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# On the smooth drop property *

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# 1 Introduction.

Let $(X, \|.\|)$ be a real Banach space and $B(\|.\|)$ be the closed unit ball of $X$. By a drop $D(x, B(\|.\|))$ determined by a point $x \in X \backslash B(\|.\|)$, we shall mean the convex hull of the set $\{x\} \cup B(\|.\|)$. If a non-void closed set $S$ of a Banach space $(X, \|.\|)$ with a positive distance from the unit ball $B(\|.\|)$ is given, then there exists a point $a \in S$, such that $D(a, B(\|.\|)) \cap S = \{a\}$, which is so called Daneš Drop Theorem [4] and it is equivalent to the Ekeland variational principle (see for example [12],[6]).

We say that the space has the drop property if the above statement is valid for all closed sets $S$, disjoint with the unit ball (see [14]).

Montesinos [11] has shown that up to equivalent norm, the drop property characterizes reflexivity. More precisely, a given norm $\|.\|$ has the drop property if and only if the space is reflexive and the norm $\|.\|$ has the Kadec-Klee property. (see also [8]).

It is clear that a drop $D(x, B(\|.\|))$ is never smooth. In the present paper we shall consider a "smooth version" of the drop property, i.e. we shall work with closed convex sets, containing the unit ball which Minkowski's functional is smooth.

In [10] it is shown a smooth drop theorem of Daneš type for spaces with smooth norm where the proof is based on the smooth variational principle of Borwein and Preiss [3].

In the proof in [11] Montesinos has constructed a decreasing sequence of drops $D(x_n, B(\|.\|))$, $x_n \in S$, where for each $n$ he either applies the Daneš drop theorem and this ends the proof or chooses the next $x_{n+1} \in D(x_n, B(\|.\|)) \cap S$ with $\|x_n\| \to 1$.

In the present paper we prove a similar result : every reflexive space admits an equivalent norm with the smooth drop property.

There are some difficulties to follow Montesinos' construction, the principal among them is when applying the smooth drop theorem, or a result of Lau. More precisely, if the point $x$ given by the construction of the smooth drop theorem is on the boundary of the previous smooth drop $D$, it is not clear how to construct the next smooth drop to be contained in $D$. The main part of our proof is to assure that $x$ may be chosen in the interior of $D$.

**Definition 1.1** *A bounded subset $D$ of $(X, \|.\|)$ is called a smooth set if there exists $\varphi : X \to \mathbf{R}$ continuous, convex, positive and Fréchet-differentiable* *function such that* $D = \{x \in X : \varphi(x) \le \alpha\}$ *and there exists* $z_0$ *with* $\varphi(z_0) < \alpha$.

The following proposirion justifies the notation smooth set.

**Proposition 1.2** *Let* $D = \{x \in X : \varphi(x) \le \alpha\}$ *be a smooth set. Then the Minkowski function* $\rho_{D_0}$ *of* $D_0 := D - z_0$ *is Fréchet-differentiable on* $X \backslash \{0\}$.

*Proof.* Put $\varphi_0(x) = \varphi(x + z_0)$. Then $D_0 = \{x : \varphi_0(x) \le \alpha\}$. Since $\varphi$ is continuous convex and $\inf \varphi < \alpha$, observe that

$$\partial D_0 = \{x \in X : \varphi_0(x) = \alpha\} = \{x \in X : \rho_{D_0}(x) = 1\}. \tag{1}$$

From the convexity of $\varphi$ it follows that $\varphi_0'(x)(x) \ge \varphi_0(x) - \varphi_0(0) = \alpha - \varphi_0(0) > 0$, when $x \in \partial D_0$. For every $0 \ne x \in X$, we have $\rho_{D_0}(x) \in \partial D_0$. Let $0 \ne \overline{x_0} \in X$ be fixed and $\overline{y_0} = \rho_{D_0}(\overline{x_0})$, $x_0 = \frac{\overline{x_0}}{\overline{y_0}}, y_0 = 1$. put $F(x,y) = \varphi_0\left(\frac{x}{y}\right)$, for $x \in X, y \ge 0$. We shall prove that $\rho_{D_0}$ is Fréchet-differentiable at $x_0$, therfore at $\overline{x_0}$, using the implicit function for the equation $F(x,y) = \alpha$, which is fulfilled for $y(x) = \rho_{D_0}(x)$. Since $\rho_{D_0}\left(\frac{\overline{x_0}}{\overline{y_0}}\right) = 1$, we have by (1) that $\varphi_0(x_0) = \alpha$. Also, we have

$$F_y'(x_0, y_0) = \varphi_0'\left(\frac{x_0}{y_0}\right)\left(\frac{x_0}{y_0^2}\right) =< \varphi_0'(x_0), -x_0 >< \varphi_0(0) - \varphi_0(x_0) < 0.$$

