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Submonotone mappings in Banach spaces and applications

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Abstract

The notions "submonotone" and "strictly submonotone" mapping, introduced by J. Spingarn in $\mathbf{R}^n$, are extended by a natural way to arbitrary Banach spaces. Some results about monotone operators are proved for submonotone and strictly submonotone ones: Rockafellar's result about locally boundedness of monotone operators; Kenderov's result about single-valuedness and upper-semicontinuity almost everywhere of monotone operators in Asplund spaces and spaces with strictly convex duals. It is shown that subdifferentials of various closses non-convex functions possess submonotone properties. Results about generic differentiability of such functions are obtained (among them is a Zajisek's and a new generalization of an Ekeland and Lebourg's theorem).

0 Introduction

A monotone mapping (Minty[21]) is a multivalued mapping T from the Banach space $(E, \|.\|)$ to its dual $(E^*, \|.\|)$ having the property that $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ whenever $y_i \in T(x_i), i = 1, 2$, where $\langle ., .\rangle$ are the dual brackets between E and E^*. With $D(T)$ we will denote the set $\{x \in E : T(x) \neq \emptyset\}$.

A multivalued mapping $T : E \to E^*$ is said to be locally bounded at $x \in E$ if there exist $C > 0$ and a neighbourhood U of x such that $z \in U$ and $z^* \in T(z)$ imply $\|z^*\| < C$. Rockafellar [24] proved that every monotone mapping $T : E \to E^*$ is locally bounded at every point of $intD(T)$.

One of the important properties of the monotone mappings is the characterization of the Asplund spaces due to Kenderov [12, 13, 14]: a Banach space E is an Asplund space iff every monotone mapping $T : E \to E^*$ is sigle-valued and norm to norm upper-semicontinuous on a dense G_δ subset of $intD(T)$. By definition, a Banach space E is said to be an Asplund space, if every continuous convex function defined on an open subset U of E is Frechet differentiable on a dense G_δ subset of U. For further information about Asplund spaces see for instance [10], [22].

A multivalued mapping $F : X \to Y$, where X and Y are topological spaces is said to be upper-semicontinuous at $x \in X$, if for every open set $V \supset F(x)$ there exists an open set $U \ni x$ such that $F(z) \subset V$ for every $z \in U$.

In this article the notions "submonotone" and "strictly submonotone" mapping, introduced by J. Spingarn [27] in $\mathbf{R}^n$ are extended by a natural, but non-trivial way to an arbitrary Banach space and it is shown that some results of Spingarn are also valid in this more general setting. We prove that the above mentioned results of R.T.Rockafellar and P. Kenderov remain valid when T is a strictly submonotone mapping. If T is submonotone, then T is locally bounded on a dense and open subset of $intD(T)$ and if, in addition T has w^*-closed graph, then T is also single-valued and norm to norm upper semiccontinuous on a dense G_δ subset of $intD(T)$.

In section 4 we present conditions, under which a function has submonotone (strictly submonotone) subdifferentials of Clarke or Pshenichniy type. With their help we obtain that the subdifferentials of non-convex functions from various classes possess submonotone properties. For example the Pshenichniy subdifferential of every continuous quasidifferentiable function, defined on a separable Banach space E is almost strictly submonotone on a dense G_δ subset of E (Theorem 4.6). Using Theorem 4.6 we obtain that every regular in sense of Clarke [5] function is generic Frechet differentiable in Asplund spaces. The Pshenichniy subdifferentials of functions, defined as pointwise supremum of special functions also have submonotone properties and we obtain that such functions are generic differentiable in appropriate spaces. In such a way we give a new proof of a Zajicek's generalization of an Ekeland and Lebourg's theorem [28] and proof a new generalization of latter theorem (Theorem 5.4). Applications of Theorem 5.4 are given to the generic differentiability of the distance function in spaces with an uniformly Gateaux differentiable norm.

Some of the results in this article are annonced in [9].

1 Preliminaries

In 1981 J. Spingan [27] introduced the notions *submonotone* and *strictly submonotone* mapping in $\mathbf{R}^n$ as follows:

A multivalued mapping $T: \mathbf{R}^n \to \mathbf{R}^n$ is said to be *submonotone* at $x \in \mathbf{R}^n$ provided

$$\liminf_{\substack{x \neq x' \to x \\ y \in T(x), y' \in T(x')}} \frac{\langle x' - x, y' - y \rangle}{\|x' - x\|} \geq 0,$$

where $\|.\|$ denote the Euclidean norm in $\mathbf{R}^n$;

A multivalued mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be strictly submonotone at $x \in \mathbf{R}^n$ provided

$$\liminf_{\substack{x_1 \neq x_2 \\ x_i \to x, i=1,2 \\ y_i \in T(x_i), i=1,2}} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} \geq 0.$$

Here we slightly change these definitions.

