since the $x_{\nu m}$ — zeros of $H_m(z)$ are simple,

$$\frac{H_{m-2}(z)}{H_m(z)}=\sum_{j=1}^{m}\frac{H_{m-2}(x_{jm})}{H_m'(x_{jm})}\cdot\frac{1}{z-x_{jm}}$$

As well-known**

---

* Turan [1] p. 283 and [2]. By a more careful estimation the strip (1.5) can be replaced by $|\operatorname{Im} z| \leqq \frac{\lambda}{\sqrt{n}}\left(1+\frac{M}{|c_{2n}|}\right)$ with a numerical λ-constant.

** This follows easily from the generator-function

$$\sum_{\nu=0}^{\infty}\frac{H_\nu(z)}{\nu!}w^\nu=e^{2zw-w^2}$$

easily after differentiation after z.

(2.1) $$H'_m(z)=2mH_{m-1}(z),$$
i. e.

$$\frac{H_{m-2}(z)}{H_m(z)}=\frac{1}{2m}\sum_{j=1}^{m}\frac{H_{m-2}(x_{jm})}{H_{m-1}(x_{jm})}\frac{1}{z-x_{jm}}$$

Using the recursion-formula*

(2.2) $$H_m(z)=2zH_{m-1}(z)-2(m-1)H_{m-2}(z),$$

we get

$$2x_{jm}H_{m-1}(x_{jm})=2(m-1)H_{m-2}(x_{jm})$$

i. e.

(2.3) $$\frac{H_{m-2}(z)}{H_m(z)}=\frac{1}{2m(m-1)}\sum_{j=1}^{m}\frac{x_{jm}}{z-x_{jm}}$$

Since** for $j=1, 2, \ldots, m$

(2.4) $$-\sqrt{2m+1}\leqq x_{jm}\leqq\sqrt{2m+1},$$

we have

(2.5) $$\left|\frac{H_{m-2}(z)}{H_m(z)}\right|\leqq\frac{\sqrt{2m+1}}{2(m-1)}\frac{1}{|\operatorname{Im} z|}.$$

This is the required inequality. For $m=2n$ this gives

(2.6) $$\left|\frac{H_{2n-2}(z)}{H_{2n}(z)}\right|\leqq\frac{\sqrt{4n+1}}{2(2n-1)}\cdot\frac{1}{|\operatorname{Im} z|},$$

further for $1\leqq l\leqq n-2$

(2.7) $$\left|\frac{H_{2l}(z)}{H_{2n}(z)}\right|=\left|\frac{H_{2l}(z)}{H_{2l+2}(z)}\right|\left|\frac{H_{2l+2}(z)}{H_{2l+4}(z)}\right|\cdots\left|\frac{H_{2n-2}(z)}{H_{2n}(z)}\right|\leqq\left(\frac{1}{2|\operatorname{Im} z|}\right)\left(\frac{1}{2|\operatorname{Im} z|}\right)\cdots\left(\frac{1}{2|\operatorname{Im} z|}\right)\times$$
$$\times\left(\frac{\sqrt{4n+1}}{2(2n-1)}\cdot\frac{1}{|\operatorname{Im} z|}\right)=\left(\frac{1}{2|\operatorname{Im} z|}\right)^{n-l}\cdot\frac{\sqrt{4n+1}}{2n-1}$$

and*** for $l=0$

---

* This can be easily verified, since differentiation of (1.2) gives

$$H'_{m-1}(z)=2zH_{m-1}(z)-H_m(z)$$

and then (2.1) gives (2.2).

** See e. g. G. Szegö [3], p. 125.

*** Using (2.6) with $n=1, 2, \ldots, N$ and multiplying we get the inequality

$$|H_{2N}(z)|\geqq\frac{2^N.1.3.5\ldots(2N-1)}{\sqrt{1.5.9\ldots(4N+1)}}\cdot|\operatorname{Im} z|^N,$$

valid for $N=1, 2, \ldots$ and all complex z-values. This method of estimation can be greatly improved if necessary.

$$\left|\frac{H_0(z)}{H_{2n}(z)}\right| = \left|\frac{H_0(z)}{H_2(z)}\right| \cdot \left|\frac{H_2(z)}{H_{2n}(z)}\right| \geqq \tag{2.8}$$

$$\leqq \sqrt{5}\left(\frac{1}{2\,|\mathrm{Im}\, z|}\right)^n \frac{\sqrt{4n+1}}{2n-1}$$

