БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ   ACADÉMIE BULGARE DES SCIENCES
ИЗВЕСТИЯ НА МАТЕМАТИЧЕСКИЯ ИНСТИТУТ
BULLETIN DE L'INSTITUT DE MATHÉMATIQUES
Том (Vol.) VIII

ON SOME INTERPOLATORY PROPERTIES OF LEGENDRE POLYNOMIALS III

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Introduction

1. In paper [8] we dealt with the interpolation of the 'type' (0, 1, 3), according to which we seek a polynomial $g(x)$ of degree $3n-1$ such that for a given system of distinct points $x_\nu (\nu=1, 2, \ldots, n)$ in $[-1,1]$ and for given values $\eta_\nu, \eta_\nu^*, \eta_\nu^{**}$ $(\nu=1, 2, \ldots, n)$ the conditions

$$g(x_\nu)=\eta_\nu,\ g'(x_\nu)=\eta_\nu^*,\ g'''(x_\nu)=\eta_\nu^{**},\ (\nu=1, 2, \ldots, n)$$

are satisfied. We proved that if n is odd and the x_ν form a symmetric system then the solution is not unique. But when n is even and x_ν are the zeros of $(1-x^2)P'_{n-1}(x)$ [$P_{n-1}(x)$ being the $(n-1)$ the Legendre polynomial with $P_{n-1}(1)=1$] then $g(x)$ is uniquely determined. In the latter case we have evaluated explicitly the fundamental polynomials of interpolation. Based on these explicit formulae we investigated in [9] the convergence of the interpolatory polynomials. Later on we solved in [10, 11] the same problems for the case (0, 1, 2, 4). Kis [6] has been able to solve the case (0, 1, 2, $\ldots$, $r-2, r$; $r\geqq 2$) when x_ν are the points $\exp i\frac{2\pi}{n}\nu$ $(\nu=1, 2,\ldots, n)$.

In this paper we solve the problem of (0, 1, 3) -interpolation when the x_ν are the zeros of $(1-x^2)P_{n-2}(x)$ where $P_{n-2}(x)$ is the $(n-1)$th Legendre polynomial with $P_{n-2}(1)=1$. Actually we seek a polynomial $R_n(x)$ of degree $\leq 3n-3$ such that for given values

$$\alpha_1, \alpha_2, \ldots, \alpha_n;\ \beta_2, \beta_3, \ldots \beta_{n-1};\ \gamma_1, \gamma_2, \ldots, \gamma_n$$

and

$$(1,1)\qquad -1=x_n<x_{n-1}<\ldots<x_2<x_1=1$$

which are the zeros of the polynomial

$$(1.2)\qquad (1-x^2)P_{n-2}(x),$$

$$(1.3)\qquad \begin{aligned} R_n(x_\nu)&=\alpha_\nu,\ \nu=1, 2,\ldots, n,\\ R'_n(x_\nu)&=\beta_\nu,\ \nu=2, 3,\ldots, n-1,\\ R'''_n(x_\nu)&=\gamma_\nu,\ \nu=1, 2,\ldots, n.\end{aligned}$$

Since the x_ν in (1.2) form symmetric system, it follows from theorem I in [8] that for n odd the solution is not unique. For n even, following exactly the same lines of proof as for theorem II in [8], it can be proved that $R_n(x)$ is uniquely determined by the values (1.3) and (1.1).[1]

The part A of this paper will be devoted to find explicitly the interpolatory polynomials when n is even.[2] In part B we shall discuss the convergence behavior of the polynomials obtained in A.

Part A

Explicit Representation of Interpolatory Polynomials

2. For $R_{2k}(x)$ we evidently have the form:

$$R_{2k}(x)=\sum_{\nu=1}^{n}\alpha_\nu u_\nu(x)+\sum_{\nu=2}^{n-1}\beta_\nu v_\nu(x)+\sum_{\nu=1}^{n}\gamma_\nu w_\nu(x) \tag{2.1}$$

where $u_\nu(x)$, $v_\nu(x)$, and $w_\nu(x)$ are the fundamental polynomials of (0, 1, 3)-interpolation. They are unique polynomials each of degree $3n-3$ determined by the following requirements:

For $\nu=1, 2, \ldots, n$,

$$u_\nu(x_j)=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j\neq\nu\\j=\nu\end{matrix}\ (j=1, 2,\ldots, n);\ u'_\nu(x_j)=0\ (j=2, 3, \ldots, n-1); \tag{2.2}$$

$$u'''_\nu(x_j)=0\,(j=1, 2,\ldots, n).$$

For $\nu=2, 3,\ldots, n-1$,

$$v_\nu(x_j)=0\ (j=1,2,\ldots,n);\ v'_\nu(x_j)=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j\neq\nu\\j=\nu\end{matrix}\ (j=2,3,\ldots,n-1); \tag{2.3}$$

$$v'''_\nu(x_j)=0\ (j=1, 2,\ldots, n).$$

For $\nu=1, 2,\ldots, n$,

$$w_\nu(x_j)=0\,(j=1, 2,\ldots, n);\ w'_\nu(x_j)=0\,(j=2, 3,\ldots, n-1); \tag{2.4}$$

$$w'''_\nu(x_j)=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j\neq\nu\\j=\nu\end{matrix}\,(j=1, 2,\ldots, n).$$

In §§ 3—5 we shall explicitly determine the polynomials $u_\nu(x)$, $v_\nu(x)$ and $w_\nu(x)$. To this we shall need the following results.

(a)

Let us denote

$$\omega(x)=(1-x^2)\,P^2_{n-2}(x) \tag{2.5}$$

which therefore gives

$$\omega(x_\nu)=0,\ \nu=1, 2, \ldots, n;\quad \omega'(x_\nu)=0,\ \nu=2, 3,\ldots n-1; \tag{2.6}$$

[1] c. f. Saxena and Sharma [8], theorem III, p. 347.

[2] Throughout this paper we shall take n to be even integer ≥ 4.

with the help of the differential equation

(2.7) $$(1-x^2)P''_{n-2}(x)-2xP'_{n-2}(x)+(n-1)(n-2)P_{n-2}(x)=0$$

satisfied by $P_{n-2}(x)$ we see that

(2.8) $$\omega'''(x_\nu)=0,\quad \nu=2,3,\ldots,n-1.$$

From (2.7) we have for n even

(2.9) $$P'_{n-2}(1)=\frac{1}{2}(n-1)(n-2)=-P'_{n-2}(-1),$$

(2.10) $$P''_{n-2}(1)=\frac{1}{8}n(n-1)(n-2)(n-3)=P''_{n-2}(-1),$$

(2.11) $$P'''_{n-2}(1)=\frac{1}{48}(n+1)n(n-1)(n-2)(n-3)(n-4)=-P'''_{n-2}(-1).$$

From (2.5) and (2.7) we can compute:

(2.12) $$\omega'(1)=-2=-\omega'(-1),$$

(2.13) $$\omega''(1)=-2-4(n-1)(n-2)=\omega''(-1),$$

(2.14) $$\omega'''(1)=-\frac{3}{2}(n-1)(n-2)[3(n-1)(n-2)+2]=-\omega'''(-1)$$

and

(2.15) $$\omega''(x_\nu)=2(1-x_\nu^2)P'^2_{n-2}(x_\nu),\quad \nu=2,3,\ldots,n-1.$$

(b)

We denote by

(2.16) $$\lambda_\nu(x)=\frac{1-x^2}{1-x_\nu^2}\cdot\frac{P_{n-2}(x)}{P'_{n-2}(x_\nu)(x-x_\nu)},\quad \nu=2,3,\ldots,n-1,$$

the fundamental polynomials (of degree $n-1$) of Lagrange interpolation based on our x_j-points for which

(2.17) $$\lambda_\nu(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \begin{matrix}j=\nu\\ j\neq\nu\end{matrix}\quad (j=1,2,\ldots,n;\ \nu=2,3,\ldots,n-1).$$

For these polynomials we have

(2.18) $$\lambda'_\nu(x_\nu)=-\frac{x_\nu}{1-x_\nu^2},\quad \lambda''_\nu(x_\nu)=-\frac{1}{3}\left[\frac{4}{(1-x_\nu^2)^2}+\frac{(n-1)(n-2)}{1-x_\nu^2}\right],$$

(2.19) $$\lambda'''_\nu(x_\nu)=-\frac{4x_\nu}{(1-x_\nu^2)^3},$$

and

(2.20) $$\lambda'_\nu(1)=-\frac{2}{(1-x_\nu)(1-x_\nu^2)P'_{n-2}(x_\nu)},$$

$$\lambda_\nu'(-1)=\frac{2}{(1+x_\nu)(1-x_\nu^2)P_{n-2}'(x_\nu)}$$

(c)

We further denote by[3]

$$r_\nu(x)=\frac{1-x_\nu^2}{1-x^2}\frac{P_{n-2}^2(x)}{(x-x_\nu)^2P_{n-2}'^2(x_\nu)},\quad \nu=2,3,\dots n-1, \tag{2.21}$$

the fundamental polynomials (of degree $2n-4$) of first kind of Hermite-Fejér interpolation based on our x_j-points satisfying the requirements:

$$r_\nu(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \begin{matrix}j=\nu\\j\neq\nu\end{matrix}\ (j=1,2,\dots,n);\quad r_\nu'(x_j)=0\ (j=2,\dots,n-1). \tag{2.22}$$

From (2.21) using (2.6) and (2.8) we have

$$r_\nu''(x_j)=\frac{\omega''(x_j)}{(1-x_\nu^2)P_{n-2}'^2(x_\nu)(x_j-x_\nu)^2}, \tag{2.23}$$

$$r_\nu''(x_\nu)=-\frac{2}{3}\left[\frac{(n-1)(n-2)}{1-x_\nu^2}+\frac{1}{(1-x_\nu^2)^2}\right]. \tag{2.24}$$

For these polynomials we shall also need

$$r_\nu'(1)=-\frac{2}{(1-x_\nu)^2(1-x_\nu^2)P_{n-2}'^2(x_\nu)}, \tag{2.25}$$

$$r_\nu'(-1)=\frac{2}{(1+x_\nu)^2(1-x_\nu^2)P_{n-2}'^2(x_\nu)}$$

and

$$r_\nu''(1)=\frac{1}{(1-x_\nu)^2}\left[\frac{\omega''(1)}{(1-x_\nu^2)P_{n-2}'^2(x_\nu)}+\frac{8}{(1-x_\nu)(1-x_\nu^2)P_{n-2}'^2(x_\nu)}\right],$$

(2.26)

$$r_\nu''(-1)=\frac{1}{(1+x_\nu)^2}\left[\frac{\omega''(-1)}{(1-x_\nu^2)P_{n-2}'^2(x_\nu)}+\frac{8}{(1+x_\nu)(1-x_\nu^2)P_{n-2}'^2(x_\nu)}\right].$$

(d)

The expression

$$\frac{(1-x^2)P_{n-2}'(x)-[(1-x_\nu^2)+x_\nu(x-x_\nu)]P_{n-2}'(x_\nu)\lambda_\nu(x)}{(x-x_\nu)^2}\equiv\frac{q_n(x)}{(x-x_\nu)^2} \tag{2.27}$$

is indeed a polynomial of degree $n-2$ and

$$\lim_{x=x_\nu}\frac{q_n(x)}{(x-x_\nu)^2}=-\frac{1}{3}\left[n(n-3)-\frac{5x_\nu^2}{1-x_\nu^2}\right]P_{n-2}'(x_\nu). \tag{2.28}$$

We also have

$$q_n'(1)=-\frac{(n-1)(n-2)}{(1-x_\nu)^2}+\frac{4}{(1-x_\nu)^3}-\frac{2}{(1-x_\nu)^2(1-x_\nu^2)}, \tag{2.29}$$