Let $(E, \|.\|)$ be a Banach space, S be its unit sphere, E^* be its dual space and denote $U(x_0, e, \delta) = \{x \in E : x \neq x_0, \|x - x_0\| < \delta, \left\| \frac{x - x_0}{\|x - x_0\|} - e \right\| < \delta\}$.

Definition 1.1 *A multivalued mapping $T : E \to E^*$ will be called submonotone at $x_0 \in E$ provided*

$$(1) \qquad \liminf_{\substack{x_0 \neq x \to_e x_0 \\ y \in T(x), y_0 \in T(x_0)}} \frac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} \geq 0, \text{ for every } e \in S,$$

where $x \to_e x_0$ denotes that x converges to x_0 in direction e.

(1) is equivalent to the following: for every $e \in S$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $\frac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} > -\epsilon$ whenever $x \in U(x_0, e, \delta)$, $y \in T(x)$ and $y_0 \in T(x_0)$.

Definition 1.2 *A multivalued mapping $T : E \to E^*$ will be called strictly submonotone at $x_0 \in E$ provided*

$$(2) \qquad \liminf_{\substack{x_1 \neq x_2 \\ x_i \to x_0, i=1,2 \\ y_i \in T(x_i), i=1,2 \\ x_1 - x_2 \to_e 0}} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} \geq 0 \text{ for every } e \in S.$$

(2) means that for every $e \in S$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $\frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} > -\epsilon$ whenever $\|x_i - x_0\| < \delta$, $y_i \in T(x_i)$, $i = 1,2$, $x_1 \neq x_2$ and $x_1 - x_2 \in U(0, e, \delta)$.

If $E = \mathbf{R}^n$, then Definitions 1.1 and 1.2 coincide with the corresponding definitions of Spingarn (due to the compactness of the unit sphere in $\mathbf{R}^n$).

The notions submonotone and strictly submonotone mappings are different, as it follows by the following.

Example 1.3 *The function* $T : \mathbf{R} \to \mathbf{R}$, $T(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$ *is submonotone at* 0*, but not strictly submonotone at* 0.

T will be called *submonotone* (resp. *strictly submonotone*) if T is submonotone (resp. strictly submonotone) at every $x_0 \in E$.

The following definition is motivated by the article of Spingarn [27].

Definition 1.4 *The multivalued mapping* $T : E \to E^*$ *will be called weak* upper semicontinuous in direction* $e \in S$ *(e-w*-d.u.s.c.) at* $x_0 \in E$ *if for every weak*-open set* $V \supset T(x_0)_e$ *there exists* $\delta > 0$ *such that* $T(x) \subset V$ *whenever* $x \in U(x_0, e, \delta)$, *where* $T(x_0)_e := \{x^* \in T(x_0) : \langle e, x^* \rangle = \max_{z \in T(x_0)} \langle e, z \rangle\}$. T *is called to be directionally weak* upper semicontinuous (w*-d.u.s.c.) if* T *is e-w*-d.u.s.c. at every* $e \in S$.

T will be called *directionally locally bounded* at x_0 if for every $e \in S$ there exists $\delta > 0$ such that $T(U(x_0, e, \delta))$ is bounded.

Proposition 1.5 *Let* $T : E \to E^*$ *be a directionally locally bounded multivalued mapping with w*-closed graph. Then the following are equivalent:*

a) T *is w*-d.u.s.c. at* $x_0 \in E$,

b) for every $e \in S$, *for every (generalized) sequence* $\{x_a\}_{a \in A}$ *with* $x_0 \neq x_a \to_e x_0$, *for every* $y_a \in T(x_a)$, $a \in A$ *with* $y_a \xrightarrow{w^*} y_0$ *it follows that* $y_0 \in T(x_0)_e$,

c) for every $e \in S$, *for every (generalized) sequence* $\{x_a\}_{a \in A}$ *with* $x_0 \neq x_a \to_e x_0$, *for every* $y_a \in T(x_a)$, $a \in A$, *it follows that* $\langle e, y_a \rangle \to \max_{z^* \in T(x_0)} \langle e, x^* \rangle$.