By the implicit function theorem for the equation $F(x,y) = \alpha$, (see p. 166 of [1]), it fllows that the function $\rho_{D_0}$ is continuosly differentiable on a neighborhood of $x_0$, which is equivalent (for convex function) to Fréchet differentiability. The proposition is proved.

We shall say that the Banach space $X$ is a *space of differentiability* if the set of all equivalent Fréchet-differentiable norm is dense in the set of all equivalent norm on $X$ endowed with the metric of uniform convergence on the unit ball of $X$.

For example if there exists one equivalent L.U.R. dual norm in $X^*$, then $X$ is a space of differentiability (see e.g. [5]). But it is not known if the space $X$ is a space of differentiability whenever there exists a Fréchet-smooth norm on it.

In [10] the following statement was proved. Here we present another proof applying directly the smooth variational principle of Borwein-Preiss [3].

**Theorem 1.3** *(smooth drop theorem). Let $(X, \|.\|)$ be a space of differentiability. Let $S$ be a closed non-void subset of $X$ such that $d := dist(S, B(\|.\|)) > 0$ and let $x_1 \in S, \|x_1\| > d + 1$. Then there exists a smooth drop $D$ such that $D \cap S$ is a singleton and $D \subset B(0, \|x_1\|)$.*

Using the proof of Borwein-Preiss variational principle in Phelps [13], we derive the following version.

**Theorem 1.4** *(Borwein-Preiss smooth variational principle). Let $(X, \|.\|)$ be a Banach space and $S$ a closed non-void subset of $X$. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a l.s.c., bounded below from $S$ and $D(f) \neq \emptyset$. Let $\varepsilon > 0,\ \lambda > 0$.*

*Then for $x_1 \in S$ there exists $\{x_n\}_{n\geq 2} \subset S$ and $\{\mu_n\}_{n\geq 1} \subset \mathbf{R}_+$ such that*

*1)* $\sum_{n=1}^{\infty} \mu_n = 1$

*2)* $f(x_n) \leq f(x_1),\ \forall n \in N$

*3)* $x_n \longrightarrow a$ *in the norm topology*

*4)* $f(a) + \frac{\varepsilon}{\lambda^2} \sum_{n\geq 1} \mu_n \|a - x_n\|^2 \ < \ f(x) + \frac{\varepsilon}{\lambda^2} \sum_{n\geq 1} \mu_n \|x - x_n\|^2,\ \forall x \in S, x \neq a$

*5)* $f(a) + \frac{\varepsilon}{\lambda^2} \sum_{n\geq 1} \mu_n \|a - x_n\|^2 \ \leq \ f(x_1).$

*Proof.* *(of the smooth drop theorem)* Taking into account that $X$ is a space of differentiability and $\|x_1\| > 1 + d$, we can assume without loss of generality, that the norm $\|.\|$ is Fréchet-differentiable. Let $\varepsilon = 1$ and $\lambda > 0$ such that $(1 + \|x_1\|)/\lambda \leq \mathrm{dist}(S, B(\|\cdot\|))$.

