№. 1 . . . .

ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

On Critical Branching Migration Processes with Predominating Emigration

George P. Yanev   Nickolay M. Yanev

БЪЛГАРСКА
АКАДЕМИЯ
НА НАУКИТЕ

1869

BULGARIAN
ACADEMY
OF SCIENCES

PREPRINT

No 1

Department of Probability and Statistics

SOFIA, May 1993

On Critical Branching Migration Processes with Predominating Emigration

George P. Yanev* Nickolay M. Yanev*

Abstract

The branching migration processes generalize the classical Bienaymé - Galton - Watson process allowing a migration component in each generation: with probability p the offspring of one particle is eliminated (family emigration) or with probability q there is not any migration or with probability r a state-dependent immigration of new particles is available, $p + q + r = 1$. The processes stopped at zero are also considered. It is investigated the critical case when the migration mean in the non-zero states is negative (predominating emigration). The asymptotic behaviour of the life-period, the probability of non-extinction and moments is obtained and limit theorems are also proved.

branching migration; stopped at zero; life-period; extinction; moments; limit theorems
60J80

1 Introduction

In this paper we investigate branching processes (with discrete time and one type of particles) which development is not isolated and admits a random migration. These migration models are particular cases of so-called controlled branching processes where in general the evolutions of the particles are not independent.

Let us have on the probability space $(\Omega, \mathcal{F}, P)$ two independent sets of non-negative integer-valued random variables $\xi = \{\xi_{jk}(t)\}$ and $\varphi = \{\varphi_j(t, n)\}$. For each j the offsprings $\{\xi_{jk}(t)\}$ are i.i.d. and φ is the set of control functions. Then the general case of a controlled branching process can be defined as follows

$$Y_{t+1} = \sum_{j\in J} \sum_{k=1}^{\varphi_j(t,Y_t)} \xi_{jk}(t), \qquad t = 0, 1, 2, ..., \tag{1.1}$$

where J is an index set and $Y_0 \geq 0$ is independent of ξ and φ. As usual $\sum_{k=1}^{0} \cdot = 0$.

The model (1.1) describes a very large class of stochastic processes (for example, all Markov chains). Particular cases are classical Bienaymé - Galton - Watson processes for which $J = \{1\}$

*Postal address for both authors: Department of Probability and Statistics, Institute of Mathematics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria.

Supported by National Fondation of Scientific Investigations, grant MM-4/91.

and $\varphi_1(t,n) \equiv n$ a.s., branching processes with immigraion where $J = \{1,2\}$, $\varphi_1(t,n) \equiv n$ a.s., and $\varphi_2(t,n) \equiv 1$ a.s. or Foster-Pakes model with $J = \{1,2\}, \varphi_1(t,n) \equiv n$ a.s. and $\varphi_2(t,n) \equiv max(0, 1-n)$ a.s. (see Athreya and Ney(1972)).

The controlled branching processes (1.1) are introduced and investigated by Sevastyanov and Zubkov(1974), Zubkov(1974) and Bagley(1986) in the case of deterministic φ. N.M.Yanev (1975) investigate the case with random φ, N.M.Yanev(1977) and G.P.Yanev and N.M.Yanev (1990) - with random φ and in random environments.

In the present paper, we consider the special case of (1.1) with $J = \{1,2,3\}$ and

$$(1.2) \qquad \begin{cases} \varphi_1(t,n) = max\{min\{n, n+\eta_t\}, 0\}, \\ \varphi_2(t,n) = max\{0, \eta_t\}(1-\delta_{n0}), \\ \varphi_3(t,n) = max\{0, \eta_t\}\delta_{n0}, \end{cases}$$

where δ_{n0} is the Kroneker delta and $\{\eta_t\}$ are i. i. d. r. v. with distribution

$$P(\eta_t = -1) = p, \quad P(\eta_t = 0) = q, \quad P(\eta_t = 1) = r, \quad p+q+r = 1.$$

Let $\{X_k(t)\}$, $\{I(t)\}$ and $\{I_0(t)\}$ are three independent sets of i. i. d. r. v. such that $X_k(t) = \xi_{1k}(t)$, $I(t) = \xi_{21}(t)$ and $I_0(t) = \xi_{31}(t)$. Then by (1.1) and (1.2) it follows the representation:

$$(1.3) \qquad Y_{t+1} = \sum_{k=1}^{Y_t} X_k(t) + M_t, \qquad t = 0,1,2,...,$$

where

$$M_t = \begin{cases} -X_1(t)1_{\{Y_t > 0\}} & \text{with probability p,} \\ 0 & \text{with probability q,} \\ I(t)1_{\{Y_t > 0\}} + I_0(t)1_{\{Y_t = 0\}} & \text{with probability r,} \qquad \text{p+q+r=1,} \end{cases}$$

Further on we will suppose that $Y_0 = 0$ a.s. (this is not an essential restriction).

Obviously, the process $\{Y_t\}$ is an homogeneous Markov chain which admits the following interpretation. In each generation t three situations are possible: with probability q the process develops like a Bienaymé - Galton - Watson branching process (without any migration) or with probability p the offspring of one particle is eliminated and does not take part in further evolution (family emigration) or with probability r in the next $(t+1)$-th generation there is an immigration of $I(t)$ or $I_0(t)$ new particles depending on the state of the process.

Let $F(s) = Es^{X_k(t)}, G(s) = Es^{I(t)}$ and $H(s) = Es^{I_0(t)}$. Then (1.3) is equivalent to the following definition: for $n, t = 0, 1, 2, \ldots$

$$(1.4) \qquad E(s^{Y_{t+1}} \mid Y_t = n) = F^n(s)[pF^{-1}(s) + q + rG(s)](1-\delta_{n0}) + [p+q+rH(s)]\delta_{n0}.$$

The definition (1.3) (or(1.4)) contains as particular cases some earlier investigated models. For example, if $q = 1$ then $\{Y_t\}$ is the classical Bienaymé - Galton - Watson process. If $r = 1$ and $I(t) \equiv I_0(t)$ a.s. then $\{Y_t\}$ will be the well-known branching process with immigration (see Athreya and Ney(1972)). The case $r = 1$ and $I(t) \equiv 0$ a.s. was considered by Foster(1971) and Pakes(1971). The process with $p = 1$ (proper emigration) is investigated by Vatutin(1977a) and Kaverin(1990). A new particular case is when $I(t) \equiv 0$ a.s. , i.e. the process with emigration which admits an immigration component only in the state zero.

The process (1.3) with $I(t) \equiv I_0(t)$ is introduced and investigated in some cases by N.Yanev and K.Mitov (1980, 1981, 1983, 1984). They considered also some processes with non-homogeneous migration when $p = p_t, q = q_t, r = r_t$ (see N.Yanev and K.Mitov(1985)).

Note that a similar model with homogeneous migration is investigated by Nagaev and Han(1980), Han(1980) and Kaverin and Atamatov(1988). A migration process with another type of emigration is announced by Grey(1988) and also by G.Yanev and N.Yanev(1991).

In this paper, we will also study the process stopped at zero which is defined as follows:

$$(1.5) \qquad Z_{t+1} = \sum_{k=1}^{Z_t} X_k(t) + M_t^0 \qquad , t = 0, 1, 2, ...,$$

where

$$M_t^0 = \begin{cases} -X_1(t) 1_{\{Z_t > 0\}} & \text{with probability p,} \\ 0 & \text{with probability q,} \\ I(t) 1_{\{Z_t > 0\}} & \text{with probability r,} \qquad \text{p+q+r=1,} \end{cases}$$

and Z_0 is independent of $\{Z_t\}, t > 0$.

Note that (1.5) is equivalent to the following definition:

$$(1.6) \quad E(s^{Z_{t+1}} \mid Z_t = n) = F^n(s)[pF^{-1}(s) + q + rG(s)](1 - \delta_{n0}), \qquad n, t = 0, 1, 2, \ldots$$

Note that the state zero is a reflexing barrier for $\{Y_t\}$ and an absorbing state for $\{Z_t\}$.

The process $\{Z_t\}$ is studied in some cases by N.Yanev and K.Mitov(1983), N.Yanev, V.Vatutin and K.Mitov(1986) and K.Mitov(1990). The particular cases of (1.5) (or 1.6) when $r = 1$ are considered by Zubkov(1972), Vatutin(1977b), Seneta and Tavare(1983) and Ivanoff and Seneta(1985).

In this paper, we investigate the processes $\{Y_t\}$ and $\{Z_t\}$ in the critical case $EX_k(t) = 1$ when the random migration is with the so-called predominating emigration (this will be precised in the Section 2 where are given all main results). In the last case the asymptotic behaviour of the life-period of $\{Y_t\}$ and the probability of non-extinction of $\{Z_t\}$ is obtained. The asymptotics of some moments are investigated and limit theorems for both processes are also proved.

Note that some of results with $I(t) \equiv I_0(t)$ are announced in G.Yanev and N.Yanev(1989).

2 Equations and basic results

For the processes (1.3) and (1.5) (or (1.4) and (1.6)) we will use the following basic notations ($\mid s \mid \leq 1$):

$$\Phi(t, s) = Es^{Y_t}, \qquad \Psi(t, s) = Es^{Z_t};$$

$$F(s) = Es^{X_k(t)} = \sum_{i=0}^{\infty} f_i s^i; \quad F_n(s) = F(F_{n-1}(s)), \quad n = 1, 2, \ldots; \quad F_0(s) = s; \quad F_n = F_n(0);$$

$$G(s) = Es^{I(t)} = \sum_{i=0}^{\infty} g_i s^i; \quad H(s) = Es^{I_0(t)} = \sum_{i=0}^{\infty} h_i s^i; \quad Q(s) = Es^{Z_0} = \sum_{i=1}^{\infty} q_i s^i;$$

$$\delta(s)=\frac{p}{F(s)}+q+rG(s)=\sum_{i=1}^{\infty}\delta_i s^i;\quad \gamma_t(s)=\prod_{j=0}^{t-1}\delta(F_j(s)),\quad \gamma_0(s)\equiv 1,\gamma_{-1}(s)\equiv 0,\gamma_t=\gamma_t(0).$$

One can obtain the following representations for $\delta(s)$:

$$\delta(s)=\frac{1}{F(s)}[p+qF(s)+rG(s)F(s)] \tag{2.1}$$

and for every $n\geq 0$

$$\begin{aligned}\delta(s) &= 1-r(1-G(s))+p\textstyle\sum_{i=1}^{n}(1-F(s))^i+p(1-F(s))^{n+1}/F(s)\\ &= 1-r(1-G(s))+p\textstyle\sum_{i=1}^{\infty}(1-F(s))^i.\end{aligned} \tag{2.2}$$

Further on we will suppose that:

$$\begin{cases}F'(1)=1, \quad 0<F''(1)=2b<\infty,\\ 0<G'(1)=\lambda<\infty, \quad 0<H'(1)=\mu<\infty, \quad 0<Q'(1)=\alpha<\infty,\\ \theta=\dfrac{\delta'(1)}{b}=\dfrac{r\lambda-p}{b}=\dfrac{E(M\mid Y>0)}{\frac{1}{2}\mathrm{Var}X}=\dfrac{E(M^0\mid Z>0)}{\frac{1}{2}\mathrm{Var}X}<0.\end{cases} \tag{2.3}$$

By (2.3) it is clear that we consider only the critical case, i.e. when the offspring mean $F'(1)$ is 1. In this case $\delta'(1)=r\lambda-p$ can be interpreted as the expectation of the migration in the non-zero states. Since $\delta'(1)<0$, i.e. in each generation the immigration mean $r\lambda$ is less than the emigration mean p, then one can say that the emigration is predominating.

The critical case with $\theta\geq 0$ (and $I(t)\equiv I_0(t)$) was investigated by N.Yanev and K.Mitov(1980, 1981, 1983), N.Yanev, V.Vatutin and K.Mitov(1986) and K.Mitov(1990). Nagaev and Han(1980) proved a limit theorem for $\{Y_t\}$ in the case $\theta=0$.

The critical case with $\theta<0$ seems to be more difficult and it was an open problem till now.

By definitions (1.3) and (1.5) (or (1.4) and (1.6)) one can obtain the functional equations for the p.g.f. of the processes $\{Y_t\}$ and $\{Z_t\}$:

$$\Phi(t+1,s)=\Phi(t,F(s))\delta(s)+\Phi(t,0)[1-\delta(s)-r(1-H(s))] \tag{2.4}$$

and

$$\Psi(t+1,s)=\Phi(t,F(s))\delta(s)+\Psi(t,0)[1-\delta(s)]. \tag{2.5}$$

Now iterating (2.4) and (2.5) one obtains for each $t=0,1,\dots$

$$\begin{aligned}\Phi(t+1,s) &= \Phi(0,F_{t+1}(s))\gamma_{t+1}(s)+\textstyle\sum_{k=0}^{t}\Phi(t-k,0)\gamma_k(s)[1-\delta(F_k(s))-\\ &\quad -r(1-H(F_k(s)))]\end{aligned} \tag{2.6}$$

and

$$\Psi(t+1,s)=\Psi(0,F_{t+1}(s))\gamma_{t+1}(s)+\sum_{k=0}^{t}\Psi(t-k,0)\gamma_k(s)[1-\delta(F_k(s))]. \tag{2.7}$$

For the process $\{Y_t\}$ we define the life-period $\tau=\tau(T)$ starting at the moment $T>0$ by the condition:

$$\tau=\inf\{n:\quad Y_{T-1}=0,\quad Y_{T+i}>0,\quad 0\leq i<n,\quad Y_{T+n}=0\}.$$

If $Z_0 \stackrel{d}{=} Y_T$ then by (1.3) and (1.5) it follows that

$$Z_t \stackrel{d}{=} Y_{T+t} 1_{\{Z_{t-1} > 0\}}, \qquad t \geq 1.$$

Therefore if

(2.8) $$Q(s) = (H(s) - h_0)/(1 - h_0),$$

then we have

(2.9) $$u_t = P(\tau > t) = P(Z_t > 0) = 1 - \Psi(t, 0).$$

Let now $\Psi_t = \Psi(t,0)$ and $U(s) = \sum_{t=0}^{\infty} u_t s^t, \quad |s| \leq 1$. Then from (2.7) and (2.9) it follows

(2.10) $$\Psi_t = Q(F_t)\gamma_t + \sum_{k=1}^{t} \Psi_{t-k}(\gamma_{k-1} - \gamma_k)$$

(2.11) $$u_t = (1 - Q(F_t))\gamma_t + \sum_{k=1}^{t} u_{t-k}(\gamma_{k-1} - \gamma_k).$$

Hence from (2.10) and (2.11) one obtains (see also (2.9))

(2.12) $$\begin{aligned} \sum_{t=0}^{\infty} \Psi_t s^t &= \frac{\sum_{t=0}^{\infty} \gamma_t Q(F_t) s^t}{1 - \sum_{t=0}^{\infty} (\gamma_{t-1} - \gamma_t) s^t} = \frac{\sum_{t=0}^{\infty} \gamma_t Q(F_t) s^t}{(1-s)\sum_{t=0}^{\infty} \gamma_t s^t} \\ &= \frac{1}{1-s} - \frac{\sum_{t=0}^{\infty} \gamma_t (1 - Q(F_t)) s^t}{(1-s)\sum_{t=0}^{\infty} \gamma_t s^t} = \frac{1}{1-s} - U(s). \end{aligned}$$

The asymptotic behaviour of the processes depends on the range of θ and on the conditions:

(2.13) $$-1 < \theta < 0, \qquad \sum_{k=2}^{\infty} f_k k^2 \log k < \infty, \qquad \sum_{k=1}^{\infty} g_k k^{1-\theta} < \infty;$$

(2.14) $$\theta = -1, \qquad \sum_{k=2}^{\infty} f_k k^2 \log^2 k < \infty, \qquad \sum_{k=2}^{\infty} g_k k^2 \log^2 k < \infty;$$

(2.15) $$\theta < -1, \qquad \sum_{k=1}^{\infty} f_k k^{1-\theta} < \infty, \qquad \sum_{k=1}^{\infty} g_k k^{1-\theta} < \infty;$$

(2.16) $$-1 < \theta < 0, \qquad \sum_{k=2}^{\infty} f_k k^2 < \infty, \qquad \sum_{k=1}^{\infty} g_k k^{1-\theta} < \infty,$$

(2.17) $$\begin{aligned} -1 < \theta < 0, &\qquad \sum_{k=2}^{\infty} h_k k^2 < \infty, \\ \theta \leq -1, &\qquad \sum_{k=1}^{\infty} h_k k^{1+|-\theta|} < \infty, \end{aligned}$$

(2.17a) $$\theta \leq -1, \qquad \sum_{k=1}^{\infty} h_k k^{1-\theta} < \infty.$$

Now we can represent the following basic results.