The proof is straightforward and is omitted.

Assertion b) from the following theorem is established by Spingarn [27], Theorem 1.1 for the case $E = \mathbf{R}^n$. Here the proof is simillar, using Propositon 1.5.

Theorem 1.6 *Let $T : E \to E^*$ be a directionally locally bounded multivalued mapping. Then*

a) T is submonotone at $x_0 \in E$ iff for every $e \in S$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$(1.1) \qquad \inf\langle e, T(x)\rangle \geq \sup\langle e, T(x_0)\rangle - \epsilon \text{ for every } x \in U(x_0, e, \delta),$$

b) if T has w^*-closed graph, then T is submonotone in $x_0 \in E$ iff T is w^*-d.u.s.c. at x_0 .

Proof. a). Let T be submonotone at x_0 . Then for every $\epsilon > 0$ and $e \in S$ there exists $\delta > 0$ such that $\frac{\langle x-x_0, y-y_0\rangle}{\|x-x_0\|} > -\frac{\epsilon}{2}$, whenever $x \in U(x_0, e, \delta_1)$, $y \in T(x)$ and $y_0 \in T(x_0)$. Since T is directionally locally bounded at x_0, there exists $\delta_2 > 0$ and $C > 0$ such that $T(x) \subset B[0; C]$ for every $x \in U(x_0, e, \delta_2)$. Let $0 < \delta < \min\{\delta_1, \delta_2, \epsilon/4C\}$ and $x \in U(x_0, e, \delta)$. Then for every $y \in T(x)$ and every $y_0 \in T(x_0)$ we have

$$\langle e, y - y_0\rangle = \langle e - \frac{x - x_0}{\|x - x_0\|}, y - y_0\rangle + \frac{\langle x - x_0, y - y_0\rangle}{\|x - x_0\|} > -2\delta C - \epsilon/2 > -\epsilon.$$

Hence $\inf\langle e, T(x)\rangle \geq \sup\langle e, T(x_0)\rangle - \epsilon$.

Conversely, let for every $e \in S$, $\epsilon > 0$ there exists δ_3 such that $\inf\langle e, T(x)\rangle \geq \sup\langle e, T(x_0)\rangle - \epsilon/2$ for every $x \in U(x_0, e, \delta_3)$. Let $0 < \delta < \min\{\delta_2, \delta_3, \epsilon/4C\}$ and $x \in U(x_0, e, \delta)$. Then for every $y \in T(x)$ and every $y_0 \in T(x_0)$ we have

$$\frac{\langle x - x_0, y - y_0\rangle}{\|x - x_0\|} = \langle \frac{x - x_0}{\|x - x_0\|} - e, y - y_0\rangle + \langle e, y - y_0\rangle > -2\delta C - \epsilon/2 > -\epsilon.$$

b) Let T be w^*-d.u.s.c. at x_0. Then obviously (1.1) is fulfilled and by part a) it follows that T is submonotone. Let now T be submonotone at x_0 and let $e \in S$, $x_a \to_e x_0$, $\alpha \in A$, $y_a \in T(x_a)$, $y_a \to y_0$ and $z \in T(x_0)_e$. Since T has w^*-closed graph, we have $y_0 \in T(x_0)$ and

$$\langle e, y_0 - z\rangle = \lim_{\alpha \in A}\langle e, y_a - z\rangle =$$

$$\lim_{a \in A}\langle \frac{x_a - x_0}{\|x_a - x_0\|}, y_a - z\rangle + \lim_{a \in A}\langle e - \frac{x_a - x_0}{\|x_a - x_0\|}, y_a - z\rangle \geq$$

$$-\lim_{a \in A}\left\| e - \frac{x_a - x_0}{\|x_a - x_0\|}\right\| \|y_a - z\| = 0,$$

bacause T is directionally locally bounded at x_0. Hence $\langle e, y_0\rangle = \max\langle e, T(x_0)\rangle$, that means $y_0 \in T(x_0)_e$ and by Proposiiton 1.5. b) it follows that T is w^*-d.u.s.c. at x_0. ∎

2 Locally boundedness of submonotone mappings

In [25] Rockafellar proved that every monotone mapping $T : E \to E^*$ is locally bounded in *intD(T)*. Here we prove that every strictly submonotone mapping $T : E \to E^*$ is locally bounded, followng Kenderov's approach [12], where the above Rockafellar's result is proved thereby a property for almost lower semicontinuity of multivalued mappings (see [15]). For submonotone mappings this is not true (unless T has w^*-closed graph), as Example 1.3 shows, but we have that every submonotone mapping $T : E \to E^*$ is locally bounded on a dense and open subset of *intD(T)*.