[3] f. Egervary and Turán [4].

$$(2.30)\qquad q_n'(-1) = -\frac{(n-1)(n-2)}{(1+x_\nu)^2} + \frac{4}{(1+x_\nu)^3} - \frac{2}{(1+x_\nu)^2(1-x_\nu^2)}.$$

(e)

The expression

$$(2.31)\qquad \frac{(1-x^2)^2 P_{n-2}'(x) - [(1-x_\nu^2)^2 + a_1(x-x_\nu)^2 + a_2(x-x_\nu)]\lambda_\nu(x) P_{n-2}'(x)}{(x-x_\nu)^3}$$

$$\equiv \frac{h_{n+1}}{(x-x_\nu)^3}$$

is a polynomial of degree $n-2$ when the values of a_1, a_2 are given to be:

$$(2.32)\qquad a_1 = -\frac{1}{3}[1+(1-x_\nu^2)(n-1)(n-2)],$$

$$(2.33)\qquad a_2 = -x_\nu(1-x_\nu^2).$$

We have

$$(2.34)\qquad \lim_{x=x_\nu} \frac{h_{n+1}(x)}{(x-x_\nu)^3} = \frac{1}{6}\left[\frac{d^3}{dx^3}\{h_{n+1}(x)\}\right]_{x=x_\nu}$$

$$= \frac{1}{6} x_\nu \left[(n-1)(n-2) - \frac{2}{1-x_\nu^2}\right] P_{n-2}'(x_\nu),$$

$$(2.35)\qquad h_{n+1}'(1) = (a_1+a_2+a_3)\lambda_\nu'(1) = -2\left[\frac{1-x_\nu}{3(1-x_\nu^2)} + \frac{(n-1)(n-2)(1-x_\nu)}{3} - 1\right],$$

$$(2.36)\qquad h_{n+1}'(-1) = (a_1-a_2+a_3)\lambda_\nu'(-1)$$

$$= -2\left[\frac{1+x_\nu}{3(1-x_\nu^2)} + \frac{(n-1)(n-2)(1+x_\nu)}{3} - 1\right].$$

3. Computation of $w_\nu(x)$ $(\nu=1, 2, \dots, n)$. The easiest to compute are the polynomials $w_\nu(x)$ in (2.1). Since these polynomials are unique, it is enough to verify the following expressions according to (2.4):

$$(3.1)\qquad w_1(x) = -\frac{1}{6}\omega(x)\left[\frac{1}{5n^2-15n+12}\int_{-1}^{x} P_{n-2}(t)\,dt + \frac{2}{(n-1)(n-2)(3n^2-9n+8)}\right],$$

$$(3.2)\qquad w_n(x) = -\frac{1}{6}\omega(x)\left[\frac{1}{5n^2-15n+12}\int_{-1}^{x} P_{n-2}(t)\,dt + \frac{2}{(n-1)(n-2)(3n^2-9n+8)}\right]$$

and for $\nu=2, 3, \ldots, n-1$,

$$(3.3)\qquad w_\nu(x)=\frac{\omega(x)}{6(1-x_\nu^2)P'^3_{n-2}(x_\nu)}\left[\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_1\int_{-1}^{x}P_{n-2}(t)dt-k_2\right]$$

where

$$(3.4)\qquad k_1=-\frac{(n-1)(n-2)(3n^2-9n+8)}{4(5n^2-15n+12)}\int_{-1}^{+1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt-\frac{x_\nu}{1-x_\nu^2}$$

$$+\frac{4}{5n^2-15n+12}\frac{x_\nu}{(1-x_\nu^2)^2},$$

$$(3.5)\qquad k_2=\frac{1}{2}\int_{-1}^{+1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+\frac{8}{(n-1)(n-2)(3n^2-9n+8)(1-x_\nu^2)^2}$$

$$-\frac{10}{(3n^2-9n+8)(1-x_\nu^2)}$$

To verify the expression for $w_\nu(x)$ $(\nu=2, 3, \ldots, n-1)$ in (3.3) we at once see that it is a polynomial of degree $3n-3$ and it obviously satisfies the first two conditions of (2.4) because of (2.6). To see that it also satisfies the third condition of (2.4) we differentiate (3.3) three times with respect to x and owing to (1.2), (2.6) and (2.8) we have

$$w_\nu'''(x_j)=0\quad(j\neq\nu,\ j=2, 3, \ldots, n-1)$$

while

$$w_\nu'''(x_\nu)=\frac{\omega''(x_\nu)}{2(1-x_\nu^2)P'^3_{n-2}(x_\nu)}\lim_{x=x_\nu}\frac{P_{n-2}(x)}{x-x_\nu}=1,$$

in consequence of (2.15). It remains to show that $w_\nu'''(\pm1)=0$. This gives us two linear equations to determine k_1 and k_2. Thus we have:

$$\omega'''(1)\left[\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_3\right]+3\omega''(1)\left[\frac{1}{1-x_\nu}+k_2\right]+3\omega'(1)\left[-\frac{1}{(1-x_\nu)^2}\right.$$

$$\left.+\frac{P'_{n-2}(1)}{(1-x_\nu)}+k_2P'_{n-2}(1)\right]=0,$$

and

$$k_3\omega'''(-1)+3\omega''(-1)\left[-\frac{1}{1+x_\nu}+k_2\right]$$

$$+3\omega'(-1)\left[-\frac{1}{(1+x_\nu)^2}-\frac{P'_{n-2}(-1)}{1+x_\nu}+k_2P'_{n-2}(-1)\right]=0.$$

From these equations owing to (2.13), (2.12), (2.11) and (2.9) we have the values for k_1 and k_2 as quoted in (3.4) and (3.5) respectively. In the same way we can verify the expressions for $w_1(x)$ and $w_n(x)$.

4. Computation of $v_\nu(x)$ $(\nu=2, 3, \ldots, n-1)$. As before we shall verify the following expression for $v_\nu(x)$ $(\nu=2, 3, \ldots, n-1)$ according to (2.3). That it is a polynomial of degree $3n-3$, follows from (2.24).

$$(4.1) \qquad v_\nu(x)=\frac{(1-x^2)P_{n-2}(x)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}r_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)}\left[\int_{-1}^{x}\frac{q_n(t)}{(t-x_\nu)^2}dt\right.$$

$$\left.+k_3\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}dt+k_4\int_{-1}^{x}P_{n-2}(t)\,dt+k_5\right]$$

where

$$(4.2) \qquad k_3=-\frac{8}{3(1-x_\nu^2)}-\frac{(n-4)(n+1)}{6}$$

$$(4.3) \qquad k_4=\frac{\omega'''(1)}{6(5n^2-15n+12)}\left[\int_{-1}^{+1}\frac{q_n(t)}{(t-x_\nu)^2}dt+k_3\int_{-1}^{+1}\frac{P_{n-2}(t)}{t-x_\nu}dt\right]$$

$$+\frac{4x_\nu}{3(1-x_\nu^2)^2}-\frac{(n-4)(n+1)}{3}\frac{x_\nu}{1-x_\nu^2}$$

$$+\frac{2x_\nu}{3(5n^2-15n+12)}\left[\frac{4(5+6x_\nu^2)}{(1-x_\nu^2)^3}+\frac{(n-4)(n+1)}{(1-x_\nu^2)^2}\right],$$

$$(4.4) \qquad k_5=-\frac{1}{2}\left[\int_{-1}^{1}\frac{q_n(t)}{(t-x_\nu)^2}dt+k_3\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}dt\right]-\frac{1}{9(1-x_\nu^2)}$$

$$-\frac{2(n-1)(n-2)}{\omega'''(1)(1-x_\nu^2)}\left(50+\frac{18}{1-x_\nu^2}\right)$$

$$+\frac{8}{\omega'''(1)(1-x_\nu^2)}\left[3+\frac{3}{1-x_\nu^2}-\frac{19}{(1-x_\nu^2)^2}\right],$$

and the $r_\nu(x)$ and $\frac{q_n(t)}{(t-x_\nu)^2}$ are as explained in (2.21) and (2.24) respectively.

We at once see in (4.1) that the first of the conditions (2.3) is obviously satisfied. For the second one, we have owing to (2.6), (1.2) and (2.22)

$$v'_\nu(x_j)=\frac{(1-x_j^2)P'_{n-2}(x_j)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}r_\nu(x_j).$$

Again from (2.22) it follows that $v'_\nu(x_j)=0$, $j\neq\nu$ $(j=2, 3, \ldots, n-1)$ and $v'_\nu(x_\nu)=1$.

To prove that the third of the conditions (2.3) is satisfied by the values k_3, k_4, and k_5 as quoted above we have on differentiating (4.1) and applying (2.6), (2.7), (2.8), (2.22), (2.24) for $j\neq\nu$

$$v_\nu'''(x_j)=\frac{3(1-x_j^2)P'_{n-2}(x_j)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}r_\nu''(x_j)-\frac{3\omega''(x_j)q_{n-2}(x_j)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)(x-x_j)^2}$$

$$=\frac{3(1-x_j^2)P'_{n-2}(x_j)\omega''(x_j)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)(x_j-x_\nu)^2}-\frac{3(1-x_j^2)P'_{n-2}(x_j)\omega''(x_j)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)(x_j-x_\nu)^2}=0.$$

From $v_\nu'''(x_\nu)=0$ we have

$$-\left[\frac{4x_\nu^2}{(1-x_\nu^2)}+(n^2-3n+6)\right]P'_{n-2}(x_\nu)+3(1-x_\nu^2)P'_{n-2}(x_\nu)r_\nu''(x_\nu)$$
$$-6\left[\lim_{x=x_\nu}\frac{q_n(x)}{(t-x_\nu)^2}+k_3\lim_{x=x_\nu}\frac{P_{n-2}(x)}{(x-x_\nu)}\right]=0. \tag{4.5}$$

Thus from (4.5), (2.24) and (2.28) we have for k_3 the value quoted in (4.2).

To determine the constants k_4 and k_5 in (4.1) we use the requirements $v_\nu'''(\pm 1)=0$. We therefore have the following two equations:

$$3[-4P'_{n-2}(1)-2]r_\nu'(1)-6r_\nu''(1)-\frac{\omega'''(1)}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}\left[\int_{-1}^{1}\frac{q_n(t)}{(t-x_\nu)^2}dt\right.$$
$$\left.k_3\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}dt+k_5\right]-\frac{3\omega''(1)}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}\left[\frac{k_3}{1-x_\nu}+k_4\right] \tag{4.6}$$
$$-\frac{3\omega'(1)}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}\left[-\frac{q_n'(1)}{(1-x_\nu)^2}+k_3\frac{P'_{n-2}(1)}{1-x_\nu}-\frac{k_3}{(1-x_\nu)^2}+k_4P'_{n-2}(1)\right]=0$$

and

$$3[4P'_{n-2}(-1)-2]r_\nu'(-1)+6r_\nu''(-1)-\frac{\omega'''(-1)k_5}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}$$
$$-\frac{3\omega''(-1)}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}\left[-\frac{k_3}{1+x_\nu}+k_4\right]-\frac{3\omega'(-1)}{(1-x_\nu^2)P'^2_{n-2}(x_\nu)}\left[\frac{q_n'(-1)}{(1+x_\nu)^2}\right. \tag{4.7}$$
$$\left.-k_3\frac{P'_{n-2}(-1)}{1+x_\nu}-\frac{k_3}{(1+x_\nu)^2}+k_4P'_{n-2}(-1)\right]=0.$$

On adding the above two equations and simplifying with the help of the formulae in § 2, we get the value of k_4 as given in (4.3) and on subtracting the two we get the value of k_5.