Theorem 2.1 *Under the conditions (2.3)*

$$E\tau = -\frac{\mu}{b\theta[1-H(0)]} = -\frac{E(I_0 \mid I_0 > 0)}{E(M \mid Y > 0)} = -\frac{EZ_0}{b\theta}.$$

(i) If additionally assume (2.13)-(2.17) then

$$u_t \sim ct^{\theta-1} \qquad c > 0, \qquad t \to \infty. \tag{2.18}$$

(ii) If one assume only (2.16) and (2.17) then (2.18) holds where c is replaced by a s.v.f. $L(t)$.

(iii) Let $\theta < -1, \sum_{k=1}^{\infty} g_k k^{1-\theta} < \infty$ and (2.18) holds for every initial p.g.f. $H(s)$. Then

$$\sum_{k=1}^{\infty} f_k k^{1-\theta} < \infty.$$

Remark. From now on c_i denote some positive constants.

Comment. In (2.18) the rate of convergence is faster than in the Bienaymé - Galton - Watson process ($q = 1, p = r = 0, \theta = 0$) for which $P(Z_t > 0) \sim \frac{EZ_0}{b} t^{-1}$.

Theorem 2.2 *Under the conditions (2.3)*

$$\lim_{t\to\infty} P(Y_t = 0) = \frac{b\theta}{b\theta - r\mu} = \frac{E(M \mid Y > 0)}{E(M \mid Y > 0) - E(M \mid Y = 0)} < 1. \tag{2.19}$$

If additionally suppose (2.13)-(2.17) then as $t \to \infty$

$$EY_t \sim \begin{cases} c_1 t^{1+\theta} & , \textit{for} \quad -1 < \theta < 0, \\ c_2 \log t & , \textit{for} \quad \theta = -1, \\ c_3 \sum_{n=1}^{\infty} n^{\theta} & , \textit{for} \quad \theta < -1. \end{cases} \tag{2.20}$$

Corollary 2.1 *If one assume only (2.16) and (2.17) then (2.20) still holds with a s.v.f. $L(t)$ instead of the constant $c_1 > 0$.*

Theorem 2.3 *Under the conditions (2.3) there exists a stationary distribution*

$$v_k = \lim_{t\to\infty} P(Y_t = k), \qquad \sum_{k=0}^{\infty} v_k = 1, \tag{2.21}$$

whose p.g.f. $V(s) = \sum_{k=0}^{\infty} v_k s^k, \mid s \mid \leq 1$, is the unique solution of the functional equation

$$V(s) = V(F(s))\delta(s) + \frac{b\theta}{b\theta - r\mu}[1 - \delta(s) - r(1 - H(s))], \tag{2.22}$$

with the initial condition

$$V(0) = b\theta/(b\theta - r\mu). \tag{2.23}$$

Assume (2.13)–(2.17). Then as $s \uparrow 1$

$$1 - V(s) \sim \begin{cases} c_1(1-s)^{-\theta} & , \textit{for} \quad -1 < \theta < 0, \\ c_2(1-s)\log\dfrac{1}{1-s} & , \textit{for} \quad \theta = -1, \\ c_3(1-s) & , \textit{for} \quad \theta < -1. \end{cases} \tag{2.24}$$

If $-1 < \theta < 0$ *then*

$$V(s) = \lim_{t\to\infty}\left\{\gamma_t(s) + \frac{b\theta}{b\theta - r\mu}\sum_{k=0}^{t-1}\gamma_k(s)[1 - \delta(F_k(s)) - r(1 - H(F_k(s)))]\right\}. \tag{2.25}$$

Corollary 2.2 *If one assume only (2.16)and (2.17), then (2.24) still holds with a s.v.f. $L(t)$ instead of the constant $c_1 > 0$.*

Comment. In the case $-1 < \theta < 0$ it follows by (2.24) and Corollary 2.2 that the stationary distribution $\{v_k\}$ belongs to the normal domain of attraction of a stable law with a parameter $(-\theta)$ (i.e. $\sum_{k=t}^{\infty} v_k \sim ct^{\theta}$) and if $\theta < -1$ then $V'(1) < \infty$.

Theorem 2.4 *Under conditions (2.3) and (2.13)–(2.15)*

$$EZ_t \sim cbt^{\theta}, \qquad t \to \infty, \tag{2.26}$$

where the constant $c > 0$ has to be replaced by a s.v.f. $L(t)$ when only (2.16)and (2.17) hold.

If additionally assume

(i) for $m =]-\theta[$: $EZ_0^{m+1} < \infty$, $F^{(m+1)}(1) < \infty$ *and* $G^{(m+1)}(1) < \infty$, θ-noninteg

(ii) for $m \geq]-\theta[+1$: $EZ_0^{m} < \infty$, $F^{(m)}(1) < \infty$ *and* $G^{(m)}(1) < \infty$,

then as $t \to \infty$

$$EZ_t^m \sim cm!b^m t^{m+\theta-1}, \qquad m = 2, 3, \ldots, \quad c > 0. \tag{2.27}$$

Comment. It is shown in the proof of the theorem that for $1 \leq m \leq [-\theta]$, the positive constant c (or s.v.f. $L(t)$) in (2.26) and (2.27) is the same as in Theorem 2.1, i.e. $c = \lim_{t\to\infty} P(Z_t > 0)t^{1-\theta}$.

Theorem 2.5 *Assume (2.3) and (2.14)–(2.17a).Then as $t \to \infty$*

$$E(Z_t \mid Z_t > 0) \sim bt \tag{2.28}$$

and

$$\lim_{t\to\infty} P\left(\frac{Z_t}{bt} \leq x \mid Z_t > 0\right) = 1 - e^{-x}, \qquad x \geq 0. \tag{2.29}$$

Comment. Although the rate of convergence (2.18) of $P(Z_t > 0)$ is faster than in the Bienaymé - Galton - Watson process, the limit results (2.28) and (2.29) on the non-extinction paths are the same for the both processes.

It is interesting to compere the presented here results with those obtained in the critical case with $\theta \geq 0$ and $I(t) \equiv I_0(t)$ (see N.Yanev and K.Mitov(1979, 1981, 1983) and N.Yanev, V.Vatutin and K.Mitov(1986)). The constants c_i below are positive and are calculated under some additional conditions:

$$u_t \sim \begin{cases} c_1, \quad 0 < c_1 < 1 & , \quad \theta > 1, \\ c_2/\log t & , \quad \theta = 1, \\ c_3 t^{\theta-1} & , \quad 0 \leq \theta < 1, \end{cases}$$

$$EZ_t \sim \begin{cases} c_4 t & , \quad \theta > 1, \\ c_5 t/\log t & , \quad \theta = 1, \\ c_6 t^{\theta} & , \quad 0 \leq \theta < 1, \end{cases}$$

$$EY_t \sim \begin{cases} c_7 t & , \quad \theta > 0, \\ c_8 t/\log t & , \quad \theta = 0, \end{cases}$$

$$\lim_{t\to\infty} P\left(\frac{Z_t}{bt} \leq x \mid Z_t > 0\right) = \begin{cases} \frac{1}{\Gamma(\theta)} \int_0^x y^{\theta-1} e^{-y} dy & , \quad \theta > 1, \\ 1 - e^{-x} & , \quad 0 \leq \theta \leq 1, \end{cases}$$

$$\lim_{t\to\infty} P\left(\frac{Y_t}{bt} \leq x\right) = \frac{1}{\Gamma(\theta)} \int_0^x y^{\theta-1} e^{-y} dy, \theta > 0,$$

$$\lim_{t\to\infty} P\left(\frac{\log Y_t}{\log t} \leq x\right) = x, \qquad 0 < x < 1, \qquad \theta = 0.$$

The last limit theorem (for $\{Y_t\}$ with $\theta = 0$) was proved by Nagaev and Han(1980). Note that in this case the asymptotic behaviour of the process $\{Y_t\}$ is similar to the Bienaymé - Galton - Watson processes with immigration only in the state zero which was investigated in the critical case by Foster(1971).

3 Preliminaries

The results of this section are of independent interest and are also used in the following sections.

Lemma 3.1 *Assume conditions (2.3). Then for every $s \in [0,1)$*

(3.1) $$\gamma_t(s) \sim t^{-\theta} L(t,s) \qquad , \quad t \to \infty,$$

where $L(t,s)$ is a s.v.f. If additionally

(3.2) $$\sum_{k=2}^{\infty} f_k k^2 \log k < \infty \qquad \text{and} \qquad \sum_{k=2}^{\infty} g_k k \log k < \infty,$$

then $L(t,s) = c(s) > 0$.

Proof. From (2.1) it follows that

$$\gamma_t(s) = \prod_{j=0}^{t-1} \delta(F_j(s)) = \prod_{j=0}^{t-1} D(F_j(s)) / \prod_{j=0}^{t-1} F(F_j(s)), \tag{3.3}$$

where $D(s) = p+qF(s)+rF(s)G(s)$ is a p.g.f. with $0 < D'(1) = r\lambda+1-p < \infty$ and $D(0) > 0$.

On the other hand, for every $s \in [0,1)$ there is $k = k(s) \geq 0$ such that

$$F_k \leq s \leq F_{k+1}. \tag{3.4}$$

Then from (3.3) and (3.4) one obtains

$$\prod_{j=0}^{t-1} D(F_j(F_k)) / \prod_{j=0}^{t-1} F(F_j(F_{k+1})) \leq \gamma_t(s) \leq \prod_{j=0}^{t-1} D(F_j(F_{k+1})) / \prod_{j=0}^{t-1} F(F_j(F_k)). \tag{3.5}$$

Let now

$$\theta_1 = \frac{1}{b}\frac{d}{ds}D(F_{k+1}(s)) \mid_{s=1} = \frac{r\lambda - p + 1}{b}, \qquad \theta_2 = \frac{1}{b}F'_{k+1}(1) = \frac{1}{b}. \tag{3.6}$$

Then by Lemma 4 of Zubkov(1972) it follows that

$$\prod_{j=0}^{t-1} D(F_j(F_{k+1})) = \prod_{j=0}^{t-1} D(F_{k+1}(F_j)) \sim \mathrm{L}_1(t,s)t^{-\theta_1} \tag{3.7}$$

and

$$\prod_{j=0}^{t-1} F(F_j(F_k)) = \prod_{j=0}^{t-1} F_{k+1}(F_j)) \sim \mathrm{L}_2(t,s)t^{-\theta_2}, \tag{3.8}$$

where $L_i(t,s)$ are s.v.f. as $t \to \infty$, such that $L_i(t,s) = c_i(s) > 0$ under the conditions (3.2).

Now (3.1) follows from (3.5)–(3.8).

Note finally that Lemma 1 of N.Yanev and K.Mitov(1983) is a particular case of Lemma 3.1 under the condition $s = 0$.

Lemma 3.2 *Let for each* $n = 1, 2, \ldots$

$$R_n^\delta(1-s) = \delta(s) - 1 - \sum_{k=1}^{n}(-1)^k \frac{\delta^{(k)}(1)}{k!}(1-s)^k, \qquad R_0^\delta(1-s) = \delta(s) - 1. \tag{3.9}$$

(i) If $n-1 < -\theta < n$, $\sum_{k=1}^\infty f_k k^{1-\theta} < \infty$ *and* $\sum_{k=1}^\infty g_k k^{1-\theta} < \infty$, *then*

$$\sum_{k=1}^{\infty} k^{-\theta} \mid R_n^\delta(\frac{1}{k}) \mid < \infty. \tag{3.10}$$

(ii) If $\sum_{k=1}^\infty f_k k^{n+1} < \infty$ *and* $\sum_{k=1}^\infty g_k k^{n+1} < \infty$, *then*

$$\sum_{k=1}^{\infty} k^n R_n^\delta(\frac{1}{k}) s^k = O(\log \frac{1}{1-s}), \qquad s \uparrow 1. \tag{3.11}$$

(iii) If $\sum_{k=2}^\infty f_k k^n \log^2 k < \infty$ *and* $\sum_{k=2}^\infty g_k k^n \log^2 k < \infty$, *then*

$$\sum_{k=2}^{\infty} k^n R_n^\delta(\frac{1}{k}) s^k = o(\frac{1}{(1-s)\log(1-s)^{-1}}). \tag{3.12}$$

Proof. Let $R_n^F(1-s)$ and $R_n^G(1-s)$ are defined by (3.9) with $F(s)$ and $G(s)$ respectively instead of $\delta(s)$. Then applying (2.2) one obtains

$$
\begin{aligned}
R_n^{\delta}(1-s) &= -r(1-G(s)) + p\sum_{i=1}^{n}(1-F(s))^i + \frac{p}{F(s)}(1-F(s))^{n+1} - \\
&- \sum_{k=1}^{n}(-1)^k\frac{\delta^{(k)}(1)}{k!}(1-s)^k \\
&= r[R_n^G(1-s) + \sum_{i=1}^{n}(-1)^i\frac{G^{(i)}(1)}{i!}(1-s)^i] + \\
&+ p\sum_{i=1}^{n}(-1)^i[R_n^F(1-s) + \sum_{j=1}^{n}(-1)^j\frac{F^{(j)}(1)}{j!}(1-s)^j]^i + \frac{p}{F(s)}(1-F(s))^{n+1} - \\
&- p\sum_{k=1}^{n}\frac{(-1)^k}{k!}\frac{d^k}{ds^k}\left(\frac{1}{F(s)}\right)\Big|_{s=1}(1-s)^k - \sum_{k=1}^{m}(-1)^k\frac{G^{(k)}(1)}{k!}(1-s)^k.
\end{aligned}
\tag{3.13}
$$

From (3.13) applying Faá di Bruno's formula (see Abramowitz and Stegun(1970), p.823) one obtains as $s \uparrow 1$

$$
\begin{aligned}
R_n^{\delta}(1-s) &= rR_n^G(1-s) - pR_n^F(1-s)(1+o(1)) + O((1-s)^{n+1}) + \\
&+ p\sum_{i=1}^{n}(-1)^i\sum_{k=i}^{n}(-1)^k(1-s)^k{\sum}'\frac{i![F'(1)]^{a_1}\dots[F^{(n)}(1)]^{a_n}}{(1!)^{a_1}(a_1!)\dots(n!)^{a_n}(a_n!)} - \\
&- p\sum_{k=1}^{n}\frac{(-1)^k}{k!}(1-s)^k\sum_{i=1}^{k}(-1)^i(i!){\sum}''\frac{k![F'(1)]^{b_1}\dots[F^{(n)}(1)]^{b_n}}{(1!)^{b_1}(b_1!)\dots(k!)^{b_k}(b_k!)} \\
&= rR_n^G(1-s) - pR_n^F(1-s)(1+o(1)) + O((1-s)^{n+1}) + \\
&+ p\sum_{i=1}^{n}\sum_{i=1}^{k}(-1)^i(i!)(-1)^k(1-s)^k{\sum}'\frac{[F'(1)]^{a_1}\dots[F^{(n)}(1)]^{a_n}}{(1!)^{a_1}(a_1!)\dots(n!)^{a_n}(a_n!)} - \\
&- p\sum_{i=1}^{n}\sum_{i=1}^{k}(-1)^i(i!)(-1)^k(1-s)^k{\sum}''\frac{[F'(1)]^{b_1}\dots[F^{(n)}(1)]^{b_n}}{(1!)^{b_1}(b_1!)\dots(k!)^{b_k}(b_k!)},
\end{aligned}
\tag{3.14}
$$

where in $\sum'$ and $\sum''$ the sums are over $(a_1,\dots,a_n)$ and $(b_1,\dots,b_k)$ such that $\{a_l\},\{b_m\}$ are positive integers, $i \le k \le n$,

$$
\left\|\begin{array}{lcl} a_1 + a_2 + \dots + a_n &=& i \\ a_1 + 2a_2 + \dots + na_n &=& k \end{array}\right.
$$

and

$$
\left\|\begin{array}{lcl} b_1 + b_2 + \dots + b_k &=& i \\ b_1 + 2b_2 + \dots + kb_k &=& k \end{array}\right.
$$

It is not difficult to see that the solution sets of these two systems of equations are identical. Hence from (3.14) for every $n = 1, 2, \dots$ as $s \uparrow 1$ one gets

$$
R_n^{\delta}(1-s) = rR_n^G(1-s) - pR_n^F(1-s)(1+o(1)) + O((1-s)^{n+1}). \tag{3.15}
$$

Now from (3.15) using Lemmas 2–4 of Vatutin(1977a) one obtains the conclusions (3.10)–(3.12).