The following lemma is a modification of a Kenderov's lemma ([15]. Lemma 1.7) for monotone mappings. Here the proof is analogous.

Lemma 2.1 *Let $T : E \to E^*$ be submonotone multivalued mapping, $A \subset E^*$ be a w^*-compact and convex subset of E^* and $T^{-1}(A) := \{x \in E : f(x0 \cap A \neq \emptyset\}$ be dense in some open set $U \subset E$. Then $T(x) \subset A$ for every $x \in U$ at which T is submonotone.*

Proof. Assume that $T(x_0) \not\subset A$ for some x_0 at which T is submonotone. Choose $x_0^* \in T(x_0) \setminus A$. Then by the separation theorem there exists $e_0 \in S$ such that

$$< e_0, x_0^* > > \max < e_0, A >$$

By the submonotonicity of T at x_0 (Theorem 1.6) for $0 < \epsilon_0 << e_0, x_0^* > - \max < e_0, A >$ there exists $\delta_0 > 0$ such that

$$\inf < e_0, T(x) > > \sup < e_0, T(x_0 > -\epsilon_0 \; for \; every \; x \in U(x; e; \delta).$$

Hence for $x_1 \in T^{-1}(A) \cap U(x_0; e_0; \delta_0)$ *and* $x_1^* \in T(x_1) \cap A$ we have

$$< e_0, x_1^* > > \inf < e_0, T(x_1) > > \sup < e_0, T(x_0) > -\epsilon_0 \geq$$

$$< e_0, x_0^* > -\epsilon_0 > \max < e_0, A > \geq < e_0, x_1^* >,$$

a contradiction.∎

Theorem 2.2 *Every submonotone mapping $T : E \to E^*$ is locally bounded at the points of some dense and open subset of $intD(T)$.*

Proof. The assertion follows by the following theorem of Kenderov ([13], Theorem 1.3) and by Lemma 2.1. ■

Theorem 2.3 *(Kenderov [15], Theorem 1.3). Let $\alpha = \{A_i\}_{i\geq 1}$ be a countable family of subsets of the set Y. Then for every mapping $F : X \to Y$ from the topological spase X into Y there exists a residual subset $X_1 \subset X$ such that for every $x \in X_1$, for every $i \geq 1$ we have: either*

1) $F(x) \cap A_i = \emptyset$ or

2) there exists an open set $U \ni x$ and a dense subset $U_1 \subset U$ such that $F(z) \cap A_i \neq \emptyset$ for every $z \in U_1$.

Now we can prove the main result of this paragraph.

Theorem 2.4 *Every strictly submonotone mapping $T : E \to E^*$ is locally bounded in $intD(T)$.*

Proof. We follow Kenderov [12], where he proved the Rockafellar result [25] for locally boundedness of monotone operators. By Theorem 2.2 if follows, that T is locally bounded at the points of some dense and open subset of $intD(T)$. Let $x_0 \in intD(T)$ be an arbitrary point and let $\epsilon > 0,\ e \in S$. By the strictly submonotonisity of T at x_0(applied for e and $-e$) it follows that there exists $\delta > 0$ such that the conditions

$$x_1 \neq x_2,\ \| x_i - x_0 \| < \delta,\ y_i \in T(x_i),\ i = 1,2 \ either \left\| \frac{x_1 - x_2}{\|x_1 - x_2\|} - e \right\| < \delta \ or$$

$$(2.1) \qquad \left\| \frac{x_1 - x_2}{\|x_1 - x_2\|} + e \right\| < \delta,\ imply\ \frac{< x_1 - x_2, y_1 - y_2 >}{\|x_1 - x_2\|} > -\epsilon.$$