5. Computation of $u_\nu(x)$ $(\nu=1, 2, \ldots, n)$. As before we shall now verify the following expressions for $u_\nu(x)$ according to (2.2).

$$u_1(x)=\left[P_{n-2}(x)-\frac{1}{2}(1-x^2)P'_{n-2}(x)\right]\lambda_1^2(x)$$
$$+\omega(x)\left[k_6\int_{-1}^{x}P_{n-2}(t)\,dt+k_7\right], \tag{5.1}$$

(5.2)
$$u_n(x)=\left[P_{n-2}(x)-\frac{1}{2}(1-x^2)P'_{n-2}(x)\right]\lambda_n^2(x)+\omega(x)\left[k_8\int_{-1}^{x}P_{n-2}(t)\,dt+k_9\right],$$

where

(5.3)
$$k_6=\frac{(n-1)(n-2)[9(n-1)(n-2)(n^2-3n+4)+40]}{16(5n^2-15n+12)}=-k_8,$$

(5.4)
$$k_7=\frac{9(n-1)(n-2)(n^2-3n+4)+16}{8(3n^2-9n+8)}=k_9$$

and for $2\leq\nu\leq n-1$,

(5.5)
$$u_\nu(x)=\lambda_\nu^3(x)+\frac{3x_\nu}{(1-x_\nu^2)}v_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\int_{-1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_{11}\int_{-1}^{x}P_{n-2}(t)\,dt+k_{12}\right],$$

where

(5.6)
$$k_{10}=\frac{1}{6}x_\nu(5n^2-15n+16)+\frac{8x_\nu}{6(1-x_\nu^2)},$$

(5.7)
$$k_{11}=\frac{\omega'''(1)}{6(5n^2-15n+12)}\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt\right]$$
$$-\frac{1}{3(5n^2-15n+12)}\left[3x_\nu k_{10}\left\{\frac{5n^2-15n+12}{1-x_\nu^2}-\frac{4}{(1-x_\nu^2)^2}\right\}-\frac{8(5+14x_\nu^2)}{(1-x_\nu^2)^3}-\frac{4(n-1)(n-2)(1+x_\nu^2)}{(1-x_\nu^2)^2}\right],$$

(5.8)
$$k_{12}=-\frac{1}{2}\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt\right]+\frac{1}{\omega'''(1)}\left[3k_{10}\left\{\frac{5n^2-15n+12}{1-x_\nu^2}-\frac{2}{(1-x_\nu^2)^2}\right\}-\frac{4x_\nu(29+9x_\nu^2)}{(1-x_\nu^2)^3}-\frac{8(n-1)(n-2)x_\nu}{(1-x_\nu^2)^2}\right].$$

We first give the proof of (5.5). We observe that owing to (2.31) $u_\nu(x)$ is a polynomial of degree $3n-3$ and owing to (2.16), (2.3), (2.6) the first of the conditions (2.2) is easily seen to be true. For the second condition we have on differentiation

$$u'_\nu(x_j)=3\lambda_\nu^2(x_j)\,\lambda'_\nu(x_j)+\frac{3x_\nu}{1-x_\nu^2}v'_\nu(x_j)$$

$$=0,\ j\neq\nu \qquad \text{(use (2.17) and (2.3))},$$

$$u_\nu(x_\nu)=3\lambda'_\nu(x_\nu)+\frac{3x_\nu}{1-x_\nu}\ 0 \qquad \text{(use (2.18))}.$$

It remains to show that the third condition is also satisfied with the values of k_{10}, k_{11}, k_{12} quoted above. We have on differentiating three times the right hand side of (5.5) and using (2.17), (2.3)

$$u'''_\nu(x_j)=6\lambda'^3_\nu(x_j)+3\lambda'''_\nu(x_j)\lambda_\nu^2(x_j)+18\lambda(x_j)\lambda'_\nu(x_j)\lambda''_\nu(x_j)$$

$$-\frac{3\omega''(x_j)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\frac{h_{n+1}(x_j)}{(x_j-x_\nu)^3}+k_{10}\frac{P_{n-2}(x_j)}{x_j-x_\nu}\right].$$

For $j\neq\nu$ we have

$$v'''_\nu(x_j)=6\lambda'^3_\nu(x_j)-\frac{3\omega''(x_j)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\frac{(1-x_j^2)^2P'_{n-2}(x_j)}{(x_j-x_\nu)^3}\right]$$

$$=6\lambda'^3_\nu(x_j)-\frac{6(1-x_j^2)^3P'^3_{n-2}(x_j)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)(x_j-x_\nu)^3}=0,$$

$$u'''_\nu(x_\nu)=6\lambda'^3_\nu(x_\nu)+3\lambda'''_\nu(x_\nu)+18\lambda'_\nu(x_\nu)\lambda''_\nu(x_\nu)$$

$$-\frac{3\omega''(x_\nu)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\lim_{x=x_\nu}\frac{h_{n+1}(x)}{(x-x_\nu)^3}+k_{10}\lim_{x=x_\nu}\frac{P_{n-2}(x)}{(x-x_\nu)}\right]$$

$$=-\frac{6x_\nu^3}{(1-x_\nu^2)^3}+\frac{14x_\nu}{(1-x_\nu^2)^3}+\frac{5x_\nu(n-1)(n-2)}{(1-x_\nu^2)^2}-\frac{6}{(1-x_\nu^2)^2}k_{10}=0,$$

when k_{10} is chosen to be (5.6).

For the determination of k_{11} and k_{12} we use the two requirements $u'''_\nu(\pm1)=0$. These give two linear equations which are:

$$6\lambda'^3_\nu(1)-\frac{\omega'''(1)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}dt+k_{12}\right]$$

(5.6) $$-\frac{3\omega''(1)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[k_{10}\frac{1}{1-x_\nu}+k_{11}\right]-\frac{3\omega'(1)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[\frac{h'_{n+1}(1)}{(1-x_\nu)^3}\right.$$

$$\left.+k_{10}\frac{P'_{n-2}(1)}{1-x_\nu}-k_{10}\frac{1}{(1-x_\nu)^2}+k_{11}P'_{n-2}(1)\right]=0$$

and

$$6\lambda'^3_\nu(-1)-\frac{\omega'''(-1)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}k_{12}-\frac{3\omega''(-1)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\left[-k_{10}\frac{1}{1+x_\nu}+k_{11}\right]$$

(5.7)

$$-\frac{3\omega'(-1)}{(1-x_\nu^2)^3 P'^3_{n-2}(x_\nu)}\left[-\frac{h'_{n+1}(-1)}{(1+x_\nu)^3}-k_{10}\frac{P_{n-2}(-1)}{1+x_\nu}\right.$$

$$\left.-k_{10}\frac{1}{(1+x_\nu)^2}+k_{11}P'_{n-2}(\ 1)\right]=0.$$

Using (2.20), (2.35) and (2.36) we can rewrite them as

$$-\frac{36}{(1-x_\nu)^3}-\omega'''(1)\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_{12}\right]$$

(5.8)

$$-3\omega''(1)\left[\frac{k_{10}}{1-x_\nu}+k_{11}\right]\cdot\frac{4}{(1-x_\nu^2)(1-x_\nu)^2}\quad\frac{4(n-1)(n-2)}{(1-x_\nu)^2}$$

$$6\left[k_{10}\frac{P'_{n-2}(1)}{1-x_\nu}-\frac{k_{10}}{(1-x_\nu)^2}+k_{11}P'_{n-2}(1)\right]=0$$

and

$$-\frac{36}{(1+x_\nu)^3}-\omega'''(\ 1)k_{12}-3\omega''(-1)\left[-\frac{k_{10}}{1-x_\nu}+k_{11}\right]$$

(5.9)

$$-\frac{4}{(1+x_\nu)^2(1-x_\nu^2)}-\frac{4(n-1)(n-2)}{(1+x_\nu)^2}-6\left[-k_{10}\frac{P'_{n-2}(-1)}{1+x_\nu}\right.$$

$$\left.-\frac{k_{10}}{(1+x_\nu)^2}\quad k_{11}P'_{n-2}(-1)\right]=0.$$

Considering the relation of $\omega'''(1)$, $\omega''(1)$ and $P'_{n-2}(1)$ with those of $\omega'''(-1)$, $\omega''(-1)$ and $P'_{n-2}(-1)$, we have on adding the above two equations

$$-36\left[\frac{1}{(1-x_\nu)^3}+\frac{1}{(1+x_\nu)^3}\right]-\omega'''(1)\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt\right]$$

(5.10) $$-6\,\omega''(1)\left[\frac{x_\nu}{1-x_\nu^2}k_{10}+k_{11}\right]-4\left[\frac{1}{1-x_\nu^2}+(n-1)(n-2)\right]\left[\frac{1}{(1-x_\nu)^2}\right.$$

$$\left.+\frac{1}{(1+x_\nu)^2}\right]+\frac{12x_\nu}{1-x_\nu^2}P'_{n-2}(1)\,k_{10}-6k_{10}\left[\frac{1}{(1-x_\nu)^2}-\frac{1}{(1+x_\nu)^2}\right]+12k_{11}P'_{n-2}(1)=0$$

and on subtracting we get:

$$-36\left[\frac{1}{(1-x_\nu)^3}-\frac{1}{(1+x_\nu)^3}\right]-\omega'''(1)\left[\int_{-1}^{1}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt+k_{10}\int_{-1}^{1}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+2k_{12}\right]$$

(5.11) $$-6\omega''(1)\frac{k_{10}}{1-x_\nu^2}-4\left[\frac{1}{(1-x_\nu)^2}-\frac{1}{(1+x_\nu)^2}\right]\left[\frac{1}{(1-x_\nu^2)}+(n-1)(n-2)\right]$$

$$+\frac{12}{1-x_\nu^2}\frac{k_{10}}{1-x_\nu^2}P'_{n-2}(1)-6\,k_{10}\left[\frac{1}{(1-x_\nu)^2}+\frac{1}{(1+x_\nu)^2}\right]=0.$$

Thus from (5.10) and (5.11) using the value of k_{10} from (5.6) and simplifying with the help of (2.9), (2.13), (3.6) and (2.14) we have for the constants k_{11} and k_{12} the values (5.7) and (5.8).