Comment. The Lemmas 2 and 3 of Vatutin(1977a) are only formulated without any proofs. The careful proof of Lemma 3 shows that the lemma is true under the condition (iii) instead of $\sum_{k=1}^{\infty} k^n g_k \log k < \infty$.

Corollary 3.1 *Assume* $\sum\limits_{k=1}^{\infty} g_k k^{1-\theta} < \infty$.

(i) If $n-1 < -\theta < n$, *and* $\sum\limits_{k=1}^{\infty} k^{-\theta} \mid R_n^\delta(\frac{1}{k}) \mid < \infty$, *then* $\sum\limits_{k=1}^{\infty} f_k k^{1-\theta} < \infty$.

(ii) If $\theta = -n$ *and* $\sum\limits_{k=1}^{\infty} k^{-\theta} R_n^\delta(\frac{1}{k}) s^k \sim c \log(1-s)^{-1}, s \uparrow 1$, *then* $\sum\limits_{k=1}^{\infty} f_k k^{1-\theta} < \infty$.

The proof follows from (3.15) applying again Lemmas 2 and 3 of Vatutin(1977a).

Further we will use also the following result.

Lemma 3.3 (Vatutin(1977a), Corollary 2) *If* $\sum\limits_{k=t+1}^{\infty} \alpha_k \sim t^{-\beta} L(t), \quad \alpha_t \downarrow 0, \beta > 0$, *then* $\alpha_t \sim c t^{-(1+\beta)} L(t)$, *where* $L(t)$ *is a s.v.f. as* $t \to \infty$ *and* c *is a positive constant.*

4 Migration processes with bounded immigration in the state zero

In this section we will consider the process (1.3) with bounded immigration in the state zero, i.e. for some $N \geq 1$

$$\tilde{I}_0(t) = I_0(t) 1_{\{I_0(t) < N\}} + N 1_{\{I_0(t) \geq N\}} \quad \text{a.s.}, \qquad t = 0, 1, 2, \ldots \tag{4.1}$$

Further we will assume also (2.8) which implies (2.9).

Note that in this case we will use the notations $\{\tilde{Y}_t\}$ and $\{\tilde{Z}_t\}$ for the processes (1.3) and (1.5). Also for the corresponding parameters we will use the same notations adding only " ~ ". If it is necessary we will point out the dependence on N. It is clear that some characteristics of $\{\tilde{Y}_t\}, \{\tilde{Z}_t\}$ and $\{Y_t\}, \{Z_t\}$ are identical. For example, $\tilde{\gamma}_t = \gamma_t$, $\tilde{\theta} = \frac{r\lambda - p}{b} = \theta$, etc.

Hence from (2.12) one has

$$\tilde{U}(s) = \sum_{t=0}^{\infty} \tilde{u}_t s^t = \frac{\sum_{t=0}^{\infty} \gamma_t (1 - \tilde{Q}(F_t)) s^t}{(1-s) \sum_{t=0}^{\infty} \gamma_t s^t}, \tag{4.2}$$

where $\tilde{Q}(s) = E s^{\tilde{Z}_0} = \frac{\tilde{H}(s) - \tilde{h}_0}{1 - \tilde{h}_0}$ and $\tilde{H}(s) = \sum_{i=0}^{N} \tilde{h}_i s^i, \quad \tilde{h}_i = h_i, \quad 0 \leq i \leq N-1$, $\tilde{h}_N = \sum_{k=N}^{\infty} h_k$.

We will use also the following relations

$${}_1\tilde{u}_t = \sum_{k=t+1}^{\infty} \tilde{u}_k, \qquad {}_{n+1}\tilde{u}_t = \sum_{k=t+1}^{\infty} {}_n\tilde{u}_k, \qquad n = 1, 2, \ldots \tag{4.3}$$

As usual $[x] = \sup\{n : \quad x \geq n, \quad n\text{-integer}\}$ and $]x[= \inf\{n : \quad x \leq n, \quad n\text{-integer}\}$.

Lemma 4.1 *Assume (2.3) and (2.13)-(2.15). Then*

(4.4) $$\sum_{t=0}^{\infty} {}_k\tilde{u}_t < \infty, \qquad k = 0,1,2,\ldots,]-\theta[-1,$$

and for $s \uparrow 1$

(4.5) $$\sum_{t=0}^{\infty} {}_{]-\theta[}\tilde{u}_t s^t = \begin{cases} O((1-s)^{-(]-\theta[+\theta)}) & \textit{if } \theta \textit{ is not integer,} \\ O(\log(1-s)^{-1}) & \textit{if } \theta \textit{ is integer .} \end{cases}$$

If one assume only (2.3) and (2.16) then

(4.6) $$\sum_{t=0}^{\infty} {}_1\tilde{u}_t s^t = O((1-s)^{-(1+\theta)})L(\frac{1}{1-s}), \qquad s \uparrow 1,$$

where $L(x)$ is a s.v.f. as $x \to \infty$.

Proof. Let $\nu = [-\theta] + 1$ and

(4.7) $$R_\nu^k(s) = F_k(s) - 1 - \sum_{i=1}^{\nu}(-1)^i \frac{F_k^{(i)}(1)}{i!}(1-s)^i, \quad k = 1,2,\ldots; R_\nu^0(1-s) = s-1,$$

where $F_k(s) = F(F_{k-1}(s)) = \sum_{n=0}^{\infty} f_n(k)s^n$ and $F_k^{(i)}(1) = \frac{d^i}{ds^i}F_k(s)\mid_{s=1}$.

It is well-known (see Sevastyanov(1971), Th.6, p.66) that $\sum_{n=0}^{\infty} f_n n^{1-\theta} < \infty$ implies

(4.8) $$\sum_{n=0}^{\infty} f_n(k) n^{1-\theta} < \infty, \qquad k = 1,2,\ldots; \quad \theta \le 0.$$

From (4.8) and Lemmas 2 and 3 of Vatutin(1977a) it follows that for every $k = 1,2,\ldots$

(4.9) $$\sum_{n=2}^{\infty} n^{-\theta} \mid R_\nu^k(\frac{1}{n}) \mid < \infty, \qquad \theta\text{–non-integer,}$$

and

(4.10) $$\sum_{n=2}^{\infty} n^{-\theta} \mid R_\nu^k(\frac{1}{n}) \mid s^n \sim c \log \frac{1}{1-s}, \qquad s \uparrow 1, \quad \theta\text{–integer.}$$

On the other hand, for $t \ge \nu$ and $i = 0,1,\ldots,\nu$ one obtains from (4.7) that

(4.11) $$F_{t-i} = F_{\nu-i}(F_t - \nu) = 1 + \sum_{k=1}^{\nu}(-1)^k \frac{F_{\nu-i}^{(k)}(1)}{k!}(1 - F_{t-\nu})^k + R_\nu^{\nu-i}(1 - F_{t-\nu}).$$

From (4.2) one has

(4.12) $$\tilde{U}(s) = \frac{W_1(s)}{W_2(s)},$$

where

(4.13) $$\begin{aligned} W_1(s) &= \sum_{t=0}^{\infty} \gamma_t (1 - \tilde{Q}(F_t))s^t = \sum_{t=0}^{\infty} \gamma_t s^t \sum_{i=1}^{N} \tilde{q}_i (1 - F_t^i) \\ &= \sum_{t=0}^{\infty} \gamma_t s^t \sum_{i=1}^{N} \tilde{q}_i \sum_{j=1}^{i} \binom{i}{j} (-1)^{j-1}(1-F_t)^j \\ &= \sum_{t=0}^{\infty} \gamma_t s^t \sum_{j=1}^{N} a_j (1-F_t)^j. \end{aligned}$$

Here $a_j = \frac{(-1)^{j-1}}{j!} E\{\dot{Z}_0(\dot{Z}_0 - 1)\dots(\dot{Z}_0 - j + 1)\}$.

Now using (4.11) with $i = 0$ and (4.7) one obtains from (4.13)

$$
\begin{aligned}
W_1(s) &= \textstyle\sum_{t=0}^{\nu-1} \gamma_t s^t \sum_{j=1}^{N} a_j (1 - F_t)^j + \\
&+ \textstyle\sum_{t=\nu}^{\infty} \gamma_t s^t \sum_{j=1}^{N} (-1)^j a_j [R_\nu^\nu(1 - F_{t-\nu}) + \sum_{k=1}^{\nu} \frac{(-1)^k}{k!} F_\nu^{(k)}(1)(1 - F_{t-\nu})^k]^j \\
&= \textstyle\sum_{t=0}^{\nu-1} \gamma_t s^t \sum_{j=1}^{N} a_j (1 - F_t)^j + \\
&+ \textstyle\sum_{t=\nu}^{\infty} \gamma_t s^t [\sum_{k=1}^{\nu} A_k^{(0)} (1 - F_{t-\nu})^k + O(R_\nu^\nu(1 - F_{t-\nu}) + (1 - F_{t-\nu})^{\nu+1})],
\end{aligned} \tag{4.14}
$$

where $A_k^{(0)} = A_k^{(0)}(N) = \frac{(-1)^k}{k!} F_\nu^{(k)}(1) \sum_{j=1}^{N} (-1)^j a_j, \quad k = 1, 2, \dots, \nu$.

From (4.2) and (4.12) applying (3.9),(4.7) and (4.11) with $i = 1$ one gets

$$
\begin{aligned}
W_2(s) &= (1 - s) \sum_{t=0}^{\infty} \gamma_t s^t = 1 - \sum_{t=0}^{\infty} \gamma_t \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})} s_t \\
&= 1 - \sum_{t=1}^{\nu-1} \gamma_t s^t \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})} - \sum_{t=\nu}^{\infty} \gamma_t s^t [\sum_{k=1}^{\nu} (1 - \delta(F_{t-1})^k + \frac{(1 - \delta(F_{t-1}))^{\nu+1}}{\delta(F_{t-1})}] \\
&= 1 - \sum_{t=1}^{\nu-1} \gamma_t s^t \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})} - \sum_{t=\nu}^{\infty} \gamma_t s^t \{\sum_{k=1}^{\nu} (-1)^k [R_\nu^\delta(1 - F_{t-1}) + \\
&+ \sum_{i=1}^{\nu} (-1)^i \frac{\delta^{(i)}(1)}{i!} (1 - F_{t-1})^i]^k + \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})}\} \\
&= 1 - \sum_{t=1}^{\nu-1} \gamma_t s^t \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})} - \\
&- \sum_{t=\nu}^{\infty} \gamma_t s^t \{\sum_{k=1}^{\nu} (-1)^k [R_\nu^\delta(1 - F_{t-1}) + \sum_{i=1}^{\nu} (-1)^i \frac{\delta^{(i)}(1)}{i!} (-1)^i + \\
&+ (R_\nu^{\nu-1}(1 - F_{t-\nu}) + \sum_{j=1}^{\nu} \frac{(-1)^j}{j!} F_{\nu-1}^{(j)}(1)(1 - F_{t-\nu})^j)^i]^k + \frac{(1 - \delta(F_{t-1}))^{\nu+1}}{\delta(F_{t-1})}\} \\
&= 1 - \sum_{t=1}^{\nu-1} \gamma_t s^t \frac{1 - \delta(F_{t-1})}{\delta(F_{t-1})} + \sum_{t=\nu}^{\infty} \gamma_t s^t \{\sum_{k=1}^{\nu} B_k^{(0)} (1 - F_{t-\nu})^k + \\
&+ O(R_\nu^\delta(1 - F_{t-\nu}) + R_\nu^{\nu-1}(1 - F_{t-\nu}) + (1 - F_{t-\nu})^{\nu+1})\},
\end{aligned} \tag{4.15}
$$

where $B_k^{(0)}, k = 1, 2, \dots, \nu$, are some constants.

Applying Lemma 3.1 (see (3.1) for $s = 0$) and the well-known fact that $1 - F_t \sim (bt)^{-1}$, $t \to \infty$, one can obtain as $s \uparrow 1$ (see Feller(1971), Ch.VIII, §9, Th.1 and Ch.XIII, §5, Th.5) that

$$
\sum_{t=0}^{\infty} \gamma_t (1 - F_t)^j s_t \sim (1 - s)^{j+\theta-1} L_1\left(\frac{1}{1 - s}\right), \qquad j = 0, 1, 2, \dots, \nu, \tag{4.16}
$$

where $L_1(x)$ is a s.v.f. as $x \to \infty$ such that $L_1(x) \to c_1 > 0$ under the conditions (2.13).

Now from (4.12)–(4.16) it is not difficult to obtain that

$$
\lim \hat{U}(s) = \sum_{t=0}^{\infty} \hat{u}_t = \frac{A_1^{(0)}}{B_1^{(0)}} = -\frac{\alpha}{b\theta} > 0, \quad s \uparrow 1. \tag{4.17}
$$

From (4.3) applying (4.12)-(4.15) and (4.17) one can show that

$$\sum_{t=0}^{\infty} {}_1\tilde{u}_t s^t = \frac{1}{1-s}\Big(\sum_{t=0}^{\infty}\tilde{u}_t - \sum_{t=0}^{\infty}\tilde{u}_t s^t\Big)$$

$$(4.18)\qquad = \frac{1}{(1-s)^2\sum_{t=0}^{\infty}\gamma_t s^t}\{\phi_1(s) + \sum_{t=\nu}^{\infty}\gamma_t s^t[\sum_{k=2}^{\nu} A_k^{(1)}(1-F_{t-\nu}^k) +$$

$$+O(R_\nu^\delta(1-F_{t-1}) + \sum_{i=0}^{\min(1,\nu-1)} R_\nu^{\nu-i}(1-F_{t-\nu}) + (1-F_{t-\nu})^{\nu+1})]\},$$

where $\varphi_1(s) \to c_1 > 0$ as $s \uparrow 1$ and $A_k^{(1)} = A_k^{(1)}(N)$ are some constants.

(i) *Let* $-1 < \theta < 0$, *i.e.* $\nu = 1$. Then from (4.18) applying Lemma 3.2, (4.9) and (4.16) one obtains as $s \uparrow 1$ that

$$(4.19)\qquad \sum_{t=0}^{\infty} {}_1\tilde{u}_t s^t = O((1-s)^{-(1+\theta)}L_2(\frac{1}{1-s})),$$

where $L_2(x)$ is a s.v.f. as $x \to \infty$ such that $L_2(x) \to c_2 > 0$ under the conditions (2.13).

Note that (4.17) and (4.18) prove the lemma in the case $-1 < \theta < 0$.

(ii) *Let* $\theta = -1$, *i.e.* $\nu = 2$. Then by Lemma 3.2, (4.10) and (4.16) it is not difficult to obtain that for $s \uparrow 1$

$$(4.20)\qquad \begin{array}{l} \sum_{t=\nu}^{\infty}\gamma_t s^t[A_2^{(1)}(1-F_{t-\nu}^2) + O(R_\nu^\delta(1-F_{t-\nu}) + R_\nu^\nu(1-F_{t-\nu}) + \\ +R_\nu^{\nu-1}(1-F_{t-\nu}) + (1-F_{t-\nu})^\nu)] = O(\log\frac{1}{1-s}). \end{array}$$

On the other hand, using Lemma 3.1 with $s = 0$ it is not difficult to show that

$$(1-s)^2\sum_{t=0}^{\infty}\gamma_t s^t = O(1), \qquad s \uparrow 1,$$

Hence from (4.18) applying also (4.20) one has

$$\sum_{t=0}^{\infty} {}_1\tilde{u}_t s^t = O(\log\frac{1}{1-s}), \qquad s \uparrow 1,$$

which proves the lemma in this case.