We put $f(x) = \mathrm{dist}^2(x, B(\|\cdot\|))$ and then apply Borwein-Preiss variational principle to S. There exists $a \in S$ such that

$$(*)\ \ f(x)+\frac{1}{\lambda^2}\sum_{n\geq 1}\mu_n\|x-x_n\|^2 \ > \ f(a)+\frac{1}{\lambda^2}\sum_{n\geq 1}\mu_n\|a-x_n\|^2,\ \ \forall x \in S,\ x \neq a,$$

where $\{x_n\} \subset S,\ x_n \longrightarrow a,\ \sum_{n\geq 1} \mu_n = 1,\ \mu_n \geq 0$ and

$$f(a) + \frac{1}{\lambda^2} \sum_{n\geq 1} \mu_n \|a - x_n\|^2 \leq f(x_1)$$

after that, consider the set

$$D = \left\{ x \in X : \; f(x) + \frac{1}{\lambda^2} \sum_{n \geq 1} \mu_n \|x - x_n\|^2 \leq f(a) + \frac{1}{\lambda^2} \sum_{n \geq 1} \mu_n \|a - x_n\|^2 \right\}$$

Then $(*)$ implies $D \cap S = \{a\}$.

Let $x \in B(\|.\|)$, then

$$f(x) + \frac{1}{\lambda^2} \sum_{n \geq 1} \mu_n \|x - x_n\|^2 \; \leq \frac{1}{\lambda^2}(1 + \|x_1\|)^2 \leq \text{dist}^2(S, B(\|.\|))$$

by the definition of $\lambda$ we deduce that

$$f(x) + \frac{1}{\lambda^2} \sum_{n \geq 1} \mu_n \|x - x_n\|^2 \leq \text{dist}^2(a, B(\|.\|)),$$

then $B(\|.\|) \subset D$.

Let $x \in D$, then

$$f(x) = \text{dist}^2(x, B(\|.\|)) \leq \; f(a) + \frac{1}{\lambda^2} \sum_{n \geq 1} \mu_n \|a - x_n\|^2 \leq \; f(x_1)$$

$$= \text{dist}^2(x_1, B(\|.\|)),$$

then $\|x\| \leq \|x_1\|$ and $D \subset B\left(0, \|x_1\|\right)$. ■

**Proposition 1.5** *Let $(X, \|.\|)$ be a Banach space. We assume that there exists a smooth drop $D$ in $X$. Then there exists an equivalent Fréchet-differentiable norm in $X$.*

*Proof.* Let $D = \{x \in X : \; f(x) \leq \alpha\}$ be a smooth drop. We consider the set $C = \{x \in X : \; f(x) + f(-x) \leq 2\alpha\}$, then $C$ is symmetric, convex, bounded and $B(\|.\|) \subset C$. Therefore $C$ is a unit ball of an equivalent norm $\|.\|_1$ on $X$. Moreover, the function $g(x) = f(x) + f(-x)$ is Fréchet-differentiable, convex and $g(0) < 2\alpha$, then by Proposition 1.2, we deduce that $\|.\|_1$ is Fréchet-differentiable on $X \backslash \{0\}$. ■

**Proposition 1.6** *Let $X$ be a Banach space. We assume that the set of all equivalent norms $\|.\|$ on $X$ which satisfies : $(*)$ for all closed subset $S$ of $X$ at positive distance of $B(\|.\|)$, there exists a smooth drop $D$ such that $D \cap S$ is a singleton, is dense in the set of all equivalent norms. Then $X$ is a space of differentiability.*

*Proof.* Let $\|.\|$ be one norm in $X$ which satisfies $(*)$ and let $\varepsilon > 0$. We consider the set $S = \{x \in X : \ \|x\| = 1 + \varepsilon\}$. Then $S$ is closed at positive distance of $B(\|.\|)$. Then by hypothesis there exists a smooth drop $D$ such that $D \cap S$ is a singleton. Put $D = \{x \in X : f(x) \leq \alpha\}$ and concider $C = \{x \in X : \ f(x) + f(-x) \leq 2\alpha\}$. Thus $B(\|.\|) \subset C$ and $C \subset S$. Therefore $C$ is a unit ball of new Fréchet-differentiable norm $\|.\|_1$ (c.f. Proposition 1.5) satisfying $\|.\| \leq \|.\|_1 \leq (1 + \varepsilon)\|.\|$. ∎

# 2 The smooth drop property.

**Lemma 2.1** *Let $(X, \|.\|)$ be a Banach space and assume that the norm $\|.\|$ is Fréchet-differentiable on $X \setminus \{0\}$. Let $C = \{x \in X : f(x) \leq f(x_0)\}$ be a smooth convex set. Then for all $\varepsilon > 0$ there exists a smooth convex set $D$ such that:*