For the verification of the expressions (5.1) and (5.2) for $u_1(x)$ and $u_n(x)$ we need the following results:

(5.12) $$\lambda_1(x)=\frac{1}{2}\,(1+x)\,P_{n-2}(x);\quad \lambda_n(x)=\frac{1}{2}\,(1-x)\,P_{n-2}(x).$$

With the help of (2.9), (2.10) and (2.11) we can easily calculate for even n's:

(5.13) $$\lambda'_1(1)=\frac{1}{2}\,[1+(n-1)(n-2)]=-\lambda'_n(-1),$$

(5.14) $$\lambda'_1(-1)=\frac{1}{2}=-\lambda'_1(1),$$

(5.15) $$\lambda''_1(1)=\frac{1}{2}\left[1+\frac{1}{4}\,n(n-3)\right](n-1)(n-2)=\lambda''_n(-1),$$

(5.16) $$\lambda''_1(-1)=\frac{1}{2}\,(n-1)(n-2)=\lambda''_n(1),$$

(5.17) $$\lambda'''_n(1)=P'''_{n-2}(1)+\frac{3}{2}\,P''_{n-2}(1)=-\lambda'''_n(-1),$$

(5.18) $$\lambda'''_1(-1)=\frac{3}{2}\,P''_{n-2}(-1)=-\lambda'''_n(1).$$

Now, obviously we observe in (5.1) that $u_1(1)=1$, $u_1(x_j)=0$, $(j=2, 3, \ldots, n)$ and $u'_1\,(x_j)=0$ $(j=2, 3, \ldots, n-1)$. To see that $u'''_1(x_j)=0$, $j=1, 2, \ldots, n$, we have on differentiating three times the right hand side of (5.1) and using (2.6), (2.8) and the fact that $\lambda_1(x_j)=0$, $j=2, 3, \ldots, n-1$

$$u'''_1(x_j)=-3\lambda'_1(x_j)\,P'_{n-2}(x_j)\,[(1-x_j^2)\,\lambda''_1(x_j)-2\lambda'_1(x_j)].$$

Now from (5.12) we have

$$\lambda'_1(x_j)=\frac{1}{2}\,(1+x_j)\,P'_{n-2}(x_j)$$

and

$$\lambda''_1(x_j)=\frac{1}{2}\,(1+x_j)\,P''_{n-2}(x_j)+P'_{n-2}(x_j)$$

so that

$$(1-x_j^2)\,\lambda''_1(x_j)-2\lambda'_1(x_j)=0.$$

Thus

$$u'''_1(x_j)=0,\quad j=2,\ 3, \ldots, n-1.$$

To have $u_1'''(\pm 1)=0$, we shall make our choice for the constants k_6 and k_7. For these we have as before the two linear equations:

(5.19) $$J_1'''(1)+k_6\,\omega'''(1)+3k_7\,\omega''(1)-6k_7\,P'_{n-2}(1)=0$$

and

(5.20) $$J_1'''(-1)+k_6\,\omega'''(-1)+3k_7\,\omega''(-1)+6k_7\,P'_{n-2}(-1)=0,$$

where

$$J_1'''(1)=\left[\lambda_1^2(x)\left\{P_{n-2}(x)-\frac{1}{2}(1-x^2)P'_{n-2}(x)\right\}\right]'''_{x=1}$$

(5.21) $$=6P'''_{n-2}(1)+6P''_{n-2}(1)+6\lambda_1'(1)\lambda_1''(1)+12\lambda_1''(1)P'_{n-2}(1)$$
$$+12\lambda_1'^2(1)P'_{n-2}(1)+18\lambda_1'(1)P''_{n-2}(1)+6\lambda_1'(1)P'_{n-2}(1)$$

and

$$J_1'''(-1)=\left[\lambda_1^2(x)\left\{P_{n-2}(x)-\frac{1}{2}(1-x^2)P'_{n-2}(x)\right\}\right]'''_{x=-1}$$

(5.22)

$$=6\lambda_1'(-1)\lambda_1''(-1).$$

From (5.27) and (5.28) we at once have on using (2.12) and (2.13):

(5.23) $$k_6=-\frac{J_1'''(1)+J_1'''(-1)}{6\,[\omega''(1)-2P'_{n-2}(1)]},$$

and

(5.24) $$k_7=-\frac{J_1'''(1)-J_1'''(-1)}{2\omega'''(1)}.$$

Thus on using (2.9) to (2.13) and (5.13) to (5.17) we have on simplifying the values (5.3) and (5.4) for k_6 and k_7.

The verification of $u_n(x)$ runs similarly.

Part B

6. In this part we shall deal with the following problem of convergence: Let us consider the sequence of points

(6.1) $$1=x_{1n}>x_{2n}>\cdots>x_{n-1,n}>x_{nn}=-1$$

which stand for the zeros of $(1-x^2)P_{n-2}(x)$. We form the interpolation polynomials for each $n=2k$ and write the fundamental polynomials as $u_{\nu n}(x)$, $v_{\nu n}(x)$ and $w_{\nu n}(x)$. Let $f(x)$ be a function defined in $[-1, 1]$; we shall consider the following sequence of polynomials:

(6.2) $$R_n(x,f)=\sum_{\nu=1}^{n}f(x_{\nu n})\,u_{\nu n}(x)+\sum_{\nu=2}^{n}f'(x_{\nu n})\,v_{\nu n}(x)+\sum_{\nu=1}^{n}\gamma_{\nu n}\,w_{\nu n}(x),$$

with arbitrary numbers $\gamma_{\nu n}$ and prove the following

Theorem. Let $f(x)$ be differentiable in $[-1, 1]$ and $f'(x) \in \text{Lip}\, \alpha$ $\left(\alpha > \frac{1}{2}\right)$. Supposing that for arbitrary small $\varepsilon > 0$ we have for $n > n_0(\varepsilon)$ and $\nu = 1, 2, \ldots, n$

$$\gamma_{\nu n} \leq \varepsilon n^{3/2}, \tag{6.3}$$

the sequence $R_n(x, f)$ converges to $f(x)$ uniformly in $[-1, 1]$.

The proof of this theorem depends mainly on the estimation of the fundamental polynomials and a lemma on approximating polynomial. The §§ 11—13 will be devoted to obtain such estimations. For this, we need to evaluate certain integrals involved in the expressions of the fundamental polynomials. These will help us to simplify the constants occuring in the same expressions. With the help of these evaluations we shall write in § 8 some alternative forms of the fundamental polynomials which are best suited for our estimation purposes. In § 9 we shall mention few well-known results about Legendre polynomials which we shall repeatedly make use of. The § 10 will be devoted to estimate the absolute values of some integrals. We shall further require a lemma on approximating polynomial which we shall give in § 14. Finally in § 15, we shall complete the proof of the theorem.

7. **The evaluation of certain integrals.** Lemma 7.1. **(a)** For $\nu = 2, 3, \ldots, n-1$,

$$\int_{-1}^{1} \frac{P_{n-2}(x)}{t-x_\nu}\, dt = \frac{2}{(1-x_\nu^2) P'_{n-2}(x_\nu)} \tag{7.1}$$

(b) For $\nu = 2, 3, \ldots, n-1$ and even n's,

$$\int_{-1}^{1} \frac{q_n(t)}{(t-x_\nu)^2}\, dt = -\frac{(n-1)(n-2)}{(1-x_\nu^2) P'_{n-2}(x_\nu)} + \frac{2(1+x_\nu^2)}{(1-x_\nu^2)^2 P'_{n-2}(x_\nu)}, \tag{7.2}$$

$$\int_{-1}^{1} \frac{h_{n+1}(t)}{(t-x_\nu)^3}\, dt = \frac{n(n-3)x_\nu}{(1-x_\nu^2) P'_{n-2}(x_\nu)} - \frac{4}{3} \frac{x_\nu}{(1-x_\nu^2)^2 P'_{n-2}(x_\nu)} \tag{7.3}$$

The part (a) of the above lemma has been proved in our paper [12], lemma 3.1. For the proof of part (b) we shall return elsewhere.

8. **Some alternative forms of fundamental polynomials.** We give here some alternative forms of the polynomials $u_\nu(x)$, $v_\nu(x)$ and $w_\nu(x)$. These forms, as we shall see, are most convenient for our estimation purposes. **(a)** For the expressions $w_\nu(x)$ $(\nu = 2, 3, 4, \ldots, n-1)$ in (3.3) we have

$$w_\nu(x) = \frac{\omega(x)}{6(1-x_\nu^2) P'^3_{n-2}(x_\nu)} \left[\int_{-1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\, dt + k_1 \int_{1}^{x} P_{n-2}(t)\, dt + k_2 \right] \quad \text{for } x < x_\nu < 1, \tag{8.1}$$

and

$$w_\nu(x)=\frac{\omega(x)}{6(1-x_\nu^2)P'^3_{n-2}(x_\nu)}\left[\int_1^x \frac{P_{n-2}(t)}{t-x_\nu}\,dt\right.$$

(8.2)

$$\left.+k_1\int_{-1}^x P_{n-2}(t)\,dt+k_2\right] \qquad \text{for } -1<x_\nu<x.$$

The values of k_1 and k_2 in (3.4) and (3.5) when simplified with the help of (7.1), give

$$k_1=-\frac{(n-1)(n-2)(3n^2-9n+8)}{2(5n^2-15n+12)}\cdot\frac{1}{(1-x_\nu^2)P'_{n-2}(x_\nu)}$$

(8.3)

$$-\frac{x_\nu}{1-x_\nu^2}+\frac{4}{5n^2-15n+12}\cdot\frac{x_\nu}{(1-x_\nu^2)^2}$$

and

$$k_2=-\frac{1}{(1-x_\nu^2)P'_{n-2}(x_\nu)}+\frac{10}{(1-x_\nu^2)(3n^2-9n+8)}$$

(8.4)

$$+\frac{8}{(n-1)(n-2)(3n^2-9n+8)(1-x_\nu^2)^2}$$

The value of $\bar{k}_2$ in (8.2) is given by

$$\bar{k}_2=\frac{1}{(1-x_\nu^2)P'_{n-2}(x_\nu)}+\frac{10}{(1-x_\nu^2)(3n^2-9n+8)}$$

(8.5)

$$+\frac{8}{(n-1)(n-2)(3n^2-9n+8)(1-x_\nu^2)^2}.$$

(b) For $v_\nu(x)$ ($\nu=2,3,\ldots,n-1$) in (4.1) we have

$$v_\nu(x)=\frac{(1-x^2)P_{n-2}(x)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}\,r_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)}\left[\int_{-1}^x \frac{q_n(t)}{(t-x_\nu)^2}\,dt\right.$$

(8.6)

$$\left.+k_3\int_{-1}^x \frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_4\int_{-1}^x P_{n-2}(t)\,dt+k_5\right] \quad \text{for } x<x_\nu<1$$

and

$$v_\nu(x)=\frac{(1-x^2)P_{n-2}(x)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}\,\gamma_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^2P'^3_{n-2}(x_\nu)}\left[\int_1^x \frac{q_n(t)}{(t-x_\nu)^2}\,dt\right.$$

(8.7)

$$+k_3\int_{+1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_4\int_{-1}^{x}P_{n-2}(t)\,dt+\bar{k}_5\Bigg]\qquad\text{for } -1<x_\nu<x.$$

The values of k_4 and k_5 in (4.3) and (4.4), when simplified with the help of (7.1) and (7.2), give

$$k_4=-\frac{2\omega'''(1)}{9(5n^2-15n+12)}\left[\frac{1}{(1-x_\nu^2)^2P'_{n-2}(x_\nu)}+\frac{(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}\right]$$

(8.8)

$$+\frac{2x_\nu}{3(5n^2-15n+12)}\left[\frac{44}{(1-x^2)^3}+\frac{(n-7)(n+4)}{(1-x_\nu^2)^2}\right]+\frac{x_\nu}{3}\left[\frac{4}{(1-x_\nu^2)^2}-\frac{(n-4)(n+1)}{1-x_\nu^2}\right]$$

and

$$k_5=\frac{2}{3}\left[\frac{(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}+\frac{1}{(1-x^2)^2P'_{n-2}(x_\nu)}\right]-\frac{1}{9(1-x^2)}$$

(8.9)
$$+\frac{4}{3(3n^2-9n+8)}\left[\frac{50}{1-x_\nu^2}+\frac{18}{(1-x_\nu^2)^2}\right]$$

$$-\frac{16}{3(n-1)(n-2)(3n^2-9n+8)}\left[\frac{3}{1-x_\nu^2}+\frac{3}{(1-x_\nu^2)^2}+\frac{19}{(1-x_\nu^2)^3}\right].$$

The value of $\bar{k}_5$ in (8.7) is given by

$$\bar{k}_5=-\frac{2}{3}\left[\frac{(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}+\frac{1}{(1-x_\nu^2)^2P'_{n-2}(x_\nu)}\right]-\frac{1}{9(1-x_\nu^2)}$$

(8.10)
$$+\frac{4}{3(3n^2-9n+8)}\left[\frac{50}{1-x_\nu^2}+\frac{18}{(1-x_\nu^2)^2}\right]$$

$$-\frac{16}{3(n-1)(n-2)(3n^2-9n+8)}\left[\frac{3}{(1-x_\nu^2)}+\frac{3}{(1-x_\nu^2)^2}+\frac{19}{(1-x_\nu^2)^3}\right].$$