Thus we obtain

$$|f(z)| \geqq |H_{2n}(z)| \left(|c_{2n}| - M\sum_{l=0}^{n-1}\left|\frac{H_{2l}(z)}{H_{2n}(z)}\right|\right) >$$

$$> |H_{2n}(z)| \left(|c_{2n}| - \sqrt{5}\,\frac{\sqrt{4n+1}}{2n-1}\, M \sum_{l=0}^{n-1}\left(\frac{1}{2\,|\mathrm{Im}\, z|}\right)^{n-l}\right).$$

Suppose $|\mathrm{Im}\, z| > \frac{1}{2}$; then

$$|f(z)| > |H_{2n}(z)| \left(|c_{2n}| - M\sqrt{5}\,\frac{\sqrt{4n+1}}{2n-1}\left(\frac{1}{2\,|\mathrm{Im}\, z| - 1}\right)\right).$$

Since for $n \geqq 1$ we have

$$4n+1 \leqq 5(2n-1),$$

we get

$$|f(z)| > |H_{2n}(z)| \left(|c_{2n}| - M\,\frac{5}{\sqrt{2n-1}} \cdot \frac{1}{2\,|\mathrm{Im}\, z| - 1}\right) > 0,$$

if*

$$|\mathrm{Im}\, z| > \frac{1}{2}\left\{1 + \frac{5}{\sqrt{2n-1}} \cdot \frac{M}{|c_{2n}|}\right\}. \qquad \text{Q. e. d.}$$

3. Owing to the possible application to Riemann's Ξ-function it is of interest to replace strips by equilateral hyperbolas where the real axis is an asymptota. It holds the

Theorem II. *If* $z = x + iy$ *and* $g(z) = \sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ *with* $\max_{\nu=0,1,\ldots,n-1} |c_{2\nu}| = M$, *then all zeros of* $g(z)$ *lie in the hyperbola*

$$|xy| \leqq \frac{5}{4}\left(1 + \frac{M}{|c_{2n}|}\right).$$

For the proof we start from the identity (2.3). Taking in account the symmetry of the $x_{j,2\nu}$—zeros this gives

$$\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)} = \frac{1}{2\nu(2\nu-1)} \sum_{j=1}^{\nu} \frac{x_{j,2\nu}^2}{z^2 - x_{j,2\nu}^2},$$

---

* Since (2.1) and (2.2) *characterize* the Hermite-polynomials, Theorem I. expresses a property *of the Hermite-expansion*. This was *not* the case with (1.5), since it uses *only* (2.1). One can prove however easily that property (2.1) *together with the orthogonality-property* along the real axis characterizes again the Hermite-polynomials „essentially".

and from (2.4)

$$\left|\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)}\right| \leq \frac{4\nu+1}{2\nu(2\nu-1)}\sum_{j=1}^{\nu}\frac{1}{|z^2-x^2_{j,2\nu}|}.$$

Since

$$|z^2-x^2_{j,2\nu}| = |x^2-y^2-x^2_{j,2\nu}+2xyi| \geq 2|xy|,$$

we get for $\nu \geq 1$

$$\left|\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)}\right| \leq \frac{4\nu+1}{4(2\nu-1)}\frac{1}{|xy|} \leq \frac{5}{4}\frac{1}{|xy|}$$

and thus for $k=0, 1, \ldots, n-1$

$$\left|\frac{H_{2k}(z)}{H_{2n}(z)}\right| \leq \left(\frac{5}{4}\cdot\frac{1}{|xy|}\right)^{n-k}.$$

Hence again for $|xy| > \frac{5}{4}$

$$g(z) \geq |H_n(z)| \left\{|c_{2n}| - \frac{M}{\frac{4}{5}|xy|-1}\right\},$$

which proves the theorem.

4. Next we turn to the proof of the following

Theorem III. *If the coefficients of*

$$G(z)=\sum_{\nu=0}^{n} c_\nu H_\nu(z)$$

are real and

$$(4.1)^{*} \qquad \sum_{\nu=0}^{n-2} 2^\nu \nu!\, c_\nu^2 < 2^n(n-1)!\, c_n^2$$

is fulfilled, then all zeros of $G(z)$ are real and simple.