*i) $D \subset C$*
*ii) $D \cap \{x \in X : \ f(x) = f(x_0)\} = \{x_0\}$*
*iii) $diam D \ < \ \varepsilon$.*

*Proof.* Let $\alpha > 0$ such that $2[f(x_0)/\alpha]^{1/2} < \varepsilon$ and we consider the set D such that $D = \{x \in X : \ f(x) + \alpha\|x - x_0\|^2 \leq f(x_0)\}$.∎

**Definition 2.2** *Let $D$ be a subset of Banach space $(X, \|.\|)$, such that $D = conv(B(\|.\|) \cup C)$ where $C$ is a smooth convex set and we assume that the norm $\|.\|$ is Fréchet-differentiable. Then $D$ is called a quasi-smooth drop.*

**Remark 2.3** *In general a quasi-smooth drop $D$ is not a smooth drop, [7]. But it is easy to see that if the space is reflexive then the quasi-smooth drop is a smooth drop (see also [7]).*

**Lemma 2.4** *Let $(X, \|.\|)$ be a Banach space. Let $x_1 \in X \setminus B(\|.\|)$, $x_2$ in the set $\{tx_1 : 0 < t < 1\}$, $\|x_2\| > 1$ and $\rho$ in the set $\{t \in \mathbf{R} : 1 < t < \|x_2\|\}$. Then for $\varepsilon = \dfrac{\|x_1 - x_2\|}{\|x_1\|} \dfrac{\rho}{diam D(x_1, B(\|.\|))}$, we have*

$$(D(x_2, B(\|.\|)) + \varepsilon B(\|.\|)) \backslash \rho B(\|.\|) \subset intD(x_1, B(\|.\|)).$$

*Proof.* Let $x \in D(x_2, B(\|.\|)) \backslash dB(\|.\|)$. Then $x = \lambda x_2 + (1-\lambda)b$, for some $b \in B(\|.\|)$ and $\lambda \in [0,1]$. But we have $\rho < \|x - b\| = \lambda\|x_2 - b\| < \lambda M$, where $M := \operatorname{diam} D(x_1, B(\|.\|))$. Hence $\lambda > \rho/M$. Choose $\varepsilon \in (0, \varepsilon_2 \rho/M)$, where $\varepsilon_2 = \|x_1 - x_2\|/\|x_1\|$ and let $y \in x + \varepsilon B(\|.\|)$, $y_2 = \dfrac{y}{\lambda} - \dfrac{1-\lambda}{\lambda} b$. We have $\varepsilon \geq \|y - x\| = \lambda\|x_2 - y_2\|$, whence $\|x_2 - y_2\| \leq \dfrac{\varepsilon}{\lambda} < \dfrac{M\varepsilon}{\rho} < \varepsilon_2$. Therefore $y_2 \in x_2 + \varepsilon_2 \operatorname{int} B(\|.\|)$. Since $x_2 = (1 - \varepsilon_2)x_1$, we have

$$x_2 + \varepsilon_2 \operatorname{int} B(\|.\|) = (1 - \varepsilon_2)x_1 + \varepsilon_2 \operatorname{int} B(\|.\|) \subset \operatorname{int} D(x_1, B(\|.\|)).$$

So we proved $y_2 \in \operatorname{int} D(x_1, B(\|.\|))$, then

$$y = \lambda y_2 + (1 - \lambda)b \in \operatorname{int} D(x_1, B(\|.\|)). \blacksquare$$

**Theorem 2.5** *Let $(X, \|.\|)$ be a space of differentiability and we assume that $\|.\|$ is Fréchet-smooth. Let $S$ be a closed non-void subset of $X$ at possitive distance $d$ of $B(\|.\|)$. Let $C$ be a quasi-smooth drop such that $S \cap intC \neq \emptyset$. Then there exists a quasi-smooth drop $D$ such that $D \cap S$ is a singleton and $D \subset C$.*

*Proof.* Put $S_1 = \overline{S \cap \operatorname{int} C}$.

*Case 1:* Tthere exists $x_1 \in S_1$ such that:

$$B(0, \|x_1\|) \cap \partial C \cap S_1 = \emptyset \tag{2}$$

By the smooth drop theorem to $S_1$ and $B(\|.\|)$, there exists a smooth drop $D_1$ such that $D_1 \cap S_1 = \{x_0\}$ and $D_1 \subset B(0, \|x_1\|)$.