Let $U := \{x \in E : x \neq 0, \| x \| < \frac{\delta}{2}, \left\| \frac{x}{\|x\|} - e \right\| < \delta/2\}$. Obviously U is open. There exist $y \in U$ such that $y_1 := x_0 + y \in intD(T)$, $y_2 := x_0 - y \in intD(T)$ and T is locally bounded at y_i, $i = 1,2$, i.e. there exist $\epsilon_1 > 0$ and $C > 0$ such that $y_i + \epsilon_1 B \subset D(T)$ and $\| z^* \| < C$ for every $z^* \in T(y_i + \epsilon_1 B), i = 1,2$. Let $\epsilon_2 \in \{0, \min(\delta \| y \| /4, \delta/2, \epsilon_1, \| y \|\}$. We will show that the set

$T(x_0+(\epsilon_2/2)B)$ is bounded. Let $x \in x_0+\epsilon_2 B/2$, $x^* \in T(x)$. Then $x = x_0+z$ for some $z \in \epsilon_2 B/2$. Let $v \in B$, $y_i^* \in T(y_i + \epsilon_2 v/2 + z) \subset T(y_i + \epsilon_2 B)$. We can write:

$$\left\| \frac{y+\epsilon_2 v/2}{\| y+\epsilon_2 v/2 \|} - e \right\| \leq$$

$$\leq \left\| \frac{y+\epsilon_2 v/2}{\| y+\epsilon_2 v/2 \|} - \frac{y}{\| y \|} \right\| + \left\| \frac{y}{\| y \|} - e \right\| \leq$$

$$\leq \| \frac{(y+\epsilon_2 v/2) \| y \| - y \|y+\epsilon_2 v/2\| \|}{\|y\| \|y+\epsilon_2 v/2\|} + \delta/2 =$$

$$= \frac{\| y(\| y \| - \| y+\epsilon_2 v/2 \|) + \epsilon_2 v \|y\|/2 \|}{\| y \| \| y+\epsilon_2 v/2 \|} + \delta/2 \leq$$

$$\leq \frac{\| \| y \| - \| y+\epsilon_2 v/2 \| \| + \epsilon_2/2}{\| y+\epsilon_2 v/2 \|} + \delta/2 \leq$$

$$\leq \frac{\epsilon_2}{\| y \| - \epsilon_2/2} + \delta/2 < \frac{2\epsilon_2}{\| y \|} + \delta/2 < \delta/2 + \delta/2 = \delta$$

$$\| y_1 + \epsilon_2 v/2 + z - x_0 \| \leq \| y_1 - x_0 \| + \epsilon_2 < \delta/2 + \delta/2 = \delta;$$

$$\| x - x_0 \| < \epsilon_2/2 < \delta/4.$$

By (2.1)

$$\frac{< y_1 + \epsilon_2 v/2 + z - x, y_1^* - x^* >}{\| y_1 + \epsilon_2 v/2 + z - x \|} \geq -\epsilon_0,$$

$$\frac{< y + \epsilon_2 v/2, y_1^* - x^* >}{\| y + \epsilon_2 v/2 \|} \geq -\epsilon_0,$$

hence

(2.2) $$< y + \epsilon_2 v/2, x^* > \leq < y + \epsilon_2 v/2, y_1^* > + \epsilon_0 \| y + \epsilon_2 v/2 \| \leq$$

$$\leq C(\| y \| + \epsilon_2/2) + \epsilon_0(\| y \| + \epsilon_2/2) = (C+\epsilon_0)(\| y \| + \epsilon_2/2) <$$

$$(C+\epsilon_0)(\delta/2+\epsilon_2/2).$$

Analogously we have $\left\| \frac{-y+\epsilon_2 v/2}{\|-y+\epsilon_2 v/2\|} + e \right\| < \delta$, $\| y_2 + \epsilon_2 v/2 + z - x_0 \| < \delta$ and as above we obtain

$$< -y+\epsilon_2 v/2, x^* > < \ (C+\epsilon_0)(\delta/2+\epsilon_2/2). \tag{2.3}$$

Adding (2.2) and (2.3) we get $\epsilon_2 < v, x^* > < \ (C+\epsilon_0)(\delta+\epsilon_2)$, which is true for every $v \in B$. Hence $\| x^* \| < \ (C+\epsilon_0)(\frac{\delta+\epsilon_2}{\epsilon_2})$. ∎

3 Submonotone mappings in Asplund spaces.

Kenderov [12],[13],[14] proved that a Banach space E is an Asplund space iff every monotone mapping $T : E \to E^*$ is single- valued and upper semicontinuous (in the norm topologies) on a dense G_δ subset of *intD(T)*. Here we established an analogous result for strictly submonotone mappings thereby a theorem of Christensen and Kenderov [2].