For the proof of this theorem we shall need a simple formula. We start from the well-known formula of Christoffel-Darboux*

$$(4.2) \qquad \sum_{m=0}^{n-1}\frac{H_m(x)H_m(y)}{2^m.m!} = \frac{1}{2^n(n-1)!}\cdot\frac{H_n(x)H_{n-1}(y)-H_n(y)H_{n-1}(x)}{x-y}.$$

We shall denote by

$$(4.3) \qquad x_1 > x_2 > \cdots > x_{n-1}$$

the zeros of $H_{n-1}(z)$. For an arbitrary integer l between 1 and $n-1$ putting $x=x_l$ into (4.2) we get

$$\sum_{m=0}^{n-2}\frac{H_m(x_l)}{2^m.m!}H_m(y) = \frac{1}{2^n(n-1)!}H_n(x_l)\frac{H_{n-1}(y)}{x_l-y}.$$

* See e. g. G. Szegő [3]. p. 102.

If $y \to x_l$, this gives

$$(4.4)\qquad \sum_{m=0}^{n-2} \frac{H_m^2(x_l)}{2^m \cdot m!} = -\frac{1}{2^n(n-1)!} H_n(x_l) H'_{n-1}(x_l).$$

But, as it follows from footnote(*), p. 125 for $x = x_l$,

$$H'_{n-1}(x_l) = -H_n(x_l).$$

Putting it into (4.4) we get for $l = 1, 2, \quad n-1$

$$(4.5)\qquad \sum_{m=0}^{n-2} \frac{H_m^2(x_l)}{2^m \cdot m!} = \frac{1}{2^n(n-1)!} H_n^2(x_l).$$

This is the required formula.

5. We shall prove Theorem III. by showing that the zeros of $H_{n-1}(z)$ separate in our case the zeros of $G(z)$. Without loss of generality we may suppose the leading coefficient c_n in $G(z)$ is positive. Writing

$$x_0 = + \qquad x_n = -\infty$$

we shall prove

$$(5.1)\qquad \operatorname{sign} G(x_l) = (-1)^l,$$

$$l = 0, 1, \ldots, n,$$

Since

$$H_n(x) = 2^n x^n +$$

this is for $l = 0$ and $l = n$ trivial. In order to prove it for $l = 1, 2, \quad .,$ $n-1$, we remark first, that as well known* the zeros of $H_{n-1}(z)$ separate those of $H_n(z)$,

$$\operatorname{sign} H_n(x_l) = (-1)^l, \qquad l \quad 0, 1, \ldots, n.$$

Hence

$$(-1)^l c_n H_n(x_l) = c_n \mid H_n(x_l)$$

and thus for $l = 1, 2, \ldots, n-1$ we have

$$(-1)^l G(x_l) = c_n | H_n(x_l) + \sum_{\nu=0}^{n-2} (-1)^l c_\nu H_\nu(x_l).$$

From this we obtain

$$(-1)^l G(x_l) \geqq | c_n | | H_n(x_l) | - \sum_{\nu=0}^{n-2} | c_\nu \quad H_\nu(x_l) =$$

$$= c_n | H_n(x_l) - \sum_{\nu=0}^{n-2} 2^{\frac{\nu}{2}} \sqrt{\nu!} \quad c_\nu \cdot \frac{| H_\nu(x_l)}{2^{\frac{\nu}{2}} \sqrt{\nu!}}.$$

Cauchy's inequality gives

---

* See e. g. G. Szegő [3], p. 45.

$$(-1)^l G(x_l) > |c_n H_n(x_l)| - \sqrt{\left(\sum_{\nu=0}^{n-2} 2^{\nu}.\nu!\, c_\nu^2\right)\left(\sum_{\nu=0}^{n-2} \frac{H_\nu^2(x_l)}{2^{\nu}.\nu!}\right)}$$

and using (4.5)

$$(-1)^l G(x_l) > |H_n(x_l)| \left\{ |c_n| - \frac{1}{2^{\frac{n}{2}} \sqrt{(n-1)!}} \sqrt{\sum_{\nu=0}^{n-2} 2^{\nu}.\nu!\, c_\nu^2} \right\} > 0$$

indeed owing to (4.1). Since $x_1, x_2, \ldots, x_{n-1}$ are all simple zeros, also the simplicity of the zeros of $G(z)$ follows.

One can obtain of course a similar theorem for the reality of all zeros of $G_1(z) = \sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ with real $c_{2\nu}$'s. We shall do not go into details of its proof.