Now (1) implies that $x_0 \in S_1 \backslash \partial C$ and with the Lemma 2.2, we finish this case.

*Case 2:* The case 1 is not fulfilled.

By hypothesis $\text{dist}(S_1, B(\|.\|)) > 0$. Choose $x_2 \in S \cap \text{int}C$ and $\alpha > 0$ such that $x_1 := (1+\alpha)x_2 \in C$. Then we apply Lemma 2.4with $\rho = 1 + \frac{d}{2}$ and obtain that $(D(x_2, B(\|.\|)) + \varepsilon B(\|.\|)) \cap S = \emptyset$. If we put $B_1 := \text{conv}\{\{x_2\}, \{-x_2\}, B(\|.\|)\}$ and $B_2 = B_1 + \varepsilon B(\|.\|)$, then $B_1$ and $B_2$ are unit balls of equivalent norms. Since $X$ is a space of differentiability, there exist a unit ball $B_3$ of equivalent Fréchet-differentiable norm $\|.\|_3$ such that $B_1 \subset B_3 \subset B_2$. The norm $\|.\|_3$ has the property of the first case, which finishes the proof. ∎

**Definition 2.6** *We say that the Banach space $X$ has a smooth drop property (in short SDP) if for all non-void and closed subset $S$ of $X$ disjoint of the unit ball, there exists a smooth drop $D$ such that $D \cap S$ is a singleton.*

**Theorem 2.7** *Let $(X, \|.\|)$ be a reflexive Banach space. We assume that the norm $\|.\|$ is Fréchet-differentiable on $X\backslash\{0\}$ and has the Kadec-Klee property. Then $(X, \|.\|)$ has the smooth drop property.*

*Proof. Case 1:* There exists a point $z \in S$ such that

$$\text{dist}\left(D\left(z, B\left(\|.\|\right)\right) \cap S, B\left(\|.\|\right)\right) > 0 \quad \text{and} \quad \left(\text{int}D\left(z, B\left(\|.\|\right)\right)\right) \cap S \neq \emptyset.$$

Then we apply Theorem 2.6, there exists a quasi-smooth drop $D$ such that $D \cap S$ is a singleton. Moreover the space is reflexive, by the Remark 2.4, $D$ is a smooth drop, and the proof is completed.

*Case 2:* If the case 1 is not fulfilled. Then by Theorem 2.6

$$\inf\left\{\text{dist}\left(z, B\left(\|.\|\right)\right) : \ z \in \text{conv}\left(\left(C \cup B\left(\|.\|\right)\right) \cap S\right)\right\} = 0$$

for all smooth convex $C$, suth that $C \cap S \neq \emptyset$.

Now it is easy to see that we can inductively define a sequence $\{x_n\}_{n\geq 1}$ in S with $x_1 \in S$ arbitrary, such that

$$(3) \qquad (1+\mu_{n+1})x_{n+1} \in D\left((1+\mu_n)\,x_n, B\left(\|.\|\right)\right) \cap (1+\mu_n)\,B\left(\|.\|\right),$$

where $\mu_n > 0, \mu_n \longrightarrow 0$. Moreover the sequence $\{x_n\}_{n\geq 1}$ is bounded in reflexive Banach space. Then without loss of generality we can assume that $x_n \overset{w}{\longrightarrow} x$, by (3) we have $\|x_n\| \to 1$, then $\|x\| \leq 1$. By (3) we show that $\{(1+\mu_n)x_n\}_{n\geq 1}$ is a stream (i.e. $(1+\mu_{n+1}x_{n+1} \in D(x_n, B(\|.\|))\backslash B(\|.\|)$, for all $n$) and by a Lemma of Montesinos [10]; stating that $\text{conv}\{(1+\mu_n)x_n :$

$n \in \mathbb{N}\} \cap B(\|.\|) = \emptyset$, then $\|x\| \geq 1$. And we deduce that $\|x\| = 1$. By hypothesis the norm $\|.\|$ has the Kadec-Klee property then $x_n \xrightarrow{\|\cdot\|} x$ in $S$ which is closed. Then $x \in S \cap B(\|.\|)$, a contradiction. Therefore the case 2 is impossible and the proof is finished. ■

# 3 Smooth drop property in strictly convex reflexive Banach space.

If in the conditions of Theorems 2.6 and 2.8 we impose an additional assumption that the norm is also strictly convex, then we can give a more precise form for the drop, i.e. it will be a convex hull of two balls with respect to the given norm.