(c) The alternative forms of $u_\nu(x)$ $(\nu=2, 3, \dots, n-1)$ in (5.5) are

$$u_\nu(x)=\lambda_\nu^3(x)+\frac{3x_\nu}{1-x_\nu^2}v_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\Bigg[\int_{-1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt$$

(8.11)

$$+k_{10}\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_{11}\int_{-1}^{x}P_{n-2}(t)\,dt+k_{12}\Bigg]\qquad\text{for } x<x_\nu<1$$

and

$$u_\nu(x)=\lambda_\nu^3(x)+\frac{3x_\nu}{1-x_\nu^2}v_\nu(x)-\frac{\omega(x)}{(1-x_\nu^2)^3P'^3_{n-2}(x_\nu)}\Bigg[\int_{+1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt$$

(8.12)

$$+k_{10}\int_{+1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt+k_{11}\int_{-1}^{x}P_{n-2}(t)\,dt+\overline{k_{12}}\Big]\quad \text{for } -1<x_\nu<x,$$

where $v_\nu(x)$ in (8.11) is given by (8.6) and $v_\nu(x)$ in (8.12) is given by (8.7). The values of constants k_{11} and k_{12} are simplified as before. Thus we have

$$k_{11}=\frac{2\omega'''(1)\,x_\nu}{9(5n^2-15n+12)}\left[\frac{2(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}+\frac{1}{(1-x_\nu^2)^2P'_{n-2}(x_\nu)}\right]$$

$$+\frac{1}{3(5n^2-15n+12)}\left[\frac{168}{(1-x_\nu^2)^3}+\frac{8n^2-24n-104}{(1-x_\nu^2)^2}-\frac{4n^2-12n+16}{1-x_\nu^2}\right]$$

$$-\frac{x_\nu^2}{6}\left[\frac{5n^2-15n+16}{1-x_\nu^2}+\frac{4}{(1-x_\nu^2)^2}\right],$$

and

$$k_{12}=-\frac{2}{3}\,x_\nu\left[\frac{2(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}-\frac{1}{(1-x_\nu^2)^2P'_{n-2}(x_\nu)}\right]$$

(8.14)

$$+\frac{x_\nu}{\omega'''(1)}\left[\frac{(5n^2-15n+12)(5n^2-15n+16)}{2(1-x_\nu^2)}\right.$$

$$\left.+\frac{12n^2-36n+68}{2(1-x_\nu^2)^2}-\frac{160}{(3-x_\nu^2)^3}\right].$$

The value of $\overline{k_{12}}$ in (8.12) is given by

$$\overline{k_{12}}=\frac{2}{3}\,x_\nu\left[\frac{2(n-1)(n-2)}{(1-x_\nu^2)P'_{n-2}(x_\nu)}-\frac{1}{(1-x_\nu^2)^2P'_{n-2}(x_\nu)}\right]$$

(8.15)

$$+\frac{x_\nu}{\omega'''(1)}\left[\frac{(5n^2-15n+12)(5n^2-15n+16)}{2(1-x_\nu^2)}+\frac{12n^2-36n+68}{2(1-x_\nu^2)^2}-\frac{160}{(1-x_\nu^2)^3}\right].$$

9. Some well-known results about Legendre polynomials. We shall later make use of the following well-known results about Legendre polynomials.

For $-1\leq x\leq 1$ we have[1]

(9.1) $$\sqrt{n}\sqrt[4]{1-x^2}\,|P_n(x)|\leq 4\sqrt{\frac{2}{\pi}},$$

(9.2) $$|P_n(x)-P_{n+2}(x)|<\frac{4}{\sqrt{\pi}\sqrt{n+1}}$$

and

(9.3) $$(1-x^2)\,P'_n(x)<\frac{2}{\sqrt{\pi}}\sqrt{n}.$$

[1] Sansone [13], p. 199.

We shall also need for $-1 \leq x \leq 1$,

(9.4) $$|P_m(x)| \leq 1,$$

and

(9.5) $$|P'_m(x)| \leq \frac{m(m+1)}{2}.$$

The following inequality plays an important role in our estimation work[5].

(9.6) $$|P_{n-2}(x)| \leq \sqrt{\frac{1-x^2}{1-x_\nu^2}}\,|P'_{n-2}(x_\nu)|\,|x-x_\nu|$$

valid for $-1 \leq x \leq 1$ and $\nu=2, 3, \dots, n-1$. Further let

(9.7) $$0 < \theta_2 < \theta_3 < \cdots < \theta_{n-1} < \pi$$

be the zeros of $P_{n-2}(\cos\theta)$, then we have[6]

(9.8) $$\left(\nu - \frac{1}{2}\right)\frac{\pi}{n-\frac{1}{2}} \leq \theta_\nu \leq \frac{\nu\pi}{n-1}, \quad \nu=2, 3, \dots, \left[\frac{n}{2}\right],$$

and[7]

(9.9) $$|P'_{n-2}(\cos\theta_\nu)| \sim \nu^{-\frac{3}{2}}(n-2)^2, \quad 0 < \theta_\nu < \pi/2.$$

From (9.8) we have

(9.10) $$(1-x_\nu^2) = 1-\cos^2\theta_\nu = \sin^2\theta_\nu > \frac{4}{\pi^2}\theta_\nu^2 > \frac{\nu^2}{(n-2)^2}.$$

10. The estimation of some integrals. We shall need the

Lemma 10.1

(a)

(10.1) $$\left|\int_{-1}^{x} P_{n-2}(t)\,dt\right| \leq \frac{2}{\sqrt{\pi}\,(n-2)^{3/2}}, \quad -1 \leq x \leq 1;$$

(b)

(10.2) $$\left|\int_{-1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\,dt\right| < \frac{4}{\sqrt{\pi}\,(n-2)^{3/2}}\,\frac{1}{x_\nu - x} \quad \text{for } x < x_\nu < 1,$$

(10.3) $$\left|\int_{1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\,dt\right| < \frac{4}{\sqrt{\pi}\,(n-2)^{3/2}}\cdot\frac{1}{x-x_\nu} \quad \text{for } -1 < x_\nu < x;$$

(c)

(10.4) $$\int_{-1}^{x} \frac{q_n(t)}{(t-x_\nu)^2}\,dt < \frac{1}{\sqrt{2\pi}\sqrt{n-2}}\,\frac{1}{(x-x_\nu)^2} + \frac{8\sqrt{n-2}}{\sqrt{\pi}\,(x_\nu - x)}, \quad x < x_\nu < 1,$$

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[5] Egervary and Turán [4], p. 265, formula (7.2).

[6] Szegö [14], p. 121, formula (6.21.7).

[7] Szegö [14], p. 236, formula (8.9.2).

$$(10.5)\qquad \int_1^x \frac{q_n(t)}{(t-x_\nu)^2}\,dt < \frac{1}{\sqrt{2\pi}\sqrt{n-2}}\,\frac{1}{(x-x_\nu)^2} + \frac{8\sqrt{n-2}}{\sqrt{\pi}\,(x_\nu-x)}, \quad -1<x_\nu<x;$$

(d)

$$(10.6)\qquad \int_1^x \frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt < \frac{(1-x_\nu^2)}{3}\,\frac{(1-x^2)\,|P_{n-2}(x)|}{(x_\nu-x)^3} + \frac{2\sqrt{n}}{(x_\nu-x)^2}$$

$$+ \frac{20\sqrt{n}}{x_\nu-x} + \frac{n^2(1-x_\nu^2)}{x_\nu-x} + \frac{2}{1-x_\nu^2}.$$

$$(10.7)\qquad \int_1^x \frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt < \frac{1-x_\nu^2}{3}\,\frac{(1-x^2)\,|P_{n-2}(x)|}{(x-x_\nu)^3} + \frac{2\sqrt{n}}{(x_\nu-x)^2} + \frac{20\sqrt{n}}{x-x_\nu}$$

$$+ \frac{n^2(1-x_\nu^2)}{x-x_\nu} + \frac{2}{1-x_\nu^2}.$$

P r o o f. (a) Since

$$(10.8)\qquad P'_{n-1}(x) - P'_{n-3}(x) = (2n-3)\,P_{n-2}(x),$$

we have on integration

$$(2n-3)\int_{-1}^x P_{n-2}(t)\,dt = P_{n-1}(t) - P_{n-3}(t),$$

whence by (9.2)

$$\left|\int_{-1}^x P_{n-2}(t)\,dt\right| < \frac{4}{\sqrt{\pi}\,(2n-3)\sqrt{n-1}} < \frac{2}{\sqrt{\pi}\,(n-2)^{3/2}}$$

(b) For the proof of the part (b) we find it sufficient to prove (10.2). By (10.8) we have

$$(2n-3)\int_{-1}^x \frac{P_{n-2}(t)}{t-x_\nu}\,dt = \int_{-1}^x \frac{P'_{n-1}(t) - P'_{n-3}(t)}{t-x_\nu}\,dt,$$

$$= \frac{P_{n-1}(x) - P_{n-3}(x)}{x-x_\nu} \quad \int_{-1}^x \frac{P_{n-1}(t) - P_{n-3}(t)}{(t-x_\nu)^2}\,dt,$$

which owing to (9.2) gives

$$(2n-3)\left|\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}dt\right| < \frac{4}{\sqrt{\pi}\sqrt{n-1}(x_\nu-x)}+\frac{4}{\sqrt{\pi}\sqrt{n-1}}\int_{-\infty}^{x}\frac{dt}{(t-x_\nu)^2}$$

$$<\frac{8}{\sqrt{\pi}\sqrt{n-1}}\frac{1}{x_\nu-x}$$

and this proves (10.2).

(c) On simplifying (2.27) with the help of (2.16) we can write

$$\int_{-1}^{x}\frac{q_n(t)}{(t-x_\nu)^2}dt=\int_{-1}^{x}\frac{(1-t^2)P'_{n-2}(t)}{(t-x_\nu)^2}dt-\frac{x_\nu}{1-x_\nu^2}\int_{-1}^{x}\frac{(1-t^2)P_{n-2}(t)}{(t-x_\nu)^2}dt$$

$$-\int_{-1}^{x}\frac{(1-t^2)P_{n-2}(t)}{(t-x_\nu)^3}dt.$$

Integrating by parts we have

$$\int_{-1}^{x}\frac{q_n(t)}{(t-x_\nu)^2}dt=\frac{(1-x^2)P_{n-2}(x)}{2(x-x_\nu)^2}\quad\frac{1}{2}\frac{(1-x^2)P'_{n-2}(x)}{x-x_\nu}+\frac{x_\nu}{1-x_\nu^2}\int_{-1}^{x}P_{n-2}(t)\,dt$$

$$-\frac{1}{2}(n-1)(n-2)\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}dt+\frac{1+x_\nu^2}{1-x_\nu^2}\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}dt.$$

Now from (10,1) and (10.2) we have

$$\left|\int_{-1}^{x}\frac{q_n(t)}{(t-x_\nu)}dt\right|\leqq\frac{(1-x^2)\,P_{n-2}(x)}{2(x-x_\nu)^2}+\frac{(1-x^2)|P'_{n-2}(x)|}{2(x_\nu-x)}+\frac{2}{\sqrt{\pi}(n-2)^{3/2}(1-x_\nu^2)}$$

$$+\left[\frac{(n-1)(n-2)}{2}+\frac{2}{1-x_\nu^2}\right]\frac{4}{\sqrt{\pi}(n-2)^{3/2}}\cdot\frac{1}{x_\nu-x}<\frac{8}{\sqrt{\pi}(n-2)^{3/2}(1-x_\nu^2)(x_\nu-x)}$$

$$+\frac{(1-x^2)\,P_{n-2}(x)}{2(x-x_\nu)^2}+\frac{(1-x^2)\;P'_{n-2}(x)}{2(x_\nu-x)}+\frac{2}{\sqrt{\pi}(n-2)^{3/2}(1-x_\nu^2)}+\frac{2(n-1)}{\sqrt{\pi}\sqrt{n-2}(x_\nu-x)}$$

and by (9.1) and (9.3) and (9.10)

$$\left|\int_{-1}^{x}\frac{q_n(t)}{(t-x_\nu)^2}dt\right|<\frac{1}{\sqrt{2\pi}\sqrt{n-2}(x-x_\nu)^2}+\frac{5\sqrt{n-2}}{\sqrt{\pi}(x_\nu-x)}+\frac{12\sqrt{n-2}}{\sqrt{\pi}\,\nu^2(x_\nu-x)}$$

$$<\frac{1}{\sqrt{2\pi(n-2)}}\cdot\frac{1}{(x-x_\nu)^2}+\frac{8\sqrt{n-2}}{\sqrt{\pi}(x_\nu-x)}$$

This proves (10.4). Similarly we can prove (10.5) by using (10.1) and (10.3).