6. The condition (4.1) of the previous theorem is obviously fulfilled if the coefficients c_ν do not decrease „too quickly“. Now we are going to prove as a counterpart of Theorem III that the same conclusion holds if the coefficients decrease sufficiently quickly. More exactly we shall prove the

Theorem IV. *If $f(z)$ has the form*

(6.1) $$f(z) = \sum_{\nu=0}^{n} (-1)^{\nu} c_{2\nu} H_{2\nu}(z)$$

with positive $c_{2\nu}$'s and for $\nu = 1, 2, \ldots, n-1$ we have

(6.2) $$c_{2\nu}^2 > 4 c_{2\nu-2}\, c_{2\nu+2},$$

then all zeros of $f(z)$ are real.

7. For the proof of this theorem we need two lemmas.

Lemma I. If the coefficients $c_{2\nu}$ of

$$F(z) = \sum_{\nu=0}^{n} (-1)^{\nu} c_{2\nu}\, z^{2\nu}$$

satisfie the condition (6.2), then all zeros of $F(z)=0$ are real and simple.

Proof: Let

(7.1) $$\xi_j = \sqrt{\frac{c_{2j}}{2c_{2j+2}}}$$

$$j = 0, 1, \ldots, n-1.$$

From (6.2) it follows evidently that

(7.2) $$0 < \xi_0 < \xi_1 < \cdots < \xi_{n-1}.$$

We fix any of our j's with $1 \leq j < n-1$ and consider $F(\xi_j)$. Owing to (6.2) we have for $\nu < j$

$$\sqrt{\frac{c_{2\nu}}{c_{2\nu+2}}} < \sqrt{\frac{c_{2j-2}}{c_{2j}}} < \xi_j$$

i. e.

(7.3) $$c_0 < c_2\xi_j^2 < c_4\xi_j^4 < \cdots < c_{2j}\xi_j^{2j}$$

and for $\mu > j$

$$\xi_j < \sqrt{\frac{c_{2j}}{c_{2j+2}}} < \sqrt{\frac{c_{2\mu}}{c_{2\mu+2}}},$$

i. e.

(7.4) $$c_{2j}\xi_j^{2j} > c_{2j+2}\xi_j^{2j+2} > \cdots > c_{2n}\xi_j^{2n}.$$

Hence writing

$$(-1)^j F(\xi_j) = c_{2j}\xi_j^{2j} - (c_{2j-2}\xi_j^{2j-2} - c_{2j-4}\xi_j^{2j-4} + \cdots) -$$
$$-(c_{2j+2}\xi_j^{2j+2} - c_{2j+4}\xi_j^{2j+4} + \cdots),$$

we may observe that the terms in the brackets decrease monotonically e.

$$\frac{(-1)^j}{\xi_j^{2j-2}} F(\xi_j) \geq c_{2j}\xi_j^2 - c_{2j-2} - c_{2j+2}\xi_j^4 =$$
$$= \frac{c_{2j}^2 - 4c_{2j-2}\,c_{2j+2}}{4c_{2j+2}} > 0.$$

Since $F(0) > 0$ and $(-1)^n F(+\infty) > 0$, we obtained that each of the intervals

$$(0, \xi_1), (\xi_1, \xi_2), \cdots, (\xi_{n-2}, \xi_{n-1}), (\xi_{n-1}, +\infty)$$

contain at least one zero and owing to the evenness of $F(z)$ the same holds for the intervals

$$(-\infty, -\xi_{n-1}), \ldots, (-\xi_2, -\xi_1), (-\xi_1, 0).$$

Since $F(z)$ is of $2n$th degree, Lemma I. is proved.

Lemma II. If the polynomial

$$\Phi(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

has only real zeros, then all zeros of

$$\varphi(z) = \sum_{\nu=0}^{n} a_\nu H_\nu(z)$$

are also real*.

Theorem IV follows evidently from lemmata I and II. A trivial limit-process shows that Theorem IV. holds under condition (6.2) also for infinite Hermite-developments.

* This was stated in my quoted lecture. Quite recently I observed that it was stated, as an incoherent remark, in a paper of G. Pólya [4], in part p. 242, though with another normalisation of $H_n(x)$. Since the necessary changes are obvious, we omit the proof.

8. The condition (4.1) is, as easy to see, certainly fulfilled if $f(z)$ has the form (6.1) and

$$\frac{c_{2\nu+2}}{c_{2\nu}} > \frac{1}{4} \qquad \nu=0, 1, \ldots, n-1 \tag{8.1}$$

In this case we shall prove more.