Recall that each reflexive space has an equivalent norm which is Kadec-Klee, strictly convex and Fréchet-smooth, (for more see [5]).

**Proposition 3.1** *Let $(X, \|.\|)$ be a Banach space. Let $\varepsilon$ be positive, consider the set $D$ such that $D := conv\{B(\|.\|) \cup \{x + \varepsilon B(\|.\|)\}\}$ and $D_1$ is the set defined by $D_1 := \{(\lambda + (1-\lambda)\varepsilon)b + (1-\lambda)x;\ \lambda \in [0,1],\ b \in B(\|.\|)\}$. Then $D = D_1$.*

*Proof.* Let $y \in D_1$. Then $y = (\lambda + (1-\lambda)\varepsilon)b + (1-\lambda)x$ for some $\lambda \in [0,1]$ and $b \in B(\|.\|)$. But $y = \lambda b + (1-\lambda)(x + \varepsilon b)$, which implies that y is in D. Therefore $D_1 \subset D$.

Conversely, let $y \in D$. Then $y = \sum_{i=1}^{n} \lambda_i b_i + \sum_{j=1}^{m} \mu_j c_j$, for some $\lambda_i,\ \mu_j \in [0,1]$; $\sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \mu_j = 1$; $b_i \in B(\|.\|)$ and $c_j \in x + \varepsilon B(\|.\|)$. But $y = \lambda' b + \mu' c'$, where $b = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda'} b_i$ in $B(\|.\|)$; $c' = \sum_{j=1}^{m} \frac{m_j}{\mu'} c_j$ in $x + \varepsilon B(\|.\|)$; $\lambda' = \sum_{i=1}^{n} \lambda_i$ and $\mu' = \sum_{j=1}^{m} \mu_j$.

If we denote by $b' = x + \varepsilon b$, $c = \frac{1}{\varepsilon}(c' - x)$ and $z = \frac{x}{1-\varepsilon}$, then $b' = (1 -$

$\varepsilon)z + \varepsilon b$ and there exists $d' := [b', c'] \cap [z, y]$ and for $d := \frac{1}{\varepsilon}(d' - x)$, we have $z - d' = z - x - \varepsilon d = z - (1 - \varepsilon)z - \varepsilon d = \varepsilon(z - d)$ and $b' - d' = \varepsilon(b - d)$.

Therefore $y \in (d, d')$, i.e. there exists $\lambda \in (0, 1)$ such that

$$y = \lambda d + (1 - \lambda)d' = \lambda d + (1 - \lambda)(x + \varepsilon d) = (\lambda + (1 - \lambda)\varepsilon)d + (1 - \lambda)x,$$

which is in $D_1$. Therefore $D \subset D_1$ and the proof is finished. ∎

**Proposition 3.2** *Let $(X, \|.\|)$ be a Banach space. Then the set defined by $D := conv\,(B\,(\|.\|) \cup \{x + \varepsilon B\,(\|.\|)\})$ is closed.*

*Proof.* Let $\{y_n\}_{n\geq 1}$ is a sequence in $D$, such that $y_n \longrightarrow y_0$. By Proposition 3.1, we have

$$y_n = (\lambda_n + (1 - \lambda_n)\,\varepsilon)\,b_n + (1 - \lambda_n)\,x, \tag{4}$$

where $\lambda_n \in [0, 1]$ and $b_n \in B(\|.\|)$, for all $n \in \mathbf{N}$. The sequence $\{\lambda_n\}_{n\geq 1}$ is bounded in $\mathbf{R}$, then without loss of generality we assume that $\lambda_n \longrightarrow \lambda_0$, and by (4), if we put $b_n = \dfrac{1}{\lambda_n + (1 - \lambda_n)\varepsilon} y_n - \dfrac{1 - \lambda_n}{\lambda_n + (1 - \lambda_n)\varepsilon} x$, we have

$$b_n \longrightarrow \frac{1}{\lambda_0 + (1 - \lambda_0)\varepsilon} y_0 - \frac{1 - \lambda_0}{\lambda_0 + (1 - \lambda_0)\varepsilon} x := b_0.$$

Since $B(\|.\|)$ is closed and $\{b_n\}_{n\geq 1}$ is in $B(\|.\|)$, then $b_0$ is in $B(\|.\|)$.