(iii) *Let* $\theta < -1$, *i.e.* $\nu \geq 2$. Using (3.9), (4.7) and (4.11) with $i = 1, 2$, one obtains

$$(1-s)^2\sum_{t=0}^{\infty}\gamma_t s^t = 1 + (\delta(0)-2)s +$$

$$(4.21)\qquad +\sum_{t=2}^{\infty}\gamma_t s^t\frac{[1-\delta(F_{t-2})] - [1-\delta(F_{t-1})] + [1-\delta(F_{t-2})][1-\delta(F_{t-1})]}{\delta(F_{t-2})\delta(F_{t-1})} =$$

$$= \psi(s) + \sum_{t=\nu}^{\infty}\gamma_t s^t[\sum_{k=0}^{\nu} B_k^{(1)}(1-F_{t-\nu})^k + O(R_\nu^\delta(1-F_{t-1}) +$$

$$+R_\nu^{\nu-1}(1-F_{t-\nu}) + R_\nu^{\nu-2}(1-F_{t-k}) + (1-F_{t-\nu})^{\nu+1})],$$

where $\lim\psi(s) = c > 0, s \uparrow 1$.

Since $\theta < -1$, then by Lemma 3.1, (4.16) and (4.21) it follows that $B_0^{(1)} = B_1^{(1)} = 0$ and $B_2^{(1)} \neq 0$.

On the other hand, $A_2^{(1)} \neq 0$ because if it is not true, then from (4.16), (4.18) and (4.21) one obtains that $\sum_{t=0}^{\infty} {}_1\hat{u}_t = 0$, which is not possible obviously.

Therefore, from (4.18) and (4.21) one gets

$$\lim \sum_{t=0}^{\infty} {}_1\hat{u}_t s^t = \sum_{t=0}^{\infty} \hat{u}_t = \frac{A_2^{(1)}}{B_2^{(1)}} < \infty, \qquad s \uparrow 1.$$

Repeating these procedures $(\nu - 1)$ and ν times respectively one obtains

$$\sum_{t=0}^{\infty} {}_{\nu-1}\hat{u}_t s^t = \frac{1}{(1-s)^{\nu} \sum_{t=0}^{\infty} \gamma_t s^t} \{\varphi_{\nu-1}(s) + \sum_{t=\nu}^{\infty} \gamma_t s^t [A_{\nu}^{(\nu-1)}(1 - F_{t-\nu})^{\nu} + \tag{4.22}$$
$$+ O(\sum_{i=1}^{\nu-1} R_{\nu}^{i}(1 - F_{t-i}) + \sum_{i=0}^{\nu-1} R_{\nu}^{\nu-i}(1 - F_{t-\nu}) + (1 - F_{t-\nu})^{\nu+1})]\}$$

and

$$\sum_{t=0}^{\infty} {}_{\nu}\hat{u}_t s^t = \frac{1}{(1-s)^{\nu+1} \sum_{t=0}^{\infty} \gamma_t s^t} \{\varphi_{\nu}(s) + \tag{4.23}$$
$$+ \sum_{t=\nu}^{\infty} \gamma_t s^t + O(\sum_{i=1}^{\nu} R_{\nu}^{i}(1 - F_{t-\nu}) + + \sum_{i=0}^{\nu-1} R_{\nu}^{\nu-i}(1 - F_{t-\nu}) + +(1 - F_{t-\nu})^{\nu+1})]\}$$

where $\varphi_i(s)$ are polynoms and $\lim \varphi_i(s) = c_i > 0, \quad s \uparrow 1, \quad i = \nu - 1, \nu.$

If θ *is not integer*, then by Lemmas 3.1 and 3.2, using (4.9), (4.16) and (4.23) one can show that

$$\sum_{t=0}^{\infty} {}_{\nu}\hat{u}_t s^t = O((1-s)^{-(\nu+\theta)}), \qquad s \uparrow 1.$$

If θ *is integer*, then by Lemmas 3.1 and 3.2, applying (4.10) and (4.22) one obtains

$$\sum_{t=0}^{\infty} {}_{-\theta}\hat{u}_t s^t = O(\log \frac{1}{1-s}), \qquad s \uparrow 1.$$

This means that the lemma is completely proved.

Lemma 4.2 *Assume (2.3) and (2.13)–(2.15). Then as $s \uparrow 1$*

$$\sum_{t=0}^{\infty} {}_{1-\theta[}\hat{u}_t s^t = \begin{cases} c_1(1-s)^{-(]-\theta[+\theta)} & \text{if } \theta \text{ is not integer}, \quad c_1 > 0, \\ c_2 \log(1-s)^{-1} & \text{if } \theta \text{ is integer}, \quad c_2 > 0. \end{cases} \tag{4.24}$$

If one assume only (2.3) and (2.16), then

$$\sum_{t=0}^{\infty} {}_1\hat{u}_t s^t = (1-s)^{-(1+\theta)} L_1(\frac{1}{1-s}), \qquad s \uparrow 1, \tag{4.25}$$

where $L_1(x)$ is a s.v.f. as $x \to \infty$.

Proof. *Let θ is not integer.* Then from Lemma 4.1 and the representation

$$\tilde{U}(s)=\sum_{t=0}^{\infty}\tilde{u}_t s^t=\sum_{t=0}^{\infty}\tilde{u}_t-(1-s)\sum_{t=0}^{\infty}{}_1\tilde{u}_t+\ldots+(-1)^{[-\theta]}(1-s)^{[-\theta]}\sum_{t=0}^{\infty}{}_{[-\theta]}\tilde{u}_t+(-1)^{]-\theta[}\sum_{t=0}^{\infty}{}_{]-\theta[}\tilde{u}_t s^t$$

it follows that

$$\tilde{U}(s)=\sum_{i=1}^{[-\theta]}M_i(1-s)^i+M_\theta(1-s)^{-\theta}+o((1-s)^{-\theta}),$$

where $M_i=\lim_{s\uparrow 1}\sum_{t=0}^{\infty}{}_i\tilde{u}_t s^t$, $\quad i=0,1,\ldots,[-\theta]$.

Therefore to prove the lemma in this case is equivalent to show that $M_\theta\neq 0$.

Remember that $\tilde{u}_t=\tilde{u}_t(N)$, $M_\theta=M_\theta(N)$ and $M_i=M_i(N)$ for $N\geq 1$, where $\tilde{Z}_0=Z_0 1_{\{Z_0\leq N\}}+N1_{\{Z_0>N\}}$. Since for $N_1\leq N_2$ one has $\tilde{u}_t(N_1)\leq\tilde{u}_t(N_2)$ and for $s\uparrow 1$

$$\sum_{t=0}^{\infty}{}_{]-\theta[}\tilde{u}_t(N_2)s^t\geq\sum_{t=0}^{\infty}{}_{]-\theta[}\tilde{u}_t(N_1)s^t\sim M_\theta(N_1)(1-s)^{-(]-\theta[+\theta)},$$

then $M_\theta(N_2)\neq 0$ if $M_\theta(N_1)\neq 0$.

First we wish to prove that $M_\theta(]-\theta[)\neq 0$.

Let us assume that $M_\theta(]-\theta[)=0$. Therefore $M_\theta(N)=$ for $1\leq N\leq\,]-\theta[+1$ and

$$\tilde{U}_N(s)=\sum_{i=0}^{[-\theta]}M_i(N)(1-s)^i+o((1-s)^{-\theta}). \tag{4.26}$$

On the other hand, from (4.2), (4.12) and (4.13) one has

$$\tilde{U}_N(s)=\frac{\sum_{j=1}^{N}a_j(N)\sum_{t=0}^{\infty}\gamma_t(1-F_t)^j s^t}{(1-s)\sum_{t=0}^{\infty}\gamma_t s^t}=\frac{\sum_{j=1}^{N}a_j(N)\Delta_j(s)}{(1-s)\Delta_0(s)}. \tag{4.27}$$

For N equal to 1 and 2 it follows from (4.26) and (4.27) that

$$\frac{a_1(1)\Delta_1(s)}{(1-s)\Delta_0(s)}=\sum_{i=0}^{[-\theta]}M_i(1)(1-s)^i+o((1-s)^{-\theta}) \tag{4.28}$$

and

$$\frac{a_1(2)\Delta_1(s)}{(1-s)\Delta_0(s)}+\frac{a_2(2)\Delta_2(s)}{(1-s)\Delta_0(s)}=\sum_{i=0}^{[-\theta]}M_i(2)(1-s)^i+o((1-s)^{-\theta}). \tag{4.29}$$

(i)*Let* $-1<\theta<0$, *i.e.* $]-\theta[+1=2$. Then from (4.28) and (4.29) as $s\uparrow 1$ one has that

$$\frac{\Delta_2(s)}{(1-s)\Delta_0(s)}=\sum_{i=0}^{[-\theta]}d_i(1-s)^i+o((1-s)^{-\theta}). \tag{4.30}$$

On the other hand, using Lemma 3.1 with $s=0$ one obtains that $\Delta_2(1)=\lim_{s\uparrow 1}\sum_{t=0}^{\infty}\gamma_t(1-F_t)^2s^t<\infty$ and $\frac{\Delta_2(s)}{(1-s)\Delta_0(s)}\sim c_2(1-s)^{-\theta}, c_2>0$. The obtained contradiction of (4.30) shows that $M_\theta(2)\neq 0$ for $-1<\theta<0$.

(ii) *Let* $\theta < -1$, *i.e.* $]-\theta[+1 \geq 3$. Similarly to (4.30), putting $N = 2,3,\ldots,]-\theta[+1$ and using (4.16) it is not difficult to see that for $s \uparrow 1$

$$\frac{\Delta_k(s)}{(1-s)\Delta_0(s)} = \sum_{i=0}^{[-\theta]} d_i(1-s)^i + o((1-s)^{-\theta}), \qquad k = 2,3,\ldots,]-\theta[+1.$$

Using Lemma 3.1 with $s = 0$ and the fact that $1 - F_t \sim 1/bt$ then one obtains

$$\Delta_{]-\theta[+1}(1) = \lim_{s\uparrow 1} \sum_{t=0}^{\infty} \gamma_t(1-F_t)^{]-\theta[+1} s^t < \infty$$

$$\frac{\Delta_{]-\theta[+1}(s)}{(1-s)\Delta_0(s)} \sim c_{]-\theta[+1}(1-s)^{-\theta}, \qquad c_{]-\theta[+1}(1-s)^{-\theta} > 0.$$

The obtained contradiction shows that $M_\theta(]-\theta[+1) \neq 0$.

This proves (4.24) for *non-integer* θ and $N \geq]-\theta[+1$.

Let now $N = 1$, i.e. $\hat{Z}_0 = 1$ a.s. Then for every $-\infty < \theta < 0$ there exists T such that $P(\hat{Z}_T \geq]-\theta[+1 \mid \hat{Z}_0 = 1) > 0$. Hence for $t > T$

$$\begin{aligned} \hat{u}_t(1) &= P(\hat{Z}_t > 0 \mid \hat{Z}_0 = 1) = P(\hat{Z}_t > 0, Z_T = k \mid \hat{Z}_0 = 1) \\ &\geq \textstyle\sum_{k=]-\theta[+1}^{\infty} P(\hat{Z}_t > 0 \mid \hat{Z}_T = k) P(\hat{Z}_T = k \mid \hat{Z}_0 = 1) \\ &\geq P(\hat{Z}_t > 0 \mid \hat{Z}_T =]-\theta[+1) P(\hat{Z}_T \geq]-\theta+1[| \hat{Z}_0 = 1) \\ &\geq c_0 \hat{u}_t(]-\theta[+1), \qquad\qquad c_0 > 0. \end{aligned}$$

Therefore

$$\begin{aligned} \textstyle\sum_{t=0}^{\infty} {}_{]-\theta[}\hat{u}_t(1)s^t &\geq c_0 \textstyle\sum_{t=0}^{\infty} \hat{u}_t(]-\theta[+1)s^t \\ &\sim c_0 M_\theta(]-\theta]+1)(1-s)^{(]-\theta[+\theta)}, \end{aligned}$$

which complete the proof of (4.24) for θ non-inheger.

The proof of (4.24) for *integer* θ is similar, i.e. using Lemma 4.1 and the procedure for non-integer θ one has

$$\sum_{t=0}^{\infty} \hat{u}_t s^t = \sum_{i=0}^{-\theta-1} M_i(1-s)^i + M_\theta(1-s)^{-\theta} \log\frac{1}{1-s}(1+o(1)), \quad M_\theta \neq 0.$$

Lemma 4.3 *Assume (2.3) and (2.13)–(2.15). Then*

$$\hat{u}_t \sim ct^{\theta-1} \qquad c > 0, \qquad t \to \infty. \tag{4.31}$$

If one supposes only (2.3) and (2.16) then

$$\hat{u}_t \sim L(t)t^{\theta-1} \qquad L(t) \text{ s.v.f.}, \qquad t \to \infty. \tag{4.32}$$

Proof. (i) *Let* θ *is non-integer* and $\nu =]-\theta[$. Then by Lemma 4.2 (see (4.24)) and the Tauberian theorem (see Feller(1971), Ch.XIII, §5) one has

$${}_\nu\hat{u}_t = \sum_{n=t+1}^{\infty} {}_{\nu-1}\hat{u}_n \sim c_\nu t^{\nu+\theta-1}, \qquad c_\nu > 0, \quad t \to \infty.$$

Since $\nu + \theta - 1 < 0$ then by Lemma 3.3 it follows that

$$_{\nu-1}\dot{u}_t \sim c_{\nu-1} t^{\nu+\theta-2}, \qquad c_{\nu-1} > 0, \quad t \to \infty.$$

Repeating this procedure ν times one obtains (4.31) for θ non-integer. The conclusion (4.32) can be proved similarly using (4.25).

(ii) *Let θ is integer.* Then from (4.22)-(4.24) it follows that

$$\sum_{t=0}^{\infty} {}_{-\theta}\dot{u}_t s^t = \frac{1}{(1-s)^{-\theta+1} \sum_{t=0}^{\infty} \gamma_t s^t} [\varphi_{-\theta}(s) +$$
$$+ c \sum_{t=-\theta+1}^{\infty} \gamma_t (1 - F_{t+\theta-1})^{-\theta+1} s^t (1 + o(1))] = \frac{W(s)}{V(s)},$$

where $c \neq 0$.

Note that

$$\sum_{t=1}^{\infty} {}_{-\theta}\dot{u}_t t s^{t-1} = \frac{1}{V^2(s)} [W'(s)V(s) - W(s)V'(s)]. \tag{4.33}$$

Using Lemma 3.1 with $s = 0$ and (4.16) one can prove that as $s \uparrow 1$

$$W(s) \sim c_1 \log(1-s)^{-1}, \quad W'(s) \sim c_2 (1-s)^{-1}, \quad V(s) \sim c_3, \tag{4.34}$$

where $c_i > 0, i = 1, 2, 3$.