Theorem V. If $f(z)=\sum_{\nu=0}^{n}(-1)^{\nu}c_{2\nu}H_{2\nu}(z)$ with positive $c_{2\nu}$'s and for an integer $1 \leq k \leq n$ we have

$$\frac{c_2}{c_0} > \frac{1}{4}, \quad \frac{c_4}{c_2} > \frac{1}{4}, \ldots, \frac{c_{2k}}{c_{2k-2}} > \frac{1}{4}, \tag{8.2}$$

then $f(z)$ has at least $2k$ real zeros with odd multiplicities.

For the proof we shall use the well-known formula*

$$z^m = \frac{m!}{2^m} \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{1}{r!\,(m-2r)!} H_{m-2r}(z) \qquad m=0, 1, \tag{8.3}$$

We form the integrals for $\nu=0, 1, \ldots, k$

$$J_{2\nu} = \int_0^{\infty} e^{-x^2} x^{2\nu} f(x) dx. \tag{8.4}$$

Using (8.3) we get

$$J_{2\nu} = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} x^{2\nu} f(x)\, dx =$$

$$= \frac{(2\nu)!}{2^{2\nu+1}} \sum_{r=0}^{\nu} \frac{1}{r!\,(2\nu-2r)!} \int_{-\infty}^{\infty} e^{-x^2} H_{2\nu-2r}(x) f(x) dx.$$

Since**

$$\int_{-\infty}^{\infty} e^{-x^2} H_{\mu}(x) H_{\nu}(x) dx = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 2^{\mu}\mu!\sqrt{\pi} & \text{if } \mu=\nu, \end{cases}$$

For the sake of completeness we reproduce a proof of (8.3). Using the generator-function in footnote(*) p. 124 we have

$$\sum_{m=0}^{\infty} \frac{(2z)^m}{m!} w^m = e^{2zw} = e^{w^2} \cdot e^{2zw-w^2} =$$

$$= \left(\sum_{j=0}^{\infty} \frac{w^{2j}}{j!}\right)\left(\sum_{l=0}^{\infty} \frac{H_l(z)}{l!} w^l\right),$$

from which (8.3) easily follows.

See e. g. G. Szegő [3], p. 101.

we obtain

$$J_{2\nu}=\frac{(2\nu)!}{2^{2\nu+1}}\sqrt{\pi}\sum_{r=0}^{\nu}(-1)^{\nu-r}c_{2\nu-2r}\frac{2^{2\nu-2r}}{r!}$$

or

$$(-1)^{\nu}J_{2\nu}=\frac{(2\nu)!\sqrt{\pi}}{2^{2\nu+1}}\left\{\frac{2^{2\nu}}{0!}c_{2\nu}-\frac{2^{2\nu-2}}{1!}c_{2\nu-2}+\cdots\right\}.$$

But the condition (2.2) together with the positivity of the $c_{2\nu}$'s results, that the terms in the bracket form a monotonically decreasing sequence with alternating signs and thus we get

$$(-1)^{\nu}J_{2\nu}>0$$

$$\nu=0,\ 1,\ \ldots,\ k$$

But then owing to a theorem of Fekete [5], and Fejér [6] $f(x)$ has in $(0,+\infty)$ at least k sign-changes and since $f(x)$ is an even function of x, it has on the real axis at least $2k$ sign=changes. Q. e. d.

9. Inserting in the integrals (8.4) e.g. a suitably chosen non-negative polynomial $h(x)$, Theorem V. can be greatly improved; a good choice might be suggested by the required simple form of the $d_{2\nu}$'s in (1.4). Having no such a guide we confine ourselves to the case $h(x)\equiv 1$. What is curious in Theorem V., is the fact, that apart from the reality of the coefficients and evenness of $f(z)$, restrictions are made only upon the first $k+1$ coefficients and by this the reality of at least $2k$ zeros of $f(z)$ is assured, *independently upon the other c_{2j}'s and upon n.* This reminds one to the theorems of Landau-Fejér-Montel-type, the most general of whose asserting that the polynomial