(d) We now prove the part (d) of the above lemma. From (2.31), (2.32), (2.33) and (2.16) we have

$$\int_{-1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt=-(1-x_\nu^2)\int_{-1}^{x}\frac{(1-t^2)\,P_{n-2}(t)}{(t-x_\nu)^4}\,dt+\int_{-1}^{x}\frac{(1-t^2)^2\,P'_{n-2}(t)}{(t-x_\nu)^3}\,dt$$

$$+x_\nu\int_{-1}^{x}\frac{(1-t^2)P_{n-2}(t)}{(t-x_\nu)^3}\,dt+\frac{1}{3(1-x_\nu^2)}\int_{-1}^{x}\frac{(1-t^2)\,P_{n-2}(t)}{(t-x_\nu)^2}\,dt$$

$$+\frac{(n-1)(n-2)}{3}\int_{-1}^{x}\frac{(1-t^2)\,P_{n-2}(t)}{(t-x_\nu)^2}\,dt.$$

On integrating by parts the first three integrals on the right of the above equation and simplifying among themselves we get

$$\int_{-1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt=\frac{1-x_\nu^2}{3}\Bigg[\frac{(1-x^2)\,P_{n-2}(x)}{(x-x_\nu)^3}+\frac{(1-x^2)P'_{n-2}(x)-2x\,P_{n-2}(x)}{2\,(x-x_\nu)^2}$$

$$-\frac{1}{(1-x_\nu)^2}\Bigg]-\frac{(1-x^2)^2\,P'_{n-2}(x)+x_\nu\,(1-x^2)\,P_{n-2}(x)}{2\,(x-x_\nu)^2}$$

$$+\frac{(n-1)(n-2)}{6}\int_{-1}^{x}\frac{t+x_\nu}{t-x_\nu}\,P_{n-2}(t)\,dt+\frac{1-x_\nu^2}{3}\int_{-1}^{x}\frac{tP'_{n-2}(t)+P_{n-2}(t)}{(t-x_\nu)^2}\,dt$$

$$-\int_{-1}^{x}\frac{t\,(1-t^2)\,P'_{n-2}(t)}{(t-x_\nu)^2}\,dt+\frac{x_\nu}{2}\int_{-1}^{x}\frac{(1-t^2)P'_{n-2}(t)}{(t-x_\nu)^2}\,dt$$

$$-x_\nu\int_{-1}^{x}\frac{t\,P_{n-2}(t)}{(t-x_\nu)^2}\,dt+\frac{1}{3(1-x_\nu^2)}\int_{-1}^{x}\frac{(1-t^2)P_{n-2}(t)}{(t-x_\nu)^2}\,dt.$$

We write

$$\frac{1}{3(1-x_\nu^2)}\int_{-1}^{x}\frac{(1-t^2)\,P_{n-2}(t)}{(t-x_\nu)^2}\,dt=\frac{1}{3}\int_{-1}^{x}\frac{P_{n-2}(t)}{(t-x_\nu)}\,dt-\frac{1}{3(1-x_\nu^2)}\int_{-1}^{x}\frac{t+x_\nu}{t-x_\nu}\,P_{n-2}(t)\,dt$$

$$=\frac{1}{3}\int_{-1}^{x}\frac{P_{n-2}(t)}{(t-x_\nu)^2}\,dt-\frac{2x_\nu}{3(1-x_\nu^2)}\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt-\frac{1}{3(1-x_\nu^2)}\int_{-1}^{x}P_{n-2}(t)\,dt.$$

Hence owing to (9.3), (9.4), (9.5) and (9.1) we have

$$\int_{-1}^{x} \frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt \leq \frac{(1-x_\nu^2)}{3}\cdot\frac{(1-x^2)\,P_{n-2}(x)}{(x_\nu-x)^3}+\frac{2\sqrt{n}}{(x-x_\nu)^2}+\frac{2}{1-x_\nu^2}$$

$$-\frac{(n-1)^2}{6}\int_{-1}^{x} P_{n-2}(t)\,dt+(n-1)^2\int_{-1}^{x}\frac{P_{n-2}(t)}{t-x_\nu}\,dt$$

$$+[(1-x_\nu^2)\,n^2-2\sqrt{n}]\int_{-\infty}^{x}\frac{dt}{(t-x_\nu)^2},$$

which by (10.1) and (10.2) at once gives

$$\int_{-1}^{x}\frac{h_{n+1}(t)}{(t-x_\nu)^3}\,dt \leq \frac{1-x_\nu^2}{3}\,\frac{(1-x^2)\,P_{n-2}(x)}{(x_\nu-x)^3}+\frac{2\sqrt{n}}{(x-x_\nu)^2}+\frac{20\sqrt{n}}{(x_\nu-x)}$$

$$+\frac{2}{1-x_\nu^2}+\frac{n^2(1-x_\nu^2)}{(x_\nu-x)}$$

Similarly we can prove (10.5). This completes the proof of Lemma 10.1.

11. Estimation of the polynomials $w_\nu(x)$. In this § we shall prove the following

Lemma 11.1. For $n=4, 6, 8, \ldots,$

(a) $$w_1(x)<(n-2)^{-9/2},\quad w_n(x)\leq(n-2)^{-9/2}$$

(b) $$|w_\nu(x)|<c_1\,\nu^2\,n^{-9/2}\ \text{for}\ 2\leq\nu\leq\frac{1}{2}\,n,$$

$$|w_\nu(x)|<c_1\,(n-\nu)^2\,n^{-9/2}\ \text{for}\ \frac{1}{2}\,n+1\leq\nu\leq n-1,$$

(c) $$\sum_{\nu=1}^{n}|w_\nu(x)|\leq c_2\,n^{-3/2},$$

hold uniformly in $-1\leq x\leq 1$, where c_1 and c_2 [8] are positive numerical constants.

Proof. The part (a) easily follows from (3.1) on using (10.1) and (9.1) while (c) is a consequence of (a) and (b). We thus prove the part (b) for which we find it sufficient to prove the first assertion.

Let first be $x<x_\nu<1$. In this case we use the form (8.1) and write

(11.1) $$w_\nu(x)=J_1+J_2+J_3.$$

For

$$J_3=\frac{\omega(x)}{6(1-x_\nu^2)\,P'^3_{n-2}(x_\nu)}\,k_2,$$

[8] From now onward $c_1, c_2, c_3, \ldots$ will be used to denote positive numerical constants.

we have owing to (8.4) and (9.1)

$$J_3 \leq \frac{16}{3\pi(n-2)} \cdot \frac{1}{(1-x_\nu^2)^2 P_{n-2}^4(x_\nu)} + \frac{160}{9\pi(n-2)^3} \cdot \frac{1}{(1-x_\nu^2)^2 |P'^3_{n-2}(x_\nu)}$$

$$+ \frac{128}{9\pi(n-2)^5} \cdot \frac{1}{(1-x_\nu^2)|P'^3_{n-2}(x_\nu)|}$$

Then by (9.9) and (9.10) we have

(11.2) $$|J_3| \leq c_3 \nu^2 . n^{-5}.$$

For

$$J_2 = \frac{\omega(x) k_1}{6(1-x_\nu^2) P'^3_{n-2}(x_\nu)} \int_{-1}^{x} P_{n-2}(t)\, dt$$

we have from (10.1) and (9.1)

$$J_2 \leq \frac{11}{3\pi(n-2)^{5/2}} \cdot \frac{k_1}{(1-x_\nu^2) P'^3_{n-2}(x_\nu)}$$

which owing to (8.3), (9.9) and (9.10) yields

$$J_2 \leq \frac{11}{3\pi\sqrt{n-2}(1-x_\nu^2)^2 P'^4_{n-2}(x_\nu)} + \frac{11}{3\pi(n-2)^{5/2}} \cdot \frac{1}{(1-x_\nu^2)^2 P'^3_{n-2}(x_\nu)}$$

(11.3)

$$+ \frac{11}{3\pi(n-2)^{9/2}(1-x_\nu^2)^3 P'^3_{n-2}(x_\nu)} < c_4 \nu^2 n^{-\frac{9}{2}}$$

Lastly for

$$J_1 = \frac{\omega(x)}{6(1-x_\nu^2) P'^3_{n-2}(x_\nu)} \int_{-1}^{x} \frac{P_{n-2}(x)}{t-x_\nu}\, dt$$

we use (10.2), (9.6), (9.1), (9.9), (9.10) and have

$$J_1 \leq \frac{2(1-x^2) P^2_{n-2}(x)}{3(n-2)^{3/2}(1-x_\nu^2)|P'^3_{n-2}(x_\nu)| |x-x_\nu|}$$

(11.4)

$$\leq \frac{8}{3(n-2)^2} \cdot \frac{1}{\sqrt{1-x_\nu^2}\, P^2_{n-2}(x_\nu)} < c_5 \frac{\nu^2}{n^5}.$$

Hence from (11.1), (11.2), (11.3) and (11.4) we have for $x < x_\nu < 1$

(11.5) $$w_\nu(x) \leq c_1 \nu^2 . n^{-\frac{9}{2}}$$

For $-1 < x_\nu < x$, we can prove the same inequality (11.5). But in this case we shall use the form (8.2) for $w_\nu(x)$ and the estimation (10.3) for $\left|\int_1^x \frac{P_{n-2}(t)}{t-x_\nu}\,dt\right|$.

At $x=x_\nu$, (11.5) is obvious.

12. Estimation of the polynomials $v_\nu(x)$. For the estimation of $v_\nu(x)$ we prove the following.

Lemma 12.1. For $n=4, 6, 8, \ldots$ and $-1\leq x\leq 1$ the following hold.

(a)
$$v_\nu(x)\leq c_{13}(n_\nu)^{-\frac{1}{2}}+c_{14}n^{-\frac{1}{2}},\quad 2\leq\nu\leq\frac{1}{2}n,$$

$$v_\nu(x)|\leq c_{13}(n-\nu)^{-\frac{1}{2}}n^{-\frac{1}{2}}+c_{14}n^{-\frac{1}{2}},\quad \frac{1}{2}n-1\leq\nu\leq n-1,$$

(b)
$$\sum_{\nu=2}^{n-1}| v_\nu(x)|\leq c_{15}\sqrt{n}.$$

Proof. The part (b) is immediate from (a). We therefore prove (a). It is sufficient to prove the first assertion.