Definition 3.1 *(Christensen [1]). Let $F : X \to Y$ be a multivalued mapping, where X and Y are topological spaces. F satisfy the property (*) at $x_0 \in E$, if there exists a point $y_0 \in F(x_0)$ such that for every open set $V \ni y_0$ there exists an open set $U \ni x_0$ such that $F(x) \cap V \neq \emptyset$ for every $x \in U$.*

Lemma 3.2 *Let $T : E \to (E^*, \| \,.\, \|)$ be a submonotone mapping, which satisfies the property (*) at $x_0 \in E$. Then T is single- valued and norm to norm upper-semicontinuous at x_0.*

Proof. Let $y_0 \in T(x_0)$ be the point from the definition of the property (*). By the property (*) it follows that for every $r > 0$ there exist an open set $U \ni x_0$ such that $T(x) \cap B(y_0; r) \neq \emptyset$ for every $x \in U$ and by Lemma 2.1 the proof is completed. ∎

Further we need the following

Theorem 3.3 *(Christensen and Kenderov [2]). Let E be a Banach space, E^* has the Radon-Nikodim property, $F : X \to (E^*, w^*)$ be an upper-semicontinuous multivalued mapping with w^*- compact and non-empty images, X be a Baire space. Then there exists a dense G_δ subset G of X such that F possesses the property (*) at every point of G, as E is considered with the norm topology.*

Theorem 3.4 *Every strictly submonotone mapping $T : E \to E^*$, where E is an Asplund space, is single-valued and norm to norm upper-semicontinuous on a dense G_δ subset of int $D(T)$.*

Proof. By the strictly submonotonicity of T for every $z \in D(T), \epsilon > 0, e \in S$, there exists $\delta(z, e, \epsilon, T)$ such that

$$\frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\| x_1 - x_2 \|} \geq -\epsilon \tag{3.1}$$

whenever $x_1 \neq x_2$, $\| x_i - z \| < \delta(z, e, \epsilon, T)$, $y_i \in T(x_i)$, $i = 1, 2$, $x_1 - x_2 \in U(0, e, \delta(z, e, \epsilon, T))$.
Denote
$W(T) = \{F : E \to E^* : F$ is a multivalued mapping for which $D(F) = D(T)$, and (3.1) is fulfilled whenever $z \in D(F)$, $e \in S$, $\epsilon > 0$.
Let $W_1 \subset W(T)$ be linear ordered subset with respect to the relation "$\subset$" between the graphs of elements of $W(T)$. Then the mapping $F_0 : E \to E^*$ defined by $F_0(x) = \cup\{F(x) : F \in W\}$ is a majorant of W_1. By the Zorn lemma it follows that $W(T)$ has maximal elements. Let T_0 be one of them.

Let $\{(x_a, x_a^*)\}_{a \in A} \subset (E, \| \cdot \|) \times (E^*, w^*)$ be a generalized sequence of $GrT_0 := \{(x, x^*) \in E \times E^* : x^* \in T_0(x)\}$ converging to (x_0, x_0^*), as $x_0 \in D(T_0)$. This means $x_a \to x_0$ and $x_a^* \xrightarrow{w^*} x_0^*$.

We will check that $(x_0, x_0^*) \in GrT_0$. It is easy to see that $< x_a - x, x_a^* - x^* > \to < x_0 - x, x_0^* - x^* >$ for every $x \in E$, $x^* \in E^*$. Let $\epsilon > 0$, $e \in S$ and $z \in D(T_0)$ be such that $\| x_0 - z \| < \delta(z, e, \epsilon, T)$. If for some $x_0 \neq x_0' \in D(T_0)$ it is fulfilled $\| x_0' - z \| < \delta(z, e, \epsilon, T)$ and $\| \frac{x_0 - x_0'}{\|x_0 - x_0'\|} - e \| < \delta(z, e, \epsilon, T)$, then for every $y_0' \in T_0(x_0')$ we have

$$\frac{< x_0 - x_0', x_0^* - y_0' >}{\| x_0 - x_0' \|} = \lim_{a \in A} \frac{< x_a - x_0', x_a^* - y_0' >}{\| x_a - x_0' \|} \geq -\epsilon.$$

Therefore $(x_0, x_0^*) \in GrT_0$, because T_0 is maximal.