Therefore $y_0 = (\lambda_0 + (1 - \lambda_0)\varepsilon)b_0 + (1 - \lambda_0)x$, and by Proposition 3.1, $y_0$ is in $D$. ∎

We say that the norm $\|.\|$ has the Kadec-Klee property; in short KKP; if for all sequence $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \overset{w}{\longrightarrow} x$ and $\|x_n\| \longrightarrow \|x\|$ then $x_n \overset{\|.\|}{\longrightarrow} x$.

It is easy to see that the following proposition is valid.

**Proposition 3.3** *Let $(X, \|.\|)$ be a Banach space. If $B\,(\|.\|)$ has the KKP, then the set $D := conv\,\{B\,(\|.\|) \cup \{x + \varepsilon B\,(\|.\|)\}\}$ has also the KKP.*

**Theorem 3.4** *Let $(X, \|.\|)$ be a reflexive Banach space. We assume that the norm $\|.\|$ is strictly convex with KKP. Let $S$ be a closed non-empty subset of $X$ at positive distance of $B\,(\|.\|)$. Let $\nu > 1$ and $z \in S$.*

*Then there exists $x_0$ and $\varepsilon > 0$ such that $x_0 + \varepsilon_0 B\,(\|.\|) \subset D\,(\nu z, B\,(\|.\|))$ and $conv\,\{(x_0 + \varepsilon_0 B\,(\|.\|)) \cup B\,(\|.\|)\} \cap S = \{z_0\}$ and $\|z_0 - x_0\| = \varepsilon_0$.*

*Proof.* Let $\delta_0 > 0$ be such that $3\delta_0 B' \subset \varepsilon B(\|.\|)$, where

$$B' := \operatorname{conv}\left\{(1+\varepsilon)\, B\left(\|.\|\right) \cup \left\{x_1 + \varepsilon B\left(\|.\|\right)\right\} \cup \left\{-x_1 + \varepsilon B\left(\|.\|\right)\right\}\right\},$$

$$\varepsilon = \frac{1-\nu}{\nu} \frac{d}{\operatorname{diam} D(\nu z, \delta_0)} \text{ and } d = \operatorname{dist}\left(S, B\left(\|.\|\right)\right).$$

There exists $x_1 \in [0, z]$ such that $S_1 := \operatorname{dist}(S, D(x_1, B(\|.\|))) < \delta_0$ and $S \cap D(x_1, B(\|.\|)) = \emptyset$. Now by Lemma 2.5 we have

$$(5) \quad \left(D\left(x_1, B\left(\|.\|\right)\right) + 2\varepsilon_0 B\left(\|.\|\right)\right) \setminus (1 + d/2)\, B\left(\|.\|\right) \subset \operatorname{int} D\left(\nu z, B\left(\|.\|\right)\right).$$

Choose $\alpha \in (0, \varepsilon)$ sufficiently small, such that, $S \cap D(x_1, (1+\alpha)B(\|.\|)) = \emptyset$.

Moreover $\alpha < \varepsilon$ implies

$$(6) \qquad D\left(x_1, (1+\alpha)\, B\left(\|.\|\right)\right) \subset D\left(x_1, B\left(\|.\|\right)\right) + \varepsilon B\left(\|.\|\right).$$

Let $x_2 \in (0, x_1) = \{tx_1 : \ t \in (0,1)\}$, and put $S_2 := \|x_1 - x_2\| < \delta_0 - \delta_1$ and $\varepsilon_2 = \frac{\|x_1\|}{\delta_2}(1 + d)$. Then,

$$\operatorname{dist}\left(S, \operatorname{conv}\left\{B\left(\|.\|\right) \cup \left\{x_2 + \varepsilon_2 B\left(\|.\|\right)\right\}\right\}\right) < \delta_0$$

and

$$S \cap \operatorname{conv}\left\{B\left(\|.\|\right) \cup \left\{x_2 + \varepsilon_2 B\left(\|.\|\right)\right\}\right\} = \emptyset$$

denote by

$$B_1 := \operatorname{conv}\left\{(1+\alpha)\, B\left(\|.\|\right) \cup \left\{\pm x_2 + \varepsilon_2 B\left(\|.\|\right)\right\}\right\}.$$