(a) Let now $\theta = -1$. Then from (4.21) one gets

$$\begin{aligned} V'(s) &= \frac{d}{ds}[(1-s)^2 \sum_{t=0}^{\infty} \gamma_t s^t] = (\delta(0) - 2) + \\ &+ \sum_{t=1}^{\infty} s^t (t+1) \gamma_{t-1} \{ \frac{1 - \delta(F_{t-2}) - [1 - \delta(F_{t-1})] + [1 - \delta(F_{t-2})][1 - \delta(F_{t-1})]}{\delta(F_{t-2}) \delta(F_{t-1})} \}. \end{aligned} \tag{4.35}$$

On the other hand, from (3.9) and (4.7) it follows that in the case $\theta = -1$

$$\begin{aligned} &1 - \delta(F_{t-2}) - [1 - \delta(F_{t-1})] + [1 - \delta(F_{t-2})][1 - \delta(F_{t-1})] = \\ &= -b^2 (1 - F_{t-1})(F_t - F_{t-1}) + \\ &+ O(R_2^{\delta}(1 - F_t) + R_2^{\delta}(1 - F_{t-1}) + R_2^F(1 - F_{t-1}) + (1 - F_{t-1})^3). \end{aligned} \tag{4.36}$$

Since $(F_t - F_{t-1})(1 - F_{t-1}) \sim t^{-3}$ as $t \to \infty$ (see Athreya and Ney(1974), p.23, Corollary 1), then from (4.35), (4.36) and (3.11) it follows that

$$V'(s) = o(\frac{1}{(1-s)\log(1-s)^{-1}}), \qquad s \uparrow 1. \tag{4.37}$$

Now from (4.33), (4.34) and (4.37) one obtains in the case $\theta = -1$ and $s \uparrow 1$

$$\sum_{t=1}^{\infty} {}_1\dot{u}_t t s^{t-1} \sim \frac{c}{1-s}, \qquad c > 0. \tag{4.38}$$

(b) Let $\theta = -2, -3, \ldots$ In this case one has

$$(4.39)\quad \begin{aligned} V'(s) &= \frac{d}{ds}[(1-s)^{-\theta+1}\sum_{t=0}^{\infty}\gamma_t s^t] \\ &= \theta(1-s)^{-\theta}\sum_{t=0}^{\infty}\gamma_t s^t + (1-s)^{-\theta}\frac{d}{ds}[1+\sum_{t=1}^{\infty}\gamma_{t-1}(\delta(F_{t-1})-1)s^t] \\ &= \theta(1-s)^{-\theta}\sum_{t=0}^{\infty}\gamma_t s^t + (1-s)^{-\theta}\sum_{t=0}^{\infty}(t+1)\gamma_t(\delta(F_t)-1)s^t. \end{aligned}$$

It is well-known that under the condition $F'''(1) < \infty$

$$(4.40)\qquad 1 - F_t = \frac{1}{bt} + c\frac{\log t}{t^2}(1+o(1)), \qquad c > 0.$$

Now using Lemma 3.1, (3.9), Lemma 3.2, (4.39) and (4.40) one can obtains that as $s \uparrow 1$

$$(4.41)\quad \begin{aligned} V'(s) &= \theta(1-s)^{-\theta}\sum_{t=0}^{\infty}\gamma_t s^t + (1-s)^{-\theta}\sum_{t=0}^{\infty}(t+1)\gamma_t[-\frac{\theta}{t} + \frac{c_1\log t}{t^2} + \frac{c_2(1+o(1))}{t^2} + \\ &+ R_2^\delta(1-F_t)]s^t = O(\log\tfrac{1}{1-s}). \end{aligned}$$

Hence from (4.33), (4.34) and (4.41) one has as $s \uparrow 1$

$$(4.42)\qquad \sum_{t=1}^{\infty} {}_{-\theta}\tilde{u}_t t s^{t-1} \sim \frac{c}{1-s}, \qquad c > 0.$$

From (4.38) and (4.42) applying the Tauberian theorem and Lemma 3.3 one obtains (4.31) for integer θ.

5 Proof of Theorem 2.1.

Applying Lemma 3.1 with $s = 0$ one obtains

$$(5.1)\qquad \gamma_t(1 - Q(F_t)) \sim \frac{\alpha}{b}L(t)t^{-\theta-1}, \qquad t \to \infty.$$

By (5.1) using Theorem 1 of Feller(1971), Ch.VIII, §9 it is not difficult to show that

$$(5.2)\qquad \sum_{t=0}^{n}\gamma_t(1-Q(F_t)) \sim \frac{\alpha}{b\theta}L(n)n^{-\theta}, \qquad n \to \infty.$$

Therefore (see Feller(1971), Ch.XIII, §5, Th. 5)

$$(5.3)\qquad \sum_{t=0}^{\infty}\gamma_t(1-Q(F_t)) \sim \frac{\alpha\Gamma(1-\theta)}{b\theta}(1-s)^{\theta}L(\frac{1}{1-s}), \qquad s \uparrow 1.$$

Similarly

$$(5.4)\qquad \sum_{t=0}^{\infty}\gamma_t s^t \sim \frac{\Gamma(2-\theta)}{1-\theta}(1-s)^{\theta-1}L(\frac{1}{1-s}), \qquad s \uparrow 1.$$

Now from (2.12) using (5.3) and (5.4) one has

$$E\tau = \lim_{s\uparrow 1} U(s) = \lim_{s\uparrow 1} \frac{\sum_{t=0}^{\infty} \gamma_t (1 - Q(F_t)) s^t}{(1-s)\sum_{t=0}^{\infty} \gamma_t s^t} = -\frac{\alpha}{b\theta}. \tag{5.5}$$

Using (2.3) and (2.8) it is not difficult to see that (5.5) is equivalent to (2.17).

(i) Let now $\{\tilde{Z}_t\}$ is the process defined in Section 4 (i.e. with bounded immigration (4.1) in the state zero). Then

$$\begin{aligned} P(Z_t = \tilde{Z}_t) &= \textstyle\sum_{k=0}^{\infty} P(Z_0 = k) P(Z_t = \tilde{Z}_t \mid Z_0 = k) \\ &= \textstyle\sum_{k=1}^{N} q_k + \sum_{k=N+1}^{\infty} q_k f_0^{k-N} = 1 - \sum_{k=N+1}^{\infty} q_k (1 - f_0^{k-N}). \end{aligned} \tag{5.6}$$

From (2.9) and (5.6) it follows that

$$\mid \tilde{u}_t - u_t \mid = \mid P(\tilde{Z}_t > 0) - P(Z_t > 0) \mid \leq P(\tilde{Z}_t \neq Z_t) \leq \sum_{k=N+1}^{\infty} q_k. \tag{5.7}$$

Since (2.17) hold then for $N \to \infty$

$$\sum_{k=N+1}^{\infty} q_k \leq c \begin{cases} N^{-2} \sum_{k=N}^{\infty} k^2 q_k \quad , & -1 < \theta < 0, \\ N^{-1-[-\theta]} \sum_{k=N}^{\infty} k^{1+[-\theta]} q_k \quad , & \theta \leq -1. \end{cases} \tag{5.8}$$

Now it follows from (5.7) and (5.8) that one can choose $0 < \varepsilon < 1$ and $N = O(t^{1+\varepsilon})$ such that

$$\mid \tilde{u}_t - u_t \mid = O(t^{\theta - 1}), \qquad t \to \infty. \tag{5.9}$$

From (5.9) and Lemma 4.3 (see (4.31)) one obtains (2.18).

Similarly, applying (4.32) one proves (ii).

(iii) In this case it is enough to consider the process $\{\tilde{Z}_t\}$.

Let θ is not integer and $\nu = [-\theta] + 1 \geq 2$. Then from (2.12) and (2.18) one obtains

$$\sum_{t=0}^{\infty} \tilde{\Psi}_t(N) s^t = \sum_{i=-1}^{\nu-1} \alpha_i(N)(1-s)^i + \alpha_\theta(N)(1-s)^{-\theta}(1 + o(1)), \tag{5.10}$$

where $\alpha_\theta(N) \neq 0$ and $\tilde{\Psi}_t(N) = P(\tilde{Z}_t = 0)$, $N = 1, 2, \ldots$

On the other hand, from (2.12) and (4.27) we have

$$\sum_{t=0}^{\infty} \tilde{\Psi}_t(N) s^t = \frac{1}{1-s} - \frac{\sum_{j=1}^{\infty} a_j(N) \Delta_j(s)}{(1-s)\Delta_0(s)}, \tag{5.11}$$

where $a_j(N) = \frac{(-1)^{j-1}}{j!} E\{\tilde{Z}_0(\tilde{Z}_0 - 1) \ldots (\tilde{Z}_0 - j + 1)\}$ and $\Delta_j(s) = \sum_{t=0}^{\infty} \gamma_t (1 - F_t)^j s^t$.

Putting in (5.11) $N = 1, 2, 3, \ldots$ and applying (5.10) and (4.16) it is not difficult to see that as $s \uparrow 1$

$$\frac{\Delta_k(s)}{(1-s)\Delta_0(s)} = \sum_{i=k-1}^{\nu-1} d_i (1-s)^i + d_\theta (1-s)^{-\theta}(1 + o(1)), \qquad k = 1, 2, \ldots, \tag{5.12}$$

where as usual $\sum_{i=k-1}^{\nu-1} \cdot = 0$ if $k > \nu$.

Let $N = 2$. Then by (2.12) it follows that

$$\sum_{t=0}^{\infty} \dot{\Psi}_t(2)s^t = \frac{\sum_{t=0}^{\infty}\gamma_t \dot{Q}(F_t)s^t}{(1-s)\Delta_0(s)} = \frac{\dot{q}_1\sum_{t=0}^{\infty}\gamma_t F_t s^t + \dot{q}_2\sum_{t=0}^{\infty}\gamma_t F_t^2 s^t}{(1-s)\Delta_0(s)} \tag{5.13}$$
$$= \frac{\dot{q}_1}{1-s} - \frac{\dot{q}_1\Delta_1(s)}{(1-s)\Delta_0(s)} + \dot{q}_2\sum_{t=1}^{\infty}\gamma_{t-1}F_t^2 s^t(1-s)\Delta_0(s).$$

On the other hand, one has as $t \to \infty$

$$F_t^2\delta(F_{t-1}) = [1-(1-F_{t-1}) + b(1-F_{t-1})^2 + R_2^F(1-F_{t-1})]^2[1 - b\theta(1-F_{t-1}) +$$
$$+\tfrac{\delta''(1)}{2}(1-F_{t-1})2 + R_2^\delta(1-F_{t-1})] \tag{5.14}$$
$$= 1 - (2+b\theta)(1-F_{t-1}) + (\tfrac{\delta''(1)}{2} + 1 + 2b\theta)(1-F_{t-1})^2 +$$
$$+[R_2^\delta(1-F_{t-1}) + 2R_2^F(1-F_{t-1})](1+o(1)).$$

Now from (5.12) and (5.14) one obtains

$$\sum_{t=0}^{\infty} \dot{\Psi}_t(2)s^t = s\Big[\frac{1}{1-s} - \frac{c_1\Delta_1(s)}{(1-s)\Delta_0(s)} + \frac{c_2\Delta_2(s)}{(1-s)\Delta_0(s)} + \frac{\sum_{t=0}^{\infty}\gamma_t W(1-F_t)s^t}{(1-s)\Delta_0(s)}\Big] \tag{5.15}$$

where $c_1 = \dot{q}_1 + 2\dot{q}_2 + \dot{q}_2 b\theta, c_2 = \dot{q}_2\left(\frac{1}{2}\delta''(1) + 1 + 2b\theta\right)$ and
$W(1-F_t) = [\dot{q}_2 R_2^\delta(1-F_t) + 2\dot{q}_2 R_2^F(1-F_t)](1+o(1)),\ t\to\infty.$

Let $1 < -\theta < 2$. From (5.10), (5.12) and (5.15) using also that

$$W(1-F_t) = o((1-F_t)^2), \qquad t \to \infty, \tag{5.16}$$

one obtains as $s \uparrow 1$

$$\frac{1}{(1-s)\Delta_0(s)}\Big[\sum_{t=0}^{\infty}\gamma_t R_2^\delta(1-F_t)s^t + \sum_{t=0}^{\infty}\gamma_t R_2^\delta(1-F_t)s^t\Big] \sim c(1-s)^{-\theta}, \qquad c \geq 0. \tag{5.17}$$

Now from (3.15), (4.16) and (5.17) it follows that for $s \uparrow 1$

$$\sum_{t=0}^{\infty}\gamma_t R_2^F(1-F_t)s^t - \sum_{t=0}^{\infty}\gamma_t R_2^G(1-F_t)s^t \sim c_1 L_1\Big(\frac{1}{1-s}\Big), \qquad c_1 \geq 0. \tag{5.18}$$

Using the condition $\sum_{k=1}^{\infty} k^{1-\theta} g_k < \infty$, (3.1) and Lemma 2 of Vatutin(1977a) one obtains that

$$\sum_{t=0}^{\infty}\gamma_t R_2^G(1-F_t) < \infty. \tag{5.19}$$

Therefore from (5.18) and (5.19) one has as $s \uparrow 1$

$$\sum_{t=0}^{\infty}\gamma_t R_2^F(1-F_t)s^t \sim c_2 L_1\Big(\frac{1}{1-s}\Big).$$

Since $R_2^F(1-s) < 0$, then

$$\sum_{t=0}^{\infty}\gamma_t \mid R_2^F(1-F_t) \mid s^t \sim \mid c_2 \mid L_1\Big(\frac{1}{1-s}\Big), \qquad s \uparrow 1 \tag{5.20}$$

On the other hand, $| R_2^F(1-s) |$ is a monotone function and from (5.20) and Lemma 3.1 applying the Tauberian theorem one obtains

$$\begin{aligned} | c_2 | L_1(t) &\sim \textstyle\sum_{k=0}^t \gamma_k | R_2^F(1-F_k) | \geq | R_2^F(1-F_t) | \sum_{k=0}^t \gamma_k \\ &\sim ct^{-\theta+1} L_1(t) | R_2^F(1-F_t) |, \qquad t \to \infty. \end{aligned}$$

Hence for every ε, $\quad 1 < \varepsilon < -\theta$,

$$\sum_{t=1}^{\infty} t^{\varepsilon} | R_2^F(1-F_t) | < \infty. \tag{5.21}$$

Using again the monotonicity of $R_2^F(x)$ from (5.21) it follows that

$$\sum_{t=1}^{\infty} t^{\varepsilon} | R_2^F(\frac{1}{t}) | < \infty$$

and by Lemma 2 of Vatutin(1977a) one obtains

$$\sum_{k=1}^{\infty} f_k k^{1+\varepsilon} < \infty. \tag{5.22}$$

Therefore $\sum_{k=1}^{\infty} f_k k^2 \log k < \infty$ and from Lemma 3.1 it follows that in (5.20) $L_1(t) \to c > 0$, $t \to \infty$. Hence

$$\sum_{t=0}^{\infty} t^{-\theta} | R_2^F(\frac{1}{t}) | < \infty. \tag{5.23}$$

Now from (5.23) and Corollary 3.1 one obtains the conclusion (iii) of the theorem in the case $-2 < \theta < -1$.

Let now consider the case $\theta < -2$ and θ noninteger. We will prove at first that $F'''(1) < \infty$. From (5.10), (5.12), (5.15) and (5.16) one has as $s \uparrow 1$

$$\frac{1}{(1-s)\Delta_0(s)} [\sum_{t=0}^{\infty} \gamma_t R_2^s(1-F_t)s^t + \sum_{t=0}^{\infty} \gamma_t R_2^F(1-F_t)s^t] \sim c(1-s)^2, \qquad c \geq 0. \tag{5.24}$$

Now from (5.24), Lemma 3.1 and (5.19) it follows that for $s \uparrow 1$

$$\sum_{t=0}^{\infty} \gamma_t R_2^F(1-F_t)s^t = c_1(1-s)^{2+\theta} L_1(\frac{1}{1-s}), \qquad c_1 \geq 0. \tag{5.25}$$

Assume $F'''(1) = \infty$. Therefore $(1-s)^{-3} | R_2^F(1-s) | \uparrow \infty$ as $s \uparrow 1$ and for each $M < \infty$ there is $T < \infty$ such that

$$| R_2^F(1-F_t) | \geq M t^{-3}, \qquad t \geq T. \tag{5.26}$$

But (5.26) implies a contradiction of (5.25). Hence $F'''(1) < \infty$.

Applying induction it is not difficult to obtain the conclution (iii) of the theorem for every noninteger $\theta < -2$.

The proof in the case of an *integer* $\theta < -1$ is similar using the representation

$$\sum_{t=0}^{\infty} \bar{\Psi}_t(N)s^t = \sum_{i=-1}^{-\theta-1} \alpha_i(N)(1-s)^i + [\alpha_\theta(N)(1-s)^{-\theta} \log \frac{1}{1-s}](1+o(1)), \qquad s \uparrow 1,$$

and Corollary 3.1.

6 Proofs of Theorems 2.2 and 2.3.

Further we will use the taboo probabilities for the migration process $\{Y_t\}$, where zero is the taboo state:

(6.1) $$\begin{cases} {}_0p_n(t) &= P(Y_t = n, Y_i > 0, 1 \le i < t \mid Y_0 = 0), \qquad t = 1, 2, \dots \\ {}_0p_n(t) &= 0, \qquad n = 0, 1, 2, \dots \end{cases}$$

Proof of Theorem 2.2. Applying (6.1) and (2.9) one gets

(6.2) $$P(\tau = t) = u_{t-1} - u_t = \frac{{}_0p_0(t+1)}{1 - {}_0p_0(1)} = \frac{{}_0p_0(t+1)}{r(1-h_0)},$$

because of ${}_0p_0(1) = P(Y_1 = 0 \mid Y_0 = 0) = 1 - r(1-h_0)$.