(9.1) $$e_0+e_1z+\cdots+e_pz^p+e_{p+1}z^{n_{p+1}}+\cdots+e_kz^{n_k}$$

with $e_p\neq 0$ and integer

$$p<n_{p+1}<\cdots<n_k$$

has at least p zeros in a circle

$$|z|<\varrho_0=\varrho_0(e_0,\ldots,e_p,\ k)$$

and suggest the existence of a $\varrho_1=\varrho_1(c_0,\ldots,c_p)$ such that (perhaps if $c_p\neq 0$) any

$$f(z)=\sum_{\nu=0}^{n}c_\nu H_\nu(z)$$

polynomial with $n\geq p$ has at least p zeros in the strip $|\mathrm{Im}z|\leq\varrho_1$. The somewhat weaker assertion about the existence of a $\varrho_2=\varrho_2(c_0,c_1,\ldots,c_p,n)$ such that with $c_p\neq 0$ any

$$f(z)=\sum_{\nu=0}^{n}c_\nu H_\nu(z)$$

polynomial with $n \geqq p$ has at least p zeros in the strip $|\operatorname{Im} z| \leq \varrho_2$, would follow from the above quoted theorem (9.1) of Montel with

$$\varrho_2 = \varrho_0 (c_0, \quad c_p, n),$$

if the following assertion is true. For any $A > 0$ the polynomial

$$F(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$$

has in the strip $|\operatorname{Im} z| \leq A$ at most as many zeros (counted with multiplicity) as

$$f(z) = \sum_{\nu=0}^{n} c_\nu H_\nu(z).$$

As I mentioned in my quoted congress-lecture, I can prove this at the present only for $A = 0$ and for such A's, for whose *all* zeros of $F(z)$ are contained in our strip $|\operatorname{Im} z| \leq A$.

Received 2. 2. 1957

REFERENCES

1. P. Turán. Sur l'algèbre fonctionelle, Compt. Rend. du Premier congrès des math. hongr, 1950.
2. P. Turán. Hermite-expansion and strips for zeros of polynomials, Archiv der Math. 5 (1954), 148—152.
3. G. Szegö. Orthogonal polynomials, 1939.
4. G. Pólya. Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins, Journ. für die reine und angew. Math. Bd. 145, 224—249.
5. M. Fekete. Sur une limite inférieure des changements de signe d'une fonction dans un intervalle. Compt. Rend. Ac. Sc. Raris, 158 (1914), 1555—1558.
6. L. Fejér. Nombre de changement de signe d'une fonction dans un intervalle et ses moments. Compt. Rend. Ac. Sc. Paris, 158 (1914), 1328—1331.

ВЪРХУ АНАЛИТИЧНАТА ТЕОРИЯ НА АЛГЕБРИЧНИТЕ УРАВНЕНИЯ

П. Туран (Будапеща)

РЕЗЮМЕ

Авторът разгледа в една своя лекция от 1950 [1] причините' поради които е желателно да се създаде аналитична теория на алгебричните уравнения, основана върху развития по полиномите на Ермит. Една такава причина бе това, че при въпросите за реалност на всички нули или при намирането на ивици, съдържащи всички нули, развитията по полиномите на Ермит се явяват по-подходящи, отколкото тейлоровите развития. Както бе отбелязано от автора, една такава теория би била особено интересна, ако $d_{2\nu}$ — коефициентите на ермитовото развитие на римановата Ξ-функция, което има вида

$$\Xi(t) \sim \sum_{\nu=0}^{\infty} (-1)^{\nu} d_{2\nu} H_{2\nu}(t), \quad d_{2\nu} > 0,$$

биха могли да бъдат изразени явно. В настоящата работа авторът дава подробни доказателства за някои от твърденията, дадени в поменатата лекция. Ако ν-тия ермитов полином $H_{\nu}(z)$ е дефиниран с

$$e^{-z^2} H_{\nu}(z) = (-1)^{\nu} (e^{-z^2})^{(\nu)},$$

той доказва следните теореми. Нека $z = x + iy$. Тогава имаме

Теорема I. Ако

$$f(z) \quad \sum_{\nu=0}^{n} a_{2\nu} H_{2\nu}(z)$$

и

$$\max_{\nu=0,1,\dots,n-1} \quad a_{2\nu} \quad = M,$$

тогава всички нули на $f(z)$ лежат в ивицата

$$y < \frac{\lambda}{\sqrt{n}} \left(1 + \frac{M}{c_{2n}}\right),$$

дето λ е числена константа, независеща от n.