Then $B_1$ is convex, symmetric and closed (by Proposition 3.2). So $B_1$ is the unit ball of a new equivalent norm $\|.\|_1$ with KKP this by Proposition 3.3. Now we can apply a Theorem of Lau [9] (see also Theorem 5.11 of [2]) staiting that there exists a dense $G_\delta$ subset $\Gamma$ of $X \setminus S$ such that every $x \in \Gamma$ has a nearest point to $S$ with respect to the norm $\|.\|_1$. So we can choose a point $a_1$ in $\delta_0 B(\|.\|)$, such that $a_1$ has a nearest point $z_0$ to $S$ with respect to the norm $\|.\|_1$. Since $a_1 + (1+2\delta_0)B_1 \supset (1+\delta_0)B_1 \supset B_1 + \delta_0 B(\|.\|)$, we have

$$(7) \qquad \left(a_1 + (1+2\delta_0)\, B_1\right) \cap S \supset \left(B_1 + \delta_0 B\left(\|.\|\right)\right) \cap S \neq \emptyset.$$

Therefor $d_1 := \operatorname{dist}_{\|.\|_1}(a_1, S) < 1 + 2\delta_0$. Since $3\delta_0 B_1 \subset 3\delta_0 B' \subset \varepsilon B(\|.\|)$, we have also

$$(8) \quad a_1 + (1+2\delta_0)\, B_1 \subset B_1 + \varepsilon B\left(\|.\|\right) \subset D\left(\pm x_1, (1+\alpha)\, B\left(\|.\|\right)\right) + \varepsilon B\left(\|.\|\right)$$

Then by (5),(6),(7) and (8) we obtain

$$(a_1 + (1+2\delta_0) B_1) \cap S \subset [D(x_1, (1+\alpha) B(\|.\|)) + \varepsilon B(\|.\|)] \cap S \subset$$
$$\subset [D(x_1, B(\|.\|)) + 2\varepsilon B(\|.\|)] \cap S \subset D(\nu z, B(\|.\|)).$$

Hence $(a_1 + d_1 B_1) \cap S \subset D(\nu z, B(\|.\|))$.

Let $x_1' := (x_1 + a_1)d_1$, $x_2' := (x_2 + a_1)d_1$ and $\varepsilon_2' = \varepsilon_2 d_1$.

If $\|z_0 - x_2'\| = \varepsilon_2'$, we take $x_0 \in (z_0, x_2')$ and $\varepsilon_0 = \|x_0 - z_0\|$, which finishies the proof.

If $\|z_0 - x_2'\| > \varepsilon_2'$, then $z_0 = \lambda x_1' + (1-\lambda)b_0$ for some $\lambda \in [0,1]$ and $b_0 \in a_1 + (1+\alpha)B(\|.\|)$. We take $x_0' := \lambda x_1' + (1-\lambda)a_1$, $x_0 \in (z_0, x_0')$ and $\varepsilon_0 = \|x_0 - z_0\|$, which finishes the proof. ∎

**Theorem 3.5** *Let $(X, \|.\|)$ be a reflexive Banach space with a strictly convex norm $\|.\|$ and having the KKP. Let $S$ be a closed non-void subset of $X$ such that $S \cap B(\|.\|) = \emptyset$. Then there exists a point $x_0$ and $\varepsilon_0 > 0$ such that conv $\{B(\|.\|) \cup \{x_0 + \varepsilon_0 B(\|.\|)\}\} \cap S = \{z_0\}$ and $\|z_0 - x_0\| = \varepsilon_0$. Moreover, if the norm is Fréchet-differentiable, then conv $\{B(\|.\|) \cup \{x_0 + \varepsilon_0 B(\|.\|)\}\}$ is a smooth drop.*

*Proof.* Using the same proof as those of Theorem 2.8. ∎

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