Let $\Phi_t = P(Y_t = 0) = \Phi(t, 0)$. Then

(6.3) $$\Phi_0 = 1, \qquad \Phi_t = \sum_{k=1}^{t} {}_0p_0(k)\Phi_{t-k}, \qquad t = 1, 2, \dots$$

Therefore

(6.4) $$\Phi(s) = \frac{1}{1 - \beta(s)},$$

where $\Phi(s) = \sum_{t=0}^{\infty} \Phi_t s^t$ and $\beta(s) = \sum_{k=0}^{\infty} {}_0p_0(k)s^k$.

On the other hand, from (6.2) one has

(6.5) $$\beta(s) = \sum_{k=0}^{\infty} {}_0p_0(k)s^k = s[1 - r(1-h_0)U(s)(1-s)].$$

Now from (6.4) applying the renewal theorem (see Feller(1979), Ch.XIII, §10, Th.1) one obtains

(6.6) $$\lim_{t\to\infty} \Phi_t = \frac{1}{\beta'(1)}.$$

Note that by Theorem 2.1 (see (2.17)) and (6.5) one has

(6.7) $$\beta'(1) = 1 - r(1-h_0)U(s) = 1 - \frac{r\mu}{b\theta}.$$

Obviously (2.19) follows from (6.6) and (6.7).

The derivation of (2.4) for $s = 1$ implies

(6.8) $$\begin{aligned} \Phi'(t+1, 1) &= \Phi'(t, 1) + b\theta(1 - \Phi(t, 0)) + r\mu\Phi(t, 0) \\ &= b\theta \textstyle\sum_{k=0}^{t} P(Y_k > 0) + r\mu \sum_{k=0}^{t} P(Y_k = 0). \end{aligned}$$

On the other hand, from (6.1) it follows that for $n \ge 1$

(6.9) $${}_0p_n(t+1) = P(Y_{t+1} = n, Y_i > 0, 1 \le i < t \mid Y_1 > 0, Y_0 = 0) = \frac{{}_0p_n(t+1)}{r(1-h_0)},$$

because of $P(Y_1 >| Y_0 = 0) = r(1-h_0)$.

Now from (6.9) using that $u_t = \sum_{n=1}^{\infty} {}_0\bar{p}_n(t)$ it is not difficult to obtain

$$
(6.10)\qquad \begin{aligned}
P(Y_k > 0) &= \sum_{n=1}^{\infty} P(Y_k = n) = \sum_{n=1}^{\infty}\sum_{i=1}^{k} {}_0p_n(i)P(Y_{k-i}=0) \\
&= r(1-h_0)\sum_{n=1}^{\infty}\sum_{i=1}^{k} {}_0\bar{p}_n(i-1)\Phi_{k-i} = r(1-h_0)\sum_{i=1}^{k} u_{i-1}\Phi_{k-i}.
\end{aligned}
$$

Remember that by (2.17), $\sum_{i=0}^{\infty} u_i = -\frac{\mu}{b\theta(1-h_0)}$. Then from (6.8) and (6.10) it follows

$$
(6.11)\qquad \begin{aligned}
EY_{t+1} &= b\theta r(1-h_0)\sum_{k=0}^{t}\sum_{i=1}^{k} u_{i-1}\Phi_{k-i} + r\mu\sum_{k=0}^{t}\Phi_k \\
&= b\theta r(1-h_0)\sum_{i=0}^{t-1} u_i \sum_{j=1}^{t-i-1}\Phi_j + r\mu\sum_{k=0}^{t}\Phi_k \\
&= b\theta r(1-h_0)\left(\sum_{i=0}^{\infty} u_i\sum_{j=0}^{t}\Phi_j - \sum_{i=t}^{\infty} u_i\sum_{j=0}^{t}\Phi_j - \sum_{i=0}^{t-1} u_i \sum_{j=t-i}^{t}\Phi_j\right) + \\
&+ r\mu\sum_{k=0}^{t}\Phi_k = -b\theta r(1-h_0)\sum_{j=0}^{t}\Phi_j\sum_{i=t-j}^{\infty} u_i \\
&= -b\theta r(1-h_0)\sum_{k=0}^{t}\Phi_{t-k}\sum_{i=k}^{\infty} u_i.
\end{aligned}
$$

From Theorem 2.1 (see(2.18)) it follows (see Feller(1971), Ch.VIII, §9, Th.1) that

$$
(6.12)\qquad \sum_{i=k}^{\infty} u_i \sim c_1 k^{\theta}, \qquad c_1 > 0, \quad k\to\infty.
$$

Now from (6.11) applying (2.19) and (6.12) it is not difficult to obtain (2.20).

Lemma 6.1 *Under the conditions (2.3) and (2.13)–(2.15) for* $0 \le s < 1$

$$
(6.13)\qquad \frac{1}{\gamma_t(s)}\sum_{k=0}^{t-1}\gamma_k(s)(1-H(F_k(s))) = -\frac{\mu}{b\theta} - \varphi(t)(1+o(1)), \qquad t\to\infty,
$$

where

$$
(6.14)\qquad \varphi(t) = \begin{cases} c_1(s)t^{\theta} &, \quad -1<\theta<0, \\ c_2(s)\dfrac{\log t}{t} &, \quad \theta = -1, \\ c_3(s)\dfrac{1}{t} &, \quad \theta < -1, \qquad c_i(s) > 0, \quad i = 1,2,3. \end{cases}
$$

Proof. (i) First we will consider the case $s = 0$. Note that from (2.8) and (2.11) we have

$$
(6.15)\qquad u_k = \gamma_k\frac{1-H(F_k)}{1-h_0} + \sum_{j=1}^{k} u_{k-j}(\gamma_{j-1}-\gamma_j).
$$

Hence

$$\sum_{k=0}^{t} u_k = \sum_{k=0}^{t} \gamma_k \frac{1 - H(F_k)}{1 - h_0} + \sum_{j=1}^{t} (\gamma_{j-1} - \gamma_j) \sum_{i=0}^{t} u_i - \sum_{k=1}^{t} u_k(\gamma_{t-k} - \gamma_t). \tag{6.16}$$

Now from (6.16) it is not difficult to show that

$$\sum_{k=0}^{t} u_k = \frac{1}{(1 - h_0)\gamma_t} \sum_{k=0}^{t} \gamma_k (1 - H(F_k)) - \frac{1}{\gamma_t} \sum_{k=1}^{t} u_k(\gamma_{t-k} - \gamma_t). \tag{6.17}$$

On the other hand from (6.17) it follows

$$\frac{1}{(1 - h_0)\gamma_t} \sum_{k=0}^{t} \gamma_k (1 - H(F_k)) = \sum_{k=0}^{\infty} u_k - \sum_{k=t+1}^{\infty} u_k + \frac{1}{\gamma_t} \sum_{k=1}^{t} u_k(\gamma_{t-k} - \gamma_t). \tag{6.18}$$

From (6.18) applying Theorem 2.1 (see (2.17)) one obtains

$$\frac{1}{\gamma_t} \sum_{k=0}^{t-1} \gamma_k (1 - H(F_k)) = -\frac{\mu}{b\theta} - \Psi(t), \tag{6.19}$$

where

$$\Psi(t) = 1 - H(F_t) + (1 - h_0) \sum_{k=t+1}^{\infty} u_k - \frac{1 - h_0}{\gamma_t} \sum_{k=1}^{t} u_k(\gamma_{t-k} - \gamma_t). \tag{6.20}$$

Note that as $t \to \infty$

$$1 - H(F_t) \sim \frac{\mu}{bt} \quad \text{and} \quad \sum_{k=t+1}^{\infty} u_k \sim ct^{\theta}, \quad c > 0. \tag{6.21}$$

On the other hand, using Lemma 3.1 for $s = 0$ and Theorem 2.1 (see (2.18)) it is not difficult to obtain

$$-\sum_{k=1}^{t} u_k(\gamma_{t-k} - \gamma_t) \sim c \int_1^t x^{\theta-1}[(t - x)^{-\theta} - t^{-\theta}]dx \sim \begin{cases} c_1 & , \quad -1 < \theta < 0, \\ c_2 \log t & , \quad \theta = -1, \\ c_3 t^{-\theta-1} & , \quad \theta < -1, \end{cases} \tag{6.22}$$

where $c_i > 0, i = 1, 2, 3$.

Now the conclusion of the lemma for $s = 0$ follows immediatly from (6.19)–(6.22).

(ii) We will prove now (6.13) for $0 < s < 1$. Let consider the sequence $u_k(s)$, $0 < s < 1$, $k = 1, 2, \ldots$ defined as follows

$$u_0(s) \equiv 1, \qquad u_k(s) = \gamma_k(s) \frac{1 - H(F_k(s))}{1 - h_0} + \sum_{j=1}^{k} u_{k-j}(s)[\gamma_{k-1}(s) - \gamma_j(s)]. \tag{6.23}$$

Then similarly to (6.18) it is not difficult to obtain

$$\begin{aligned} &\frac{1}{(1 - h_0)\gamma_t(s)} \sum_{k=0}^{t} \gamma_k(s)[1 - H(F_k(s))] \\ &= \sum_{k=0}^{\infty} u_k(s) - \sum_{k=t+1}^{\infty} u_k(s) + \frac{1}{\gamma_t(s)} \sum_{k=1}^{t} u_k(s)(\gamma_{t-k}(s) - \gamma_t(s)). \end{aligned} \tag{6.24}$$

Note that as $t \to \infty$

$$1 - F_t(s) = \frac{1 + \alpha_t(s)}{bt + (1-s)^{-1}}, \qquad \text{where} \quad \lim_{t\to\infty} \sup_{0\le s<1} \alpha_t(s) = 0.$$

Now applying Lemma 3.1 and following the proof of Theorem 2.1 it is not difficult to show that for $0 < s < 1$

$$\text{(6.25)} \qquad \sum_{k=0}^{\infty} u_k(s) = -\frac{\mu}{b\theta(1-h_0)} \quad \text{and} \quad u_t(s) \sim c(s)t^{\theta-1}, \quad c(s) > 0.$$

From (6.24) and (6.25) one can prove (6.13) for $0 < s < 1$ similarly to the case $s = 0$.

Proof of Theorem 2.3. Let $p_{ij}(t) = P(Y_{t+k} = j \mid Y_k = i)$. Note that $p_{00}(t) = \Phi_t$ and $p_{00}(1) = \Phi_1 = 1 - r(1-h_0) > 0$. Therefore the state zero has a period 1.

On the other hand, by the definition of the process $\{Y_t\}$ it follows that the set of its states $\mathcal{Z}$ consists of all states which can be reached by the state 0, i.e. for each $j \in \mathcal{Z}$ there exists t_j such that $p_{0j}(t_j) > 0$.

Hence for all $i, j \in \mathcal{Z}$

$$p_{ij}(t_j + 1) \ge p_{i0}(1)p_{0j}(t_j) > 0$$

because of $p_{i0}(1) = pf_0^{\max(i-1,0)} + qf_0^i + rf_0^i g_0 > 0$.

Now by Theorem 2.2 (see (2.19)) and the well-known results for Markov chains (see Feller(1979), Ch.XV) it follows that the aperiodic and non-decomposable Markov chain $\{Y_t\}$ is ergodic, i.e. there exits a stationary distribution $\{v_k\}, k = 0, 1, \ldots$

Therefore

$$\lim_{t\to\infty} \Phi(t,s) = V(s), \qquad V(s) = \sum_{k=0}^{\infty} v_k s^k, \quad |s| \le 1,$$

and putting $t \to \infty$ in (2.4) one obtains the equation

$$\text{(6.26)} \qquad V(s) = V(F(s))\delta(s) + \frac{b\theta}{b\theta - r\mu}[1 - \delta(s) - r(1 - H(s))]$$

with the initial condition (see Theorem 2.2, (2.19))

$$\text{(6.27)} \qquad V(0) = \frac{b\theta}{b\theta - r\mu}.$$

Now, we will prove that under the conditions (2.3) the equation (6.26) with initial condition (6.27) has a unique solution $V(s)$ with $\lim_{s\uparrow 1} V(s) = 1$.

Iterating (6.26) one gets

$$\text{(6.28)} \qquad V(s) = V(F_t(s))\gamma_t(s) + \frac{b\theta}{b\theta - r\mu}\sum_{k=0}^{t-1} \gamma_k(s)\Delta(F_k(s)),$$

where

$$\Delta(s) = 1 - \delta(s) - r(1 - H(s)).$$

It is not difficult to see that for $0 \le s < 1$ and $t \to \infty$

$$\Delta(F_t(s)) = \frac{b\theta - r\mu}{bt + (1-s)^{-1}}(1 + o(1)). \tag{6.29}$$

On the other hand, from (6.27) and (6.28) for $s = 0$ one has

$$V(F_t) = \frac{b\theta}{(b\theta - r\mu)\gamma_t}[1 - \sum_{k=0}^{t-1} \gamma_k \Delta(F_k)]. \tag{6.30}$$

Now, from Lemma 3.1 and (6.29) for $s = 0$ and $t \to \infty$ it follows that

$$\sum_{k=0}^{t-1} \gamma_k \Delta(F_k) \sim \frac{r\mu - b\theta}{b\theta} t^{-\theta} L(t), \tag{6.31}$$

where $L(s)$ is a s.v.f.

Hence from (6.30) applying again Lemma 3.1 and (6.31) one obtains $\lim_{t\to\infty} V(F_t) = 1$, which proves that

$$\lim_{s\uparrow 1} V(s) = V(1) = \lim_{t\to\infty} \Phi(t, 1) = 1, \tag{6.32}$$

i.e. the stationary distribution $\{v_k\}$ is a proper one.

Let now $W(s)$ is some solution of (6.26) with $W(1) = 1$. Then from (6.28) it follows

$$\frac{W(s)}{V(s)} = \frac{W(F_t(s))\gamma_t(s) + \frac{b\theta}{(b\theta - r\mu)} \sum_{k=0}^{t-1} \gamma_k(s)\Delta(F_k(s))}{V(F_t(s))\gamma_t(s) + \frac{b\theta}{(b\theta - r\mu)} \sum_{k=0}^{t-1} \gamma_k(s)\Delta(F_k(s))}. \tag{6.33}$$

Now, putting $t \to \infty$ in (6.33) and applying Lemma 3.1, (6.29) and (6.32) one gets $W(s) \equiv V(s)$ for $0 \le s \le 1$ which proves the uniqueness of the solution of (6.26).

Let now for $0 < s \le 1$ define

$$V_n(s) = \gamma_n(s) + V(0) \sum_{k=0}^{n-1} \gamma_k(s)\Delta(F_k(s)), \qquad n = 1, 2, \dots \tag{6.34}$$

To prove (2.25) it is enough to show that for $-1 < \theta < 0$ there exists $\lim_{n\to\infty} V_n(s) = W(s)$ and $W(s)$ satisfies the equation (6.26).