We shall prove that T_0 is w^*-u.s.c. in $D(T)$. Assume the contrary: let T_0 be not w^*-u.s.c. at some $x_0 \in D(T_0)$. Then there exist w^*-open set $V \supset T(x_0)$, sequences $\{x_n\}_{n \geq 1} \subset D(T_0)$, $\{y_n\}_{n \geq 1}$, $y_n \in T_0(x_n)$ such that $x_n \to x_0$ and $y_n \notin V$ for every $n \in N$. Since T_0 is locally bounded (Theorem 2.3), $\{y_n\}_{n>1}$ is bounded. By the Alaoglou-Bourbaki theorem there exists a generalized w^* convergent subsequence $\{y_a\}_{a\in D}$, $y_a \xrightarrow{w^*} y_0 \notin V$. But by the above reasonings it follows that $y_0 \in T_0(x_0)$, a contradiction.

Now we apply Theorem 3.3 and obtain that T_0 has the property (*) with respect to the norm topology in E^* on a dense G_δ subset G_0 of $intD(T)$. Lemma 3.2 shows that T_0 is single-valued and norm to norm upper-semicontinuous on G_0. Since $GrT \subset GrT_0$ (by the maximality of T_0), T is also single-valued and norm to norm upper-semicontinuous on G_0. ∎

Remark 3.5 *It is easy to see that Theorem 3.4 is valid also if T is submonotone mapping with w^*-closed graph. Then we apply Theorem 3.3 for the dense and open subset $G \subset intD(T)$ on which T is locally bounded (according to Theorem 2.2).*

The following theorem is a modification of a result of Kenderov and Robert [16]. Here we use the same idea.

Theorem 3.6 *Let $T : E \to E^*$ be a submonotone mapping on a dense G_δ subset G_1 of $intD(T)$ and let $v : E^* \to \mathbf{R}$ be a w^*-lower semicontinuous convex function. Then there exists a dense G_δ subset of $intD(T)$ at every point x of which we have $v(x_1^*) = v(x_2^*)$ for every $x_i^* \in T(x)$, $i = 1, 2$.*

Proof. By Theorem 2.3 and Lemma 2.1 it follows that there exists a dense G_δ subset G_2 of $intD(T)$ such that for every $x \in G_2$ there exist $C(x) > 0$ and $\delta(x) > 0$ for which $T(z) \subset B[0; C(x)]$ for every $z \in B(x; \delta(x)) \cap G_1$. The set $A_n := \{x^* \in E^* : v(x^*) \le r_n\}$, where r_n is the n-th rational number, is convex and w^*-closed (since v is w^*-lower semicontinuous and convex). There exists a dense G_δ subset G_3 of $intD(T)$ at every point of which it is fulfilled either a) or b) of Theorem 2.3 for the family $\{A_n\}_{n>1}$. Let $x_0 \in G_1 \cap G_2 \cap G_3$. Assume that $t_1 := \inf\ v(T(x_0)) < \sup\ v(T(x_0)) =: t_2$. Choose a rational number $r \in (t_1, t_2)$. Then $r = r_n$ for some integer n and $T(x_0) \cap A_n \neq \emptyset$, whence $T(x_0) \cap B[0; C(x_0)] \cap A_n \neq \emptyset$. The set $B[0; C(x_0)] \cap A_n$ is w^*-compact and convex and by Lemma 2.1 it follows that $T(x_0) \subset B[0; C(x_0)] \cap A_n$.

Hence $t_2 \le r_n$, which is a contradiction.∎

Corollary 3.7 *Let $T : E \to E^*$ be a submonotone mapping on a dense G_δ subset of $intD(T)$ and let there exists on E^* a w^*-lower semicontinuous strictly convex function φ . Then T is single-valued on a dense G_δ subset of $intD(T)$.*

Proof. First we will prove that the mapping $coT : E \to E^*, (coT)(x) = coT(x)$, where co denotes the convex hull, is submonotone at every point x_0 at which T is. For every $\epsilon > 0$ and $e \in S$ there exists $\delta > 0$ such that the conditions $x \in U(x_0, e, \delta),\ y \in T(x),\ y_0 \in T(x_0)$ imply

$$\frac{\langle x - x_0, y - y_0 \rangle}{\| x - x_0 \|} \geq -\epsilon.$$

Let $x \in U(x_0, e, \delta),\ y \in coT(x),\ y_0 \in coT(x_0)$. Then for some n and m we have

$$y = \sum_{i=1}^{n} \lambda_i y_i,\ \lambda_i \in [0,1],\ y_i \in T(x),\ i = 1, \ldots, n,\ \sum_{i=1}^{n} \lambda_i = 1 \text{ and}$$

$$y_0 = \sum_{j=1}^{m} \lambda_j^0 y_j^0,\ \lambda_j^0 \in [0,1],\ y_j^0 \in T(x_0),\ j = 1, \ldots, m,\ \sum_{j=1}^{m} \lambda_j^0 = 1.$$