Let first be $x<x_\nu<1$. Then using the form (8.6) for (x) we have

$$v_\nu(x)\leq\frac{(1-x^2)|P_{n-2}(x)\ r_\nu(x)}{(1-x_\nu^2)|P'_{n-2}(x_\nu)|}+\frac{(1-x^2)P^2_{n-2}(x)}{(1-x_\nu^2)^2|P'^3_{n-2}(x_\nu)}\left|\int_{-1}^x\frac{q_n(x)}{(t-x_\nu)^2}\,dt\right|$$

$$\frac{(1-x^2)P^2_{n-2}(x)|k_3}{(1-x_\nu^2)|P'^3_{n-2}(x_\nu)}\int_{-1}^x\frac{P_{n-2}(t)}{t-x_\nu}\,dt+\frac{(1-x^2)P^2_{n-2}(x)\ k_4}{(1-x_\nu^2)^2\ P'^3_{n-2}(x_\nu)}\left|\int_{-1}^x P_{n-2}(t)\,dt\right|$$

(12.1)

$$+\frac{(1-x^2)P^2_{n-2}(x)\ k_5}{(1-x_\nu^2)\ P'_{n-2}(x_\nu)|^3}=J_4+J_5+J_6+J_7+J_8.$$

As to estimate

$$J_8=\frac{(1-x^2)P^2_{n-2}(x)|k_5}{(1-x_\nu^2)^2\ P'_{n-2}(x_\nu)|^3},$$

we have from (8.9), (9.10)

$$|k_5|\leq\frac{2}{3}\left[\frac{(n-1)(n-2)}{(1-x_\nu^2)|P'_{n-2}(x_\nu)|}+\frac{1}{(1-x_\nu^2)|P'_{n-2}(x_\nu)|}\right]$$

$$+\frac{1}{9(1-x_\nu^2)}+\frac{272}{9(n-2)^2}\cdot\frac{1}{(1-x_\nu^2)^2}+\frac{400}{9(n-2)^4}\cdot\frac{1}{(1-x_\nu^2)^3}$$

$$\leq\frac{3}{2}\frac{(n-2)^4}{\nu^2|P'_{n-2}(x_\nu)|}+\frac{13}{9}\frac{(n-2)^2}{\nu^2}<c_6\frac{n^2}{\sqrt{\nu}}.$$

Hence from (9.9), (9.10) and (9.1)

$$J_8 \leq c_6 \frac{(1-x^2) P_{n-2}^2(x)}{(1-x_\nu^2)^2 \,|\, P'^3_{n-2}(x_\nu)} \cdot \frac{n^2}{\sqrt{\nu}} < c_7 n^{-1}. \tag{12.2}$$

Now from (8.8) and (2.14)

$$k_4 < \frac{(n-1)(n-2)}{(1-x_\nu^2)^2 \,|\, P'_{n-2}(x_\nu)\,|} + \frac{(n-1)^2(n-2)^2}{(1-x_\nu^2)\,|\, P'_{n-2}(x_\nu)\,|} + \frac{3}{2(1-x_\nu^2)^2} + \frac{(n-4)(n+1)}{3(1-x_\nu^2)}$$

$$+ \frac{88}{15(n-2)^2(1-x_\nu^2)^3} < c_8 \nu^{-\frac{1}{2}} n^4.$$

Hence for

$$J_7 = \frac{(1-x^2) P_{n-2}^2(x)\,|\,k_4\,|}{(1-x_\nu^2)^2 \,|\, P'_{n-2}(x_\nu)^3\,|} \left| \int_{-1}^{x} P_{n-2}(t)\,dt \right|,$$

we have owing to (10.1)

$$J_7 \leq \frac{8\sqrt{2}}{\pi} c_8 \cdot \frac{n^{\frac{3}{2}}}{\sqrt{\nu}} \cdot \frac{1}{(1-x_\nu^2)^2 \,|\, P'_{n-2}(x_\nu)\,|^3} < c_9 n^{-\frac{1}{2}} \tag{12.3}$$

As to

$$J_6 = \frac{(1-x^2)\,|\,P_{n-2}(x)\,|\,|\,k_3\,|}{(1-x_\nu^2)^2 \,|\, P'_{n-2}(x_\nu)\,|^3} \left| \int_{-1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\,dt \right|,$$

we have from (10.2), (4.2) and (9.6),

$$J_6 < \frac{25}{3\sqrt{\pi}} \cdot \frac{\sqrt{1-x^2}\,|\,P_{n-2}(x)\,|}{(1-x_\nu^2)^{3/2} P'^2_{n-2}(x_\nu)} \sqrt{n-2} < c_{10} n^{-1}. \tag{12.4}$$

For

$$J_5 = \frac{(1-x^2) P_{n-2}^2(x)}{(1-x_\nu^2)^2 \,|\, P'^3_{n-2}(x_\nu)\,|} \left| \int_{-1}^{x} \frac{q_n(t)}{(t-x_\nu)^2}\,dt \right|$$

we have owing to (10.4), (9.7)

$$J_5 = \frac{1}{\sqrt{2\pi}\sqrt{n-2}} \cdot \frac{(1-x^2) P_{n-2}^2(x)}{(1-x_\nu^2)^2 \,|\, P'_{n-2}(x_\nu)\,|^3 (x-x_\nu)^2}$$

$$+ \frac{8\sqrt{n-2}}{\sqrt{\pi}} \cdot \frac{(1-x^2) P_{n-2}^2(x)}{(1-x_\nu^2)^2 \,|\, P'^3_{n-2}(x_\nu)\,|\,|\,x-x_\nu\,|}$$

$$< \frac{1}{\sqrt{2\pi}\sqrt{n-2}} \cdot \frac{1}{(1-x_\nu^2)\,|\,P'_{n-2}(x_\nu)\,|} + \frac{8\sqrt{n-2}}{\sqrt{\pi}} \cdot \frac{\sqrt{1-x^2}\,|\,P_{n-2}(x)\,|}{(1-x_\nu^2)^{3/2} P'^2_{n-2}(x_\nu)},$$

then from (9.1), (9.9) and (9.10) we at once have

$$(12.5)\qquad J_5 < \frac{1}{\sqrt{2\pi}} \cdot \frac{(n-2)^{3/2}}{\nu^2\, |P'_{n-2}(x_\nu)|} + \frac{32\sqrt{2}}{\pi} \cdot \frac{(n-2)^3}{\nu^3 P'^2_{n-2}(x_\nu)} < \frac{c_{11}}{\sqrt{\nu n}}$$

For the last sum

$$J_4 = \frac{(1-x^2)}{(1-x_\nu^2)} \frac{P_{n-2}(x)}{P'_{n-2}(x_\nu)} r_\nu(x),$$

we have from (2.21) and (9.6)

$$J_4 = \frac{(1-x^2)^2\, |P_{n-2}(x)|^3}{(1-x_\nu^2)^2\, |P'_{n-2}(x_\nu)|^3 (x-x_\nu)^2} \leq \frac{(1-x^2)\, |P_{n-2}(x)|}{(1-x_\nu^2)\, |P'_{n-2}(x_\nu)|}$$

and from (9.1), (9.9) and (9.10) we have

$$(12.6)\qquad J_4 < \frac{4\sqrt{2}}{\sqrt{\pi}} \frac{(n-2)^{\frac{3}{2}}}{\nu^2\, |P'_{n-2}(x_\nu)|} < \frac{c_{12}}{\sqrt{\nu n}}$$

Hence from (12.1) to (12.6) we have for $x < x_\nu < 1$

$$(12.7)\qquad v_\nu(x) \leq c_{13}(n\nu)^{-\frac{1}{2}} + c_{14}\, n^{-\frac{1}{2}}$$

For $-1 < x_\nu < x$, we can prove (12.7) but starting from (8.7) and using (10.3) and (10.5). For $x = x_\nu$ the inequality is certainly true.

13. Estimation of the polynomials $u_\nu(x)$, In this § we shall prove the following

Lemma 13.1. For even n's ≥ 4

(a) $$u_1(x) \leq 5n^{3/2}, \quad u_n(x) \leq 5n^{3/2};$$

(b) $$u_\nu(x) \leq c_{23} \frac{n}{\nu} + c_{24} \frac{n}{\nu^2}, \quad 2 \leq \nu \leq \frac{n}{2}$$

$$u_\nu(x) < c_{23} \frac{n}{n-\nu} + c_{24} \frac{n}{(n-\nu)^2}, \quad \frac{1}{2} n \leq \nu \leq n-1;$$

(c) $$\sum_{\nu=1}^{n} |u_\nu(x)| \leq c_{25}\, n^{3/2},$$

hold uniformly in $-1 \leq x \leq 1$.

Proof. (a) From (5.1) and (5.12) we have on using (9.4)

$$|u_1(x)| \leq 1 + (1-x^2)\, |P'_{n-2}(x)| + (1-x^2)\, P^2_{n-2}(x)\, |k_6 \int_{-1}^{x} P_{n-2}(t)\, dt + k_7|$$

But from (5.3) and (5.4) we have for k_6 and k_7 the estimations

$$|k_6| \leq \frac{n^4}{8} \quad \text{and} \quad |k_7| \leq \frac{5}{4} n^2.$$

Therefore by (9.1), (9.3) and (10.1) we have

$$u_1(x) \leq 1 + \frac{2\sqrt{n}}{\sqrt{\pi}} + \frac{32}{\pi n}\left[\frac{n^4}{4\sqrt{\pi}(n-2)^{3/2}} + \frac{5}{4}n^2\right] < 5n^{3/2}.$$

The same follows for $u_n(x)$.

We now prove the part (b) of the above lemma. As before it is sufficient to prove the first assertion. Let first be $x < x_\nu < 1$, then using the form (8.11) for $u_\nu(x)$ we have

$$u_\nu(x) \leq |\lambda_\nu(x)|^3 + \frac{3}{1-x_\nu^2}\ (x) + \frac{\omega(x)}{(1-x_\nu^2)P'^3_{n-2}(x_\nu)} \int_{-1}^{x} \frac{h_{n+1}(t)}{(t-x_\nu)^3}\, dt$$

(13.1)
$$+ \frac{\omega(x)\, |k_{10}|}{(1-x_\nu^2)^3\, P'^3_{n-2}(x_\nu)} \int_{-1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\, dt + \frac{\omega(x)\ k_{11}}{(1-x_\nu^2)^3\, P'^3_{n-2}(x_\nu)} \int P_{n-2}(t)\, dt$$

$$+ \frac{\omega(x)\, |k_{12}|}{(1-x_\nu^2)^3\, |P'^3_{n-2}(x_\nu)|} = J_9 + J_{10} + J_{11} + J_{12} + J_{13}\ \ J_{14}.$$

The estimation of these J's is similar to that in the § 12. We shall not give the details of the working and only write the results. Thus

(13.2)
$$J_{14} = \frac{\omega(x)\ k_{12}}{(1-x_\nu^2)^3\, |P'_{n-2}(x_\nu)|^3} \leq c_{16}\frac{n}{\nu^2}$$

(13.3)
$$J_{13}\ \ \frac{\omega(x)\ k_{11}}{(1-x_\nu^2)^3\ P'_{n-2}(x_\nu)^3}\left|\int_{-1}^{x} P_{n-2}(t)\, dt\right| \leq c_{17}\frac{n^{3/2}}{\nu^2}$$

$$J_{12} = \frac{\omega(x)\ k_{10}}{(1-x_\nu^2)\ P'_{n-2}(x_\nu)^3} \int_{-1}^{x} \frac{P_{n-2}(t)}{t-x_\nu}\, dt$$

(13.4)
$$\leq c_{18}\sqrt{n}\,\frac{(1-x^2)\,P^2_{n-2}(x)}{(1-x_\nu^2)\,|P'^3_{n-2}(x_\nu)\ \ x-x_\nu)}$$

$$\leq c_{18}\sqrt{n}\,\frac{(1-x^2)^{\frac{1}{2}}\,|P_{n-2}(x)|}{(1-x_\nu^2)^{5/2}\,P'^2_{n-2}(x_\nu)} < c_{19}\frac{n}{\nu^2}.$$