Indeed, from (6.34) and (6.27) applying Lemma 3.1 and Lemma 6.1 for $-1 < \theta < 0$ it follows

$$\begin{aligned} \lim_{n\to\infty} V_n(s) &= \lim_{n\to\infty} \{\gamma_n(s) + V(0) \sum_{k=0}^{n-1} \gamma_k(s)[1 - \delta(F_k(s))] - \\ &\quad -rV(0) \sum_{k=0}^{n-1} \gamma_k(s)[1 - H(F_k(s))]\} \\ &= \lim_{n\to\infty} \{\gamma_n(s) + V(0) - V(0)\gamma_n(s) - rV(0)[-\frac{\mu}{b\theta}\gamma_n(s) - c_1(s)c(s)]\} \\ &= V(0)[1 + rc_1(s)c(s)] = W(s) < \infty. \end{aligned} \tag{6.35}$$

On the other hand, from (6.34) one gets

$$
\begin{aligned}
V_{n+1}(s) &= \gamma_{n+1}(s) + V(0)\sum_{k=0}^{n}\gamma_k(s)\Delta(F_k(s)) \\
&= \delta(s)\gamma_n(F(s)) + V(0)\Delta(s) + V(0)\delta(s)\sum_{k=0}^{n}\gamma_k(F(s))\Delta(F_k(F(s))) \\
&= \delta(s)V_n(F(s)) + V(0)\Delta(s).
\end{aligned} \tag{6.36}
$$

Since (6.35) then putting $n \to \infty$ in (6.36) one obtains that $W(s)$ satisfies (6.26). Now to prove (2.24) we note that from (6.30) and Lemma 6.1 it follows

$$
\begin{aligned}
V(F_t) &= \frac{b\theta}{(b\theta - r\mu)\gamma_t}\{1 - \sum_{k=0}^{t-1}\gamma_k[1-\delta(F_k)] + r\sum_{k=0}^{t-1}\gamma_k[1-H(F_k)]\} \\
&= 1 - \frac{rb\theta}{b\theta - r\mu}\varphi(t)(1+o(1)).
\end{aligned} \tag{6.37}
$$

Since $1 - F_t \sim (bt)^{-1}$ as $t \to \infty$ then from (6.37) and (6.14) one has

$$
1 - V(F_t) \sim \begin{cases} c_1(1-F_t)^{-\theta} & , \quad -1 < \theta < 0, \\ c_2(1-F_t)\log(1-F_t)^{-1} & , \quad \theta = -1, \\ c_3(1-F_t) & , \quad \theta < -1. \end{cases} \tag{6.38}
$$

On the other hand, for each $0 \le s < 1$ there is $k \ge 0$ such that

$$
F_k \le s \le F_{k+1}. \tag{6.39}
$$

Now, from (6.39) it follows that

$$
\frac{1 - V(F_{k+1})}{(1-F_k)\log(1-F_{k+1})^{-1}} \le \frac{1-V(s)}{(1-s)\log(1-s)^{-1}} \le \frac{1-V(F_k)}{(1-F_{k+1})\log(1-F_k)^{-1}} \tag{6.40}
$$

and

$$
\frac{1 - V(F_{k+1})}{(1-F_k)^{\rho}} \le \frac{1-V(s)}{(1-s)^{\rho}} \le \frac{1-V(F_k)}{(1-F_{k+1})^{\rho}}, \qquad \rho > 0. \tag{6.41}
$$

Since $F'(1) = 1$ and $F''(1) = 2b < \infty$ then

$$
1 - F_{k+1} = 1 - F_k - O((1-F_k)^2), \qquad k \to \infty. \tag{6.42}
$$

Now, from (6.41) and (6.42) one has

$$
\frac{1 - V(F_{k+1})}{(1-F_k)^{\rho}(1+o(1))} \le \frac{1-V(s)}{(1-s)^{\rho}} \le \frac{1-V(F_k)}{(1-F_k)^{\rho}(1+o(1))} \tag{6.43}
$$

and similarly for (6.40).

The relations (6.38)–(6.43) obviously prove (2.24).

7 Proof of Theorem 2.4.

Further we will use the notations

$$EZ_t^{[n]} = E\{Z_t(Z_t-1)\dots(Z_t-n+1)\} = \Psi^{(n)}(t,1) = \frac{\partial^n\Psi(t,s)}{\partial s^n}\,|_{s=1}, \quad n=1,2,\dots$$

Derivating (2.5) for $s=1$ and iterating it is not difficult to obtain

$$\begin{aligned} \Psi'(t+1,1) &= \Psi'(t,1) + b\theta(1-\Psi(t,0)) \\ &= EZ_0 + b\theta\textstyle\sum_{k=0}^{t} P(Z_k>0) \\ &= EZ_0 + b\theta\textstyle\sum_{k=0}^{\infty} u_k - b\theta\textstyle\sum_{k=t+1}^{\infty} u_k. \end{aligned} \tag{7.1}$$

Now, from (7.1) applying Theorem 2.1 one gets

$$EZ_t = \Psi'(t,1) = -b\theta\sum_{k=t}^{\infty} u_k \sim -\frac{b\theta}{|\,\theta\,|}L(t)t^{\theta} = bL(t)t^{\theta}, \tag{7.2}$$

where $L(t)$ is a s.v.f. and $L(t)\equiv c>0$ if additionally $\sum_{k=2}^{\infty} f_k k^2\log k<\infty$.

Let consider the inequality (see Sevastyanov(1971), Lemma IX.1.1)

$$P(\xi>0) \ge \frac{(E\xi)^2}{E\xi^2+E\xi} \ge \frac{(E\xi)^2}{2E\xi^2},$$

which is valued for non-negative integer valued r.v.

Hence from here, (7.2) and (2.18) it follows

$$\frac{EZ_t^2}{EZ_t} \ge \frac{EZ_t}{2P(Z_t>0)} \sim \frac{b}{2}t. \tag{7.3}$$

Applying the well-known Lyapounov inequality to (7.3) one obtains for $n\ge 2$ and t large enough

$$\frac{EZ_t^{n+1}}{EZ_t^n} \ge \frac{EZ_t^n}{EZ_t^{n-1}} \ge \frac{EZ_t^2}{EZ_t} \ge c_2 t, \qquad c_2>0. \tag{7.4}$$

Now, from (7.2) and (7.4) it is not difficult to show by induction that for large enough t

$$EZ_t^n \ge ct^{n+\theta-1}, \qquad c>0, \quad n=1,2,\dots \tag{7.5}$$

A formal derivation of (2.5) $(n+1)$–times in $s=1$ implies

$$\begin{aligned} &\Psi^{(n+1)}(t+1,1) = \Psi^{(n+1)}(t,F(1)) + (n+1)b\theta\Psi^{(n)}(t,F(1)) + \\ &+\sum_{k=1}^{n-1}\binom{n+1}{n+1-k}\delta^{(n+1-k)}(1)\Psi^{(k)}(t,F(1)) + \delta^{(n+1)}(1)(1-\Psi(t,0)). \end{aligned} \tag{7.6}$$

Applying Faá di Bruno's formula (see Abramowitz and Stegun(1970), p.823) one gets

$$\Psi^{(n+1)}(t,F(s))\Big|_{s=1} = \Psi^{(n+1)}(t,1) + n(n+1)b\Psi^{(n)}(t,1) + \sum_{k=1}^{n-1} c(k,n)\Psi^{(k)}(t,1). \tag{7.7}$$

Now, from (7.6) and (7.7) applying also an iteration it is not difficult to show that

$$
\begin{aligned}
EZ_{t+1}^{[n+1]} &= \Psi^{(n+1)}(t+1,1) = \Psi^{(n+1)}(t,1) + (n+1)b(n+\theta)\Psi^{(n)}(t,1)+ \\
&+ \sum_{k=1}^{n-1} c(k,n)\Psi^{(k)}(t,1) + \delta^{(n+1)}(1)(1-\Psi(t,0)) \\
&= EZ_0^{[n+1]} + (n+1)b(n+\theta)\sum_{j=0}^{t} EZ_j^{[n]} + \sum_{k=1}^{n-1} c(k,n)\sum_{j=0}^{t} EZ_j^{[k]} + \\
&+\delta^{(n+1)}(1)\sum_{j=0}^{t} P(Z_j > 0).
\end{aligned}
\tag{7.8}
$$

(i) First we will prove (2.27) in the case $2 \le m < -\theta$. Assume now (2.27) for $m = n < -\theta$. Then all series on the right side in (7.8) converge as $t \to \infty$ and therefore

$$
\lim_{t\to\infty} EZ_t^{[n+1]} = c_1 \ge 0. \tag{7.9}
$$

We will prove that $c_1 = 0$ if $n+1 < -\theta$.

If one assumes that $c_1 > 0$, then changing in (7.8) n with $n+1$ one obtains $EZ_t^{[n+2]} \to -\infty$ as $t \to \infty$, which is imposible. Therefore $c_1 = 0$ in (7.9).

Now, from (7.8), (7.9) and the induction predicate it follows that

$$
\begin{aligned}
EZ_{t+1}^{[n+1]} &= EZ_0^{[n+1]} + (n+1)b(n+\theta)\left(\sum_{j=0}^{\infty} EZ_j^{[n]} - \sum_{j=t+1}^{\infty} EZ_j^{[n]}\right) + \\
&+ \sum_{k=1}^{n-1} c(k,n)\left(\sum_{j=0}^{\infty} EZ_j^{[k]} - \sum_{j=t+1}^{\infty} EZ_j^{[k]}\right) + \delta^{(n+1)}(1)\left(\sum_{j=0}^{\infty} u_j - \sum_{j=t+1}^{\infty} u_j\right) \\
&= \lim_{t\to\infty} EZ_t^{[n+1]} - (n+1)b(n+\theta)\sum_{j=t+1}^{\infty} EZ_j^{[n]} - \sum_{k=1}^{n-1} c(k,n)\sum_{j=t+1}^{\infty} EZ_j^{[k]} - \\
&-\delta^{(n+1)}(1)\sum_{j=t+1}^{\infty} u_j \sim -\frac{(n+1)b(n+\theta)}{\mid n+\theta \mid} b^n n! c t^{n+\theta} = c(n+1)! b^{n+1} t^{n+\theta},
\end{aligned}
\tag{7.10}
$$

where $c = \lim_{t\to\infty} P(Z_t > 0)t^{1-\theta} > 0$ (see Theorem 2.1).

Since (2.27) is fulfilled for $m = n+1 < -\theta$ then by induction it is true for all $1 \le m < -\theta$.

(ii) Consider now the case $m = -\theta$. Let $n+1 = -\theta$. Then it follows that (2.27) is true for $EZ_t^{[n]}$ and from (7.8) one has

$$
\lim_{t\to\infty} EZ_t^{[n+1]} = c_1 \ge 0. \tag{7.11}
$$

On the other hand, from (7.8) for $n+1 = -\theta$ one obtains

$$
\begin{aligned}
EZ_{t+1}^{[n+2]} &= EZ_0^{[n+2]} + (n+2)b(-\theta+0)\sum_{j=0}^{t} EZ_j^{[n]} + \\
&+ \sum_{k=1}^{n} c(k,n)\sum_{j=0}^{t} EZ_j^{[k]} + \delta^{(n+2)}(1)\sum_{j=0}^{t} u_j
\end{aligned}
\tag{7.12}
$$

and similarly to (7.9) one gets

$$
\lim_{t\to\infty} EZ_t^{[n+2]} = c_2 < \infty. \tag{7.13}
$$

If one supposes now that $c_1 > 0$ in (7.11) then from (7.13) it follows that

$$\lim_{t\to\infty} \frac{EZ_t^{n+2}}{EZ_t^{n+1}} = const. \lim_{t\to\infty} \frac{EZ_t^{[n+2]}}{EZ_t^{[n+1]}} < \infty,$$

which is a contradiction with (7.4).

Therefore $c_1 = 0$ in (7.11) and similarly to (7.10) one obtains

$$EZ_t^{[n+1]} \sim c(-\theta)!b^{-\theta}t^{-1} \tag{7.14}$$

for $n+1 = -\theta$ and $c = \lim_{t\to\infty} P(Z_t > 0)t^{1-\theta}$.

(iii) Now we will prove (2.27) for $m \geq [-\theta]+1$. If θ *is an integer,* then we suppose $m = -\theta + 1$. Hence from (7.5) one has

$$\lim_{t\to\infty} EZ_t^{[m]} \geq c_1 > 0.$$

From here and (7.12) for $n+2 = m$ one gets

$$\lim_{t\to\infty} EZ_t^{[m]} = c_2 > 0. \tag{7.15}$$

One can assume $c_2 = c_1(-\theta+1)!b^{-\theta+1}, c_1 > 0$, in (7.15) and from (7.8) it is not difficult to obtain (2.27) by induction for all $m > -\theta$.

Let θ *is not integer.* As we already proved, (2.27) is true for $2 \leq m \leq [-\theta] = \nu - 1 < -\theta$. Now first we will prove (2.27) for $m = \nu$. From (7.8) one has

$$\lim_{t\to\infty} EZ_t^{[\nu]} = c_1 \geq 0. \tag{7.16}$$

Assume $c_1 > 0$. Then from (7.8) it follows

$$EZ_t^{[\nu+1]} \sim (\nu+1)b(\nu+\theta)c_1 t. \tag{7.17}$$

Since $EZ_t^{[n]} \sim EZ_t^n$ then applying Lyapounov inequality for $n \leq \nu+1$ one obtains

$$\begin{aligned} 0 &< c_1^2(1+o(1)) = (EZ_t^\nu)^2 \leq EZ_t^{\nu+1}EZ_t^{\nu-1} \\ &\sim [(\nu+1)b(\nu+\theta)ct][c_1(\nu-1)!b^{\nu-1}t^{\nu-1+\theta-1}] \\ &\sim c_2 t^{\nu-1+\theta} \to 0, \qquad t\to\infty. \end{aligned}$$

The obtained contradiction shows that $c_1 = 0$ in (7.16). Now similarly to (7.10) it is not difficult to obtain that

$$EZ_t^{[\nu]} \sim c_1\nu!b^\nu t^{\nu+\theta-1}.$$

From here and (7.8) it follows by induction that (2.27) is fulfilled for all $m \geq [-\theta]+1$, which complete the proof of the theorem.

8 Truncated processes. Proof of Theorem 2.5.

Let $C_k = \{\omega : X_i(t) = k\}, k = 0, 1, 2, \dots$ Then $P(C_k) = f_k$ and $\sum_{k=0}^{\infty} f_k = 1$.

Let now for every $0 < \varepsilon < 1$ define $D_0 \subset C_0$ such that $P(D_0) = f_0(1-\varepsilon)$. Hence we can define the i.i.d.r.v. $\{\bar{X}_i(t)\}$ as follows: for $N \geq 1$ and $0 < \varepsilon = \varepsilon(N) < 1$ we put

(8.1)
$$\bar{X}_i(t) = \begin{cases} 0 & , \ \omega \in D_0 \ , \\ X_i(t) & , \ \omega \in D_k \ , \quad k = 1, 2, \dots, N-1, \\ N & , \ \omega \in D_N \ , \end{cases}$$

where $D_k = C_k$, $k = 1, 2, \dots, N-1$ and $D_N = \bigcup_{k=N}^{\infty} C_k \bigcup (C_0 \backslash D_0)$.