Then we can write:

$$\frac{\langle x - x_0, y - y_0 \rangle}{\| x - x_0 \|} = \frac{1}{\| x - x_0 \|} \langle x - x_0, \sum_{i=1}^{n} \lambda_i y_i - \sum_{j=1}^{m} \lambda_j^0 y_j^0 \rangle =$$

$$= \frac{1}{\| x - x_0 \|} \langle x - x_0, \sum_{i=1}^{n} \lambda_i (y_i - \sum_{j=1}^{m} \lambda_j^0 y_j^0) \rangle =$$

$$= \frac{1}{\| x - x_0 \|} \langle x - x_0, \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \lambda_j^0 (y_i - y_j^0) \rangle =$$

$$= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \lambda_j^0 \frac{\langle x - x_0, y_i - y_j^0 \rangle}{\| x - x_0 \|} \geq \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \lambda_j^0 (-\epsilon) = -\epsilon.$$

By the assumption the level sets $L_c := \{x^* \in E^* : \varphi(x^*) = c\}, c \in \mathbf{R}$ do not contain non-trivial segment. By Theorem 3.6 it follows that there exists a dense G_δ subset Γ of $intD(T)$ such that for every $x \in \Gamma$ $coT(x) \subset L_c$, for some $c \in \mathbf{R}$, i.e. $co(T)$ consists only one point, therefore $T(x)$ also consists of only one point. ∎

Corollary 3.8 *Let $T : E \to E^*$ be a submonotone mapping on a dense G_δ subset of $intD(T)$, E be a separable Banach space. Then T is single-valued on a dense G_δ subset of $intD(T)$.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of E. By Theorem 3.6 it follows that for every $n \in N$ there exists a dense G_δ subset Γ_n of $intD(T)$ such that for every $x \in \Gamma_n$ it is fulfilled $\langle x, x_1^* \rangle = \langle x, x_2^* \rangle$ whenever $x_i^* \in T(x)$, $i = 1, 2$. Let $\Gamma_0 = \bigcap_n^{\infty} \Gamma_n$ and $x_0 \in \Gamma_0$. Assume that $T(x_0)$ contains two different points x_1^* and x_2^*. There exists $n_0 \in N$ such that $\langle x_{n_0}, x_1^* \rangle > \langle x_{n_0}, x_2^* \rangle$, which is a contradiction. ∎

By Corollary 3.8 we obtain the Mazur theorem [19] about Gateaux differentiability on a dense G_δ subset of a convex continuous function defined on a separable Banach space.

4 Submonotone subdifferentials of locally Lipschitz functions

The function $f^0(x; h) = \limsup_{z \to x\ t \downarrow 0} \frac{(f(z + th) - f(z))}{t}$ is said to be Clarke's derivative at the point $x \in E$ in direction $h \in E$ for a locally lipschitz function $f : E \to \mathbf{R}$(see [3]).

The function $f'(x; h) = \lim_{t \downarrow 0} \frac{(f(x + th) - f(x))}{t}$(if it exists) is said to be *derivative in direction* at the point x(in direction h). The set $\partial f(x) = \{x^* \in E^* : < h, x^* > \leq f^0(x; h), h \in E\}$ is said to be *Clarke's subdifferential* at x for f(see [3]).

The function $f : E \to \mathbf{R}$ is said to be *quasidifferentiable in sense of Pshenichniy* [22] at $x \in E$, iff there exist $f'(x; h)$ for every $h \in E$ and a w^*-closed and convex subset $\partial_p f(x) \subset E$ such that $f'(x; h) = \max < h, \partial_p f(x) >$. By the separation theorem it is clear that $\partial_p f(x) = \{x^* \in E^* : < h, x^* > \leq f'(x; h), h \in E\}$.

The locally Lipschitz function f is said to be *regular in sense of Clarke* at x, if $f'(x; .) = f^0(x; .)$. It is clear