Теорема II. Всички нули на $f(z)$ лежат също в областта

$$|xy| \leq \frac{3}{4}\left(1+\frac{M}{|c_{2n}|}\right).$$

Теорема III. Ако

$$F(z)=\sum_{\nu=0}^{\infty} b_\nu H_\nu(z),$$

дето b_ν са реални и

$$\sum_{\nu=0}^{n-2} 2^\nu \nu!\, b_\nu^2 < 2^n(n-1)!\, b_n^2,$$

всички нули на $F(z)$ са реални и прости.

Теорема IV. Ако

$$g(z)=\sum_{\nu=0}^{n} (-1)^\nu c_{2\nu} H_{2\nu}(z),$$

дето $c_{2\nu}$ са положителни и

(1) $$c_{2\nu}^2 > 4c_{2\nu-2}\, c_{2\nu+2}, \quad \nu = 1, 2, \ldots, n-1$$

всички нули на $g(z)$ са реални и прости.

Теорема V. Ако вместо (1) имаме при $1 \leq k \leq n$

$$\frac{c_2}{c_0} > \frac{1}{4}, \quad \frac{c_4}{c_2} > \frac{1}{4}, \ldots, \frac{c_{2k}}{c_{2k-2}} > \frac{1}{4},$$

$g(z)$ има поне $2k$ реални нули.

ОБ АНАЛИТИЧЕСКОЙ ТЕОРИИ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ

П. Туран (Будапешт)

РЕЗЮМЕ

Автор рассмотрел в одной своей лекции, прочитанной в 1950 г. [1], причины, делающие желательным найти аналитическую теорию алгебраических уравнений, основанной на эрмитовых развитиях полиномов. Одной такой причиной являлось то, что при рассмотрении вопросов о действительности всех нулей или о нахождении полос, содержащих все нули, развития по полиномам Эрмита являются повидимому более удобным орудием исследования, чем тейлоровое развитие. Автор тоже отметил, что такая теория имела бы особенный интерес, если удастся для функции Римана

$$\Xi(t) \sim \sum_{\nu=0}^{\infty} (-1)^{\nu} d_{2\nu} H_{2\nu}(t), \quad d_{2\nu} > 0,$$

выразить явно коэффициенты ее развития $d_{2\nu}$ по полиномам Эрмита. В этой работе он дает подробные доказательства некоторых теорем из упомянутой лекции. Определяя полиномы Эрмита $H_{\nu}(z)$ через

$$e^{-z^2} H_{\nu}(z) = (-1)^{\nu} (e^{-z^2})^{(\nu)},$$

автор доказывает следующие теоремы:

Пусть $z = x + iy$. Тогда имеет место

Теорема I. Если

$$f(z) = \sum_{\nu=0}^{n} a_{2\nu} H_{2\nu}(z)$$

и

$$\max_{\nu=0,1,\ldots,n-1} |a_{2\nu}| = M,$$

тогда все нули $f(z)$ лежат в полосе

$$|y| \leq \frac{\lambda}{\sqrt{n}} \left(1 + \frac{M}{c_{2n}}\right)$$

(λ постоянная, не зависящая от n).

Теорема II. Все нули $f(z)$ лежат тоже в области

$$|xy| \leq \frac{3}{4}\left(1+\frac{M}{|c_{2n}|}\right).$$

Теорема III. Если

$$F(z)=\sum_{\nu=0}^{n} b_\nu H_\nu(z),$$

где b_ν действительны и

$$\sum_{\nu=0}^{n-2} 2^\nu \nu!\,b_\nu^2 < 2^n (n-1)!\,b_n^2,$$

тогда все нули $F(z)$ действительны и просты.

Теорема IV. Если

$$g(z)=\sum_{\nu=0}^{n}(-1)^\nu c_{2\nu} H_{2\nu}(z),$$

$c_{2\nu}$ положительны и

(1) $$c_{2\nu}^2 > 4c_{2\nu-2}\,c_{2\nu+2}, \quad \nu=1,\ 2, \ldots,\ n-1,$$

тогда все нули $g(z)$ действительны и просты.

Теорема V. Если вместо (1) для $1 \leq k \leq n$ имеет место

$$\frac{c_2}{c_0} > \frac{1}{4},\ \frac{c_4}{c_2} > \frac{1}{4}, \ldots, \frac{c_{2k}}{c_{2k-2}} > \frac{1}{4},$$

тогда $g(z)$ имеет по крайней мере $2k$ действительных нулей.