For

$$J_{11} = \frac{\omega(x)}{(1-x_\nu^2)^3\ P'_{n-2}(x_\nu)^3} \int_{-1}^{x} \frac{h_{n+1}(t)}{(t-x_\nu)^3}\, dt$$

we have from (10.6) and (9.6),

$$J_{11} \leq \frac{(1-x^2)^2 |P_{n-2}(x)|^3}{3(1-x_\nu^2)^2 |P'_{n-2}(x_\nu)|^3 |x-x_\nu|^3} + \frac{2\sqrt{n}(1-x^2)P_{n-2}^2(x)}{(1-x_\nu^2)^3 |P'^3_{n-2}(x_\nu)| (x-x_\nu)^2}$$

$$+ \frac{20\sqrt{n}(1-x^2)P_{n-2}^2(x)}{(1-x_\nu^2)^3 |P'^3_{n-2}(x_\nu)| |x-x_\nu|^3} + \frac{n^2(1-x^2)P_{n-2}^2(x)}{(1-x_\nu^2)^3 |P'^3_{n-2}(x_\nu)| |x-x_\nu|}$$

$$+ \frac{2(1-x^2)P_{n-2}^2(x)}{(1-x_\nu^2)^4 |P'^3_{n-2}(x_\nu)|} < \frac{1}{3\sqrt{1-x_\nu^2}} + \frac{2\sqrt{n}}{(1-x_\nu^2)|P'_{n-2}(x_\nu)|}$$

$$+ \frac{20\sqrt{n}\sqrt{1-x^2}\,|P_{n-2}(x)|}{(1-x_\nu^2)^{5/2} P'^2_{n-2}(x_\nu)} + \frac{n^2\sqrt{1-x^2}\,|P_{n-2}(x)|}{(1-x_\nu^2)^{3/2} P'^2_{n-2}(x_\nu)} + \frac{2(1-x^2)P_{n-2}^2(x)}{(1-x_\nu^2)^4 |P'_{n-2}(x_\nu)|^3}.$$

Hence from (9.1), (9.9) and (9.10) we get

$$J_{11} < c_{20}\sqrt{n} + c_{21}\frac{n}{\nu}. \tag{13.5}$$

For

$$J_{10} = \frac{3}{(1-x_\nu^2)} |v_\nu(x)|$$

we have from the first part of (a) of lemma 12.1, the inequality

$$J_{10} < c_{22}\frac{n^{3/2}}{\nu^2}. \tag{13.6}$$

Curious is the estimation of

$$J_9 = |\lambda_\nu(x)|^3.$$

We write, owing to (2.16)

$$\lambda_\nu^3(x) = \frac{(1-x^2)^3 P_{n-2}^3(x)}{(1-x_\nu^2)^3 P'^3_{n-2}(x_\nu)(x-x_\nu)^3}$$

$$= \frac{(1-x^2)^2 P_{n-2}^3(x)}{(1-x_\nu^2)^2 P'^3_{n-2}(x_\nu)(x-x_\nu)^3} - \frac{(1-x^2)^2 P_{n-2}^3(x)(x+x_\nu)}{(1-x_\nu^2)^3 P'^3_{n-2}(x_\nu)(x-x_\nu)^2}.$$

Hence

$$|\lambda_\nu^3(x)| \leq \frac{(1-x^2)^2 |P_{n-2}(x)|^3}{(1-x_\nu^2)^2 |P'^3_{n-2}(x_\nu)| |x-x_\nu|^3} + \frac{(1-x^2)^2 |P_{n-2}^3(x)| |x+x_\nu|}{(1-x_\nu^2)^3 |P'_{n-2}(x_\nu)|^3 (x-x_\nu)^2},$$

and by (9.6) etc. we have

$$|\lambda_\nu^3(x)| \leq \sqrt{\frac{1-x^2}{1-x_\nu^2}} + \frac{2(1-x^2)|P_{n-2}(x)|}{(1-x_\nu^2)|P'_{n-2}(x_\nu)|} < \frac{n}{\nu} + c_{22}\, n^{3/2} . \nu^{-5/2}. \tag{13.7}$$

Thus from (13.1), (13.2), ..., (13.7) we have for $x < x_\nu < 1$ the estimation

$$|u_\nu(x)| \leq c_{23}\frac{n}{\nu} + c_{24}\frac{n^{3/2}}{\nu^2} \tag{13.8}$$

For $-1 < x_\nu < x$, we start with (8.12) for $u_\nu(x)$ and using the estimations (10.3) and (10.7) instead of (10.2) and (10.6). At $x = x_\nu$, (13.8) is obvious owing to the property of $u_\nu(x)$.

To prove (c) we have from (a) and (b)

$$\sum_{\nu=1}^{n} |u_\nu(x)| \leq 10n^{3/2} + c_{23} \left\{ \sum_{\nu=2}^{n/2} \frac{n}{\nu} + \sum_{\nu=n/2+1}^{n-1} \frac{n}{n-\nu} \right\}$$

$$+ c_{24} \left\{ \sum_{\nu=2}^{n/2} \frac{n^{3/2}}{\nu^2} + \sum_{\nu=n/2+1}^{n-1} \frac{n^{3/2}}{(n-\nu)^2} \right\}$$

which by Schwarz's inequality gives

$$\sum_{\nu=1}^{n} |u_\nu(x)| \leq c_{25} n^{3/2}.$$

14. The approximating polynomial $\varrho_n(x)$. We shall need the following

Lemma 14.1. If a function $f(x)$ is differentiable in $[-1,1]$ and $f(x) \in \mathrm{Lip}\,\alpha\,(0 < \alpha < 1)$ then there is a polynomial $\varrho_n(x)$ of degree at most n possessing the following properties:

(a) $$|f(x) - \varrho_n(x)| \leq c_{26} n^{-1-\alpha} \{(\sqrt{1-x^2})^{\alpha+1} + n^{-1-\alpha}\},$$

(b) $$|f'(x) - \varrho_n'(x)| \leq c_{27} n^{-\alpha} \{(\sqrt{1-x^2})^{\alpha} + n^{-\alpha}\}$$

and

(c) $$|\varrho_n'(x)| \leq c_{28} n^{-\alpha}$$

uniformly in $-1 \leq x \leq 1$.

The parts (a) and (b) of the above lemma have been proved in lemma 9.1 of our paper [12] while (c) is a consequence of (a). We shall in the following prove the part (c). We shall follow the same method of proof as for the lemma 9.1 in [12].

We put[9]

$$n_0 = n \quad n_1 = \left[\frac{n_0}{2}\right], \dots, n_{j+1} = \left[\frac{n_j}{2}\right], \dots, n_r = 1,$$

$$r = \left[\frac{\log n}{\log 2}\right] + 1.$$

Then we have

$$(14.1) \qquad \varrho_n(x) = \sum_{j=1}^{r-1} [\varrho_{n_j}(x) - \varrho_{n_{j+1}}(x)] + \varrho_1(x).$$

[9] G. Freud, [5].

From (a) we have

(14.2) $$|\varrho_{n_j}(x)-\varrho_{n_{j+1}}(x)|\leq\frac{c_{29}}{n_j^{a+1}}\left[(\sqrt{1-x^2})^{a+1}+\frac{1}{n^{a+1}}\right].$$

Now using the inequality of Dzyadik[10] we have from (14.2),

(14.3) $$|\varrho'_{n_j}(x)-\varrho'_{n_{j+1}}(x)|\leq c_{30}\, n_j^{-a}[(\sqrt{1-x^2})^a+n_j^{-a}]$$
$$\leq c_{30}\left(\frac{n}{2^j}\right)^{-a}\left[(\sqrt{1-x^2})^a+\left(\frac{n}{2^j}\right)^{-a}\right].$$

Hence from (4.1)

$$|\varrho'_n(x)|\leq\sum_{j=0}^{r-1}|\varrho'_{n_j}(x)-\varrho'_{n_{j+1}}(x)|$$
$$\leq c_{31}\, n^{-a}\left[(\sqrt{1-x^2})^a+n^{-a}\right]\leq c_{28}\, n^{-a}.$$

From (c) on using Bernstein's inequality we have for $-1<x<1$,

(d) $$|\varrho'''_n(x)|<c_{32}\frac{n^2}{1-x^2},$$

while Markov's inequality gives

(e) $$|\varrho'''_n(\pm 1)|<c_{33}\, n^{4-a}.$$

15. The main proof of the theorem. We follow the usual technique of proofs [2, 9, 11]. There holds the relation:

$$R_n(x,f)-f(x)=R_n(x,f-\varrho_n)+\varrho_n(x)-f(x)$$

which owing to (6.2) gives

$$|R_n(x;f)-f(x)|\leq|f(x)-\varrho_n(x)|+\left|\sum_{\nu=1}^{n}[f(x_{\nu n})-\varrho_n(x_{\nu n})]\,u_{\nu n}(x)\right.$$
$$+\sum_{\nu=2}^{n-1}[f'(x_{\nu n})-\varrho'_n(x_{\nu n})]\,v_{\nu n}(x)+\left.\sum_{\nu=1}^{n}[\gamma_{\nu n}-\varrho'''_n(x_{\nu n})]\,w_{\nu n}(x)\right|$$
$$\leq|f(x)-\varrho_n(x)|+\sum_{\nu=1}^{n}|u_{\nu n}(x)|\max_{-1\leq u\leq 1}|f(u)-\varrho_n(u)|$$
$$+\sum_{\nu=2}^{n-1}|v_{\nu n}(x)|\max_{-1\leq v\leq 1}|f'(v)-\varrho'_n(v)|+\varepsilon\, n^{3/2}\sum_{\nu=1}^{n}|w_{\nu n}(x)|+\sum_{\nu=1}^{n}|\varrho'''_n(x_{\nu n})|\,|w_{\nu n}(x)|$$

Owing to the part (a) of lemma 14.1 and the part (c) of lemma 13.1, the first two terms in (15.1) tend to zero uniformly for $-1\leq x\leq 1$; so does the third term in consequence of the part (b) of lemma 14.1 and the part

[10] Dzyadyk, V. K., [3].

(b) of lemma 12.1. The fourth term similarly tends to zero owing to part (c) of lemma 11.1. We now estimate the last sum in (15.1)

$$\sum_{\nu=1}^{n} \varrho_n'''(x_{\nu n}) \, w_{\nu n}(x) \equiv S. \tag{15.2}$$

We write owing to (6.1)

$$S = \varrho_n'''(1) \quad w_{1n}(x) + \varrho_n'''(-1) \quad w_{nn}(x)$$
$$+ \sum_{\nu=2}^{n-1} \varrho_n'''(x_{\nu n}) \quad w_{\nu n}(x) \tag{15.3}$$

From the part (d) of lemma 14.1 and the part (a) of lemma 11.1 we have for the first two terms in (15.3) the upper bound $c_{33} n^{-\frac{1}{2}-\alpha}$.

To estimate the remaining sum

$$S_1 \quad \sum_{\nu=2}^{n-1} \varrho_n'''(x_{\nu n}) \quad w_{\nu n}(x) \tag{15.4}$$

we use the part (b) of lemma 14.1 and the part (b) of lemma 11.1. These give:

$$S_1 \leq \sum_{\nu=2}^{n-1} \frac{n}{1-x_{\nu n}^2} \quad \frac{\nu^2}{n^{\frac{9}{2}}} = c_{34} \sum_{\nu=2}^{n-1} \frac{\nu^2}{1-x_{\nu n}^2} \cdot n^{\ldots 2-\alpha}$$
$$< c_{34} \sum_{\nu=2}^{n-1} n^{-\frac{1}{2}-\alpha} \leq c_{35} n^{\frac{1}{2}-\alpha} \tag{15.5}$$

Thus (15.1) to (15.5) complete the proof of the theorem.

This work is part of the Research-programme of National Research Fellowship awarded to the author by the Government of India.

Received August 13, 1963

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