One can also consider

(8.2)
$$\bar{Z}_0 = Z_0 1_{\{Z_0 \leq N-1\}} + N 1_{\{Z_0 \geq N\}}$$

and

(8.3)
$$\bar{I}(t) = I(t) 1_{\{I(t) \leq N-1\}} + N 1_{\{I(t) \geq N\}}.$$

Now, we can define the truncated process $\{\bar{Z}_t\}$ by the following recurent formula:

(8.4)
$$\bar{Z}_{t+1} = \sum_{k=1}^{\bar{Z}_t} \bar{X}_k(t) + \bar{M}_t^0 \qquad , t = 0, 1, 2, \dots,$$

where

$$\bar{M}_t^0 = \begin{cases} -\bar{X}_1(t) 1_{\{\bar{Z}_t > 0\}} & \text{with probability p,} \\ 0 & \text{with probability q,} \\ \bar{I}(t) 1_{\{\bar{Z}_t > 0\}} & \text{with probability r,} \end{cases} \qquad \text{p+q+r=1,}$$

$$p + q + r = 1, \quad t = 0, 1, 2, \dots; \qquad \bar{Z}_0 > 0.$$

From (8.1) one gets

$$E\bar{X}_i(t) = \sum_{k=1}^{N} k f_k + N \left(\varepsilon f_0 + \sum_{k=N+1}^{\infty} f_k \right) = 1 - \sum_{k=N+1}^{\infty} (k-N) f_k + N f_0 \varepsilon.$$

Further on we will assume that

(8.5)
$$0 < \varepsilon = \varepsilon(N) = \frac{1}{N f_0} \sum_{k=N+1}^{\infty} (k-N) f_k.$$

Note that $\varepsilon(N) \leq \frac{1}{N f_0}$ and $\varepsilon(N) \to 0, N \to \infty$. Now, from (8.5) it follows that

(8.6)
$$E\bar{X}_i(t) = 1 = EX_i(t).$$

Now, from (8.5) and (8.6) one has

$$
(8.7)\quad
\begin{aligned}
2\bar{b} &= Var\bar{X}_i(t) = \sum_{k=1}^{N} k^2 f_k + N^2\left(\varepsilon f_0 + \sum_{k=N+1}^{\infty} f_k\right) - 1 \\
&= 2b - \sum_{k=N+1}^{\infty} k^2 f_k + N^2 \varepsilon f_0 + N^2 \sum_{k=N+1}^{\infty} f_k \\
&= 2b - \sum_{k=N+1}^{\infty} f_k k(k-N) < 2b.
\end{aligned}
$$

Similarly from (8.3) one gets

$$
(8.8)\quad
\begin{aligned}
\bar{\lambda} &= E\bar{I}(t) = \sum_{k=1}^{N} k g_k + N \sum_{k=N+1}^{\infty} g_k = \lambda - \sum_{k=N+1}^{\infty} k g_k + N \sum_{k=N+1}^{\infty} g_k \\
&= \lambda - \sum_{k=N+1}^{\infty} (k-N) g_k < \lambda.
\end{aligned}
$$

Now, from (8.5)–(8.8) it follows that $\{\bar{Z}_t\}$ is a critical branching migration process and for every $N \geq 1$ one has

$$
(8.9)\quad \bar{\theta} = \frac{r\bar{\lambda} - p}{\bar{b}} < \frac{r\lambda - p}{b} = \theta < 0.
$$

Lemma 8.1 *Assume (2.3), (2.14)–(2.17a) and (8.5). Then there exists $N = N(t) = o(t) \to \infty$ as $t \to \infty$ such that*

$$
\lim_{t\to\infty} \left| P\left(\frac{Z_t}{bt} \leq x \mid Z_t > 0\right) - P\left(\frac{\bar{Z}_t}{\bar{b}t} \leq x \mid \bar{Z}_t > 0\right) \right| = 0.
$$

Proof. Note that

$$
(8.10)\quad
\begin{aligned}
P(Z_t > 0) &= P(Z_t > 0, Z_t = \bar{Z}_t) + P(Z_t > 0, Z_t \neq \bar{Z}_t) \\
&\leq P(\bar{Z}_t > 0) + P(Z_t \neq \bar{Z}_t).
\end{aligned}
$$

Similarly to (8.10) one gets

$$
(8.11)\quad \left| \frac{P(Z_t > 0)}{P(\bar{Z}_t > 0)} - 1 \right| = \frac{\mid P(Z_t > 0) - P(\bar{Z}_t > 0) \mid}{P(\bar{Z}_t > 0)} \leq \frac{P(Z_t \neq \bar{Z}_t)}{P(\bar{Z}_t > 0)}.
$$

Now, similarly to (8.10) and (8.11) one obtains

$$
(8.12)\quad -2\frac{P(Z_t \neq \bar{Z}_t)}{P(\bar{Z}_t > 0)} \leq P\left(\frac{Z_t}{bt} \leq x \mid Z_t > 0\right) - P\left(\frac{\bar{Z}_t}{\bar{b}t} \leq x \mid \bar{Z}_t > 0\right) \leq 2\frac{P(Z_t \neq \bar{Z}_t)}{P(Z_t > 0)}
$$

Let now

$$
\begin{aligned}
A_j(t) &= \{\omega : X_j(t) \neq \bar{X}_j(t)\} &&, \quad j = 1, 2, \ldots; \quad t = 0, 1, 2, \ldots; \\
B_t &= \{\omega : I(t) \neq \bar{I}(t), Z_t > 0\} &&, \quad t = 0, 1, 2, \ldots; \\
C &= \{\omega : \bar{Z}_0 < Z_0\} &&.
\end{aligned}
$$

From here, (1.5), (8.4) and (8.1) it follows that

$$P(Z_t \neq \bar{Z}_t) \leq P(C) + \sum_{i=0}^{t-1} E\{E[1_{\{\bigcup_{j=1}^{Z_i} A_j(i)\}} \mid Z_i]\} + \sum_{k=0}^{t-1} P(B_k). \tag{8.13}$$

On the other hand

$$\begin{aligned} E\{E[1_{\{\bigcup_{j=1}^{Z_i} A_j(i)\}} \mid Z_i]\} &\leq E\textstyle\sum_{j=1}^{Z_i} P(A_j(i)) \\ &= EZ_i P(X_j(i) \neq \bar{X}_j(i)), \end{aligned} \tag{8.14}$$

where from (8.1) and (8.5) one gets

$$P(X_j(i) \neq \bar{X}_j(i)) = 1 - \sum_{k=0}^{N} P(D_k) = \frac{1}{N} \sum_{k=N+1}^{\infty} (k-N) f_k + \sum_{k=N+1}^{\infty} f_k = O(U_N(f)), \tag{8.15}$$

where

$$U_N(f) = \begin{cases} N^{-2} \sum_{k=N+1}^{\infty} k^2 f_k & , \quad -1 < \theta < 0, \\ (N^2 \log N)^{-1} \sum_{k=N+1}^{\infty} k^2 f_k & , \quad \theta = -1, \\ N^{-(1-\theta)} \sum_{k=N+1}^{\infty} k^{1-\theta} f_k & , \quad \theta < -1. \end{cases}$$

Similarly

$$P(B_k) = P(I(k) \neq \bar{I}(k)) P(Z_k > 0) = P(Z_k > 0) O(U_N(g)). \tag{8.16}$$

and

$$P(C) = P(Z_0 \geq N+1) = \sum_{N+1}^{\infty} q_k = O(V_N(q)). \tag{8.17}$$

where

$$V_N(q) = \begin{cases} N^{-2} \sum_{k=N+1}^{\infty} k^2 q_k & , \quad -1 \leq < \theta < 0, \\ N^{-(1-\theta)} \sum_{k=N+1}^{\infty} k^{1-\theta} q_k & , \quad \theta < -1. \end{cases}$$

Now from (8.13)-(8.17) one obtains

$$P(Z_t \neq \bar{Z}_t) = O(V_N(q)) + O(U_N(f)) \sum_{i=0}^{t-1} EZ_i + O(V_N(g)) \sum_{k=0}^{t-1} P(Z_k > 0). \tag{8.18}$$

Note that from Theorem 2.4 (see (2.26)) it follows that

$$\sum_{i=0}^{t-1} EZ_i \sim W(t) = \begin{cases} c_1 t^{1+\theta} & , \quad -1 < \theta < 0, \\ c_2 \log t & , \quad \theta = -1, \\ c_3 & , \quad \theta < -1. \end{cases} \tag{8.19}$$

Similarly from Theorem 2.1 (see (2.18)) one has

$$\sum_{k=0}^{t-1} P(Z_k > 0) = O(1), \qquad t \to \infty. \tag{8.20}$$

Hence from (8.18)-(8.20) and (2.18) one obtains

$$P(Z_t \neq \bar{Z}_t)/P(Z_t > 0) = O\left(t^{1-\theta}[V_N(q) + U_N(g) + U_N(f)W(t)]\right). \tag{8.21}$$

Now one can choose $N = N(t) = o(t) \to \infty$, such that

$$t^{1-\theta}[V_N(q) + U_N(g) + U_N(f)W(t)] \to 0, \qquad t \to \infty. \tag{8.22}$$

Therefore, from (8.22) and (8.23) one gets

$$\lim_{t\to\infty} P(Z_t \neq \bar{Z}_t \mid Z_t > 0) = 0. \tag{8.23}$$

Since $\{\bar{Z}_t\}$ is also a critical migration process with $\bar{\theta} < 0$, (see (8.9)) then

$$\lim_{t\to\infty} P(Z_t \neq \bar{Z}_t \mid \bar{Z}_t > 0) = 0. \tag{8.24}$$

Hence (8.11), (8.12), (8.24) and (8.25) implies the lemma.

Proof of Theorem 2.5. (2.28) follows from Theorem 2.1 and Theorem 2.4 because

$$E(Z_t \mid Z_t > 0) = \frac{EZ_t}{P(Z_t > 0)} \sim \frac{bL(t)t^{\theta}}{L(t)t^{\theta}} = bt.$$

First we will prove a similar relation to (2.29) for the process $\{\bar{Z}_t\}$, when N is fixed i.e.

$$\lim_{t\to\infty} P\left(\frac{\bar{Z}_t}{bt} \leq x \mid \bar{Z}_t > 0\right) = 1 - e^{-x}, \qquad x \geq 0. \tag{8.25}$$

Note first that from Theorem 2.1 (see (2.18)) and Theorem 2.4 (see (2.27)) applied to the process $\{\bar{Z}\}$ (with (8.4) and (8.5)) it is not difficult to obtain that

$$\lim_{t\to\infty} E\left\{\left(\frac{\bar{Z}_t}{bt}\right)^n \mid \bar{Z}_t > 0\right\} = \begin{cases} n! & , \ 1 \leq n \leq [-\theta], \\ \frac{c_1}{c} n! = \delta n! & , \ n \geq [-\theta] + 1, \end{cases} \tag{8.26}$$

where $c_1 > 0$ and $c = \lim_{t\to\infty} P(\bar{Z}_t > 0)t^{1-\bar{\theta}} > 0$.

We will prove that $\delta = 1$. Let $\varphi_t(\alpha)$ be the Laplace transformation of the function $W_t(x) = P(\frac{\bar{Z}_t}{bt} \leq x \mid \bar{Z}_t > 0), x \geq 0$.

It is well-known (see Feller(1971)) that for $\alpha \geq 0$

$$\sum_{k=0}^{2n-1} \frac{(-1)^k}{k!}\varphi_t^{(k)}(0)\alpha^k \leq \varphi_t(\alpha) \leq \sum_{k=0}^{2n} \frac{(-1)^k}{k!}\varphi_t^{(k)}(0)\alpha^k. \tag{8.27}$$

Now, from (8.25) and (8.26) it follows that for $0 < \alpha < 1 - u$, $0 < u < 1$,

$$\begin{aligned} \lim_{t\to\infty} \varphi_t(\alpha) &= \varphi(\alpha) = \sum_{k=0}^{[-\theta]}(-1)^k\alpha^k + \delta \sum_{k=[-\theta]+1}^{\infty} (-1)^k\alpha^k \\ &= \frac{\delta}{1+\alpha} + (1-\delta)\sum_{k=0}^{[-\theta]}(-1)^k\alpha^k. \end{aligned} \tag{8.28}$$

From (8.27) it follows that $\lim_{t\to\infty}\varphi_t(\alpha) = \varphi(\alpha)$ for $\alpha > 0$ because $\varphi_t(\alpha)$ and $\varphi(\alpha)$ are analytic functions. Since $\varphi(\alpha) \to 1, \alpha \downarrow 0$, then $\varphi(\alpha)$ is a Laplace transformation of a proper distribution (see Feller(1971)).

If now assume that $\delta \neq 1$, then as $\alpha \to \infty$

$$| \varphi(\alpha) | \sim | 1-\delta | \alpha^{[-\theta]},$$

which is a contradiction with well-known properties of the Laplace transformations.

Therefore $\delta = 1$ and

$$\varphi(\alpha) = \frac{1}{1+\alpha},$$

which implies (8.24) by the Continuity theorem (see Feller(1971)).

Since from (8.7) $\bar{b} \uparrow b$, then from (8.24) one obtains for some N

$$(8.29) P\left(\frac{\bar{Z}_t}{bt} \leq x \mid \bar{Z}_t > 0\right) = P\left(\frac{\bar{Z}_t}{\bar{b}t}\frac{\bar{b}}{b} \leq x \mid \bar{Z}_t > 0\right) \to 1 - e^{-x}, \qquad x \geq 0, \quad t \to \infty.$$

Now (2.29) follows from (8.28) and Lemma 8.1.

Let now $N = N(t) = o(t)$ as in Lemma 8.1. Using Lemma 4 and Lemma 5 of Kaverin (1990) one can obtain (8.25) for $N(t) = o(t)$. Therefore (8.28) is fulfiled for $N(t) = o(t)$.

Now (2.29) follows from Lemma 8.1.

References

Abramowitz, M. and Stegun, I.A. (1970) *Handbook of Math. Functions with Formulas, Graphs and Math. Tables.* Dover, New York.

Athreya, K.B. and Ney, P.E. (1972) *Branching Processes.* Springer-Verlag, Berlin.

Bagley, J.H. (1986) On the almost sure convergence of controlled branching processes. *J. Appl. Prob.* **23**, 827-831.

Feller W. (1969) *An Introduction to Probability Theory and its Applications, Vol 1.* Wiley, New York.

Feller W. (1971) *An Introduction to Probability Theory and its Applications, Vol 2.* Wiley, New York.

Foster, J.H. (1971) A limit theorem for a branching process with state-dependent immigration. *Ann. Math. Statist.* **42**, 1773-1776.

Grey, D.R. (1988) Supercritical branching processes with density independent catastrophes. *Math. Proc. Camb. Phil. Soc.* **104**, 413-416.

Han, L.V. (1980) Limit theorems for Galton-Watson branching processes with migration. *Siberian Math. J.*, **21**, 183-194 (In Russian).

Ivanoff, G. and Seneta, E. (1985) The critical branching process with immigration stopped at zero. *J. Appl. Prob.* **22**, 223-227.

Kaverin, S.V. (1990) A Refinement of Limit Theorems for Critical Branching Processes with an Emigration. *Theory Prob. Appl.*, **35**, 574-580.

Kaverin, S. and Atamatov, A. (1988) Branching random processes with unbounded migration of particles. *Izv. AN UzSSR*, No1, 22-27 (In Russian).

Mitov, K.V. (1990) Critical branching process with migration stopped at zero. *Proc. 19th Spring Conf. Union of Bulg. Math.* 367-371 (In Bulgarian).

Nagaev, S.V. and Han, L.V. (1980) Limit theorems for critical Galton-Watson branching process with migration. *Theory Prob. Appl.* **25**, 523-534.

Pakes, A.G. (1971) A branching process with a state-dependent immigration component. *Adv. Appl. Prob.* **3**, 301-314.

Seneta, E. and Tavare, S. (1983) A note on models using the branching processes with immigration stopped at zero. *J. Appl. Prob.* **20**, 11-18.

Sevastyanov, B.A. (1971) *Branching processes.* Nauka, Mos ow (In Russion).

Sevastyanov, B.A. and Zubkov, A.M. (1974) Controlled branching processes. *Theory Prob. Appl.* **19**, 15-25.

Vatutin, V.A. (1977a) A critical Galton-Watson branching process with emigration. *Theory Prob. Appl.* **22**, 482-497.

Vatutin, V.A. (1977b) A conditional limit theorem for a critical branching process with immigration. *Math. Zametki* **21**, 727-736 (In Russian).

Yanev, G.P. and Yanev, N.M. (1989) On the critical branching migration process with predominant emigration. *C. R. Acad. Bulg. Sci.,* **42**, No12, 47-48.

Yanev, G.P. and Yanev, N.M. (1990) Extinction of Controlled Branching Processes in Random Environments. *Mathematica Balkanica, New series,* **4**, Fasc.4, 368-380.

Yanev, G.P. and Yanev, N.M. (1991) On a new model of branching migration processes. *C. R. Acad. Bulg. Sci.,* **44**, No3, 19-22.

Yanev, N.M. (1975) Conditions for extinction of φ -branching processes with random φ. *Theory Prob. Appl.,* **20**, 433-440.

Yanev, N.M. (1977) Controlled branching processes in random environment. *Math. Balkanica,* **7**, 137-156 (In Russian).

Yanev, N.M. and Mitov, K.V. (1980) Controlled branching processes: The case of random migration. *C. R. Acad. Bulg. Sci.* **33**, No4, 433-435.

Yanev, N.M. and Mitov, K.V. (1981) Critical branching migration processes. *Proc. 10th Spring Conf. Union of Bulg. Math.,* 321-328 (In Russian).

Yanev, N.M. and Mitov, K.V. (1983) The life-periods of critical branching processes with random migration. *Theory Prob. Appl.,* **28**, 458-467.

Yanev, N.M. and Mitov, K.V. (1984) Subcritical branching migration processes. *PLISKA, Bulg. Math. Publ.,* **7**, 75-82.

Yanev, N.M. and Mitov, K.V. (1985) Critical branching processes with nonhomogeneous migration. *Ann. Prob.,* **13**, 923-933.

Yanev, N.M., Vatutin, V.A. and Mitov, K.V. (1986) Critical branching processes with random migration stopped at zero. *Proc. 15th Spring Conf. Union of Bulg. Math.,* 511-517 (In Russian).

Zubkov, A.M. (1972) Life-periods of a branching process with immigration. *Theory Prob. Appl.,* **17**, 179-188.

Zubkov, A.M. (1974) Analogues between Galton-Watson processes and φ -branching processes. *Theory Prob. Appl.,* **19**, 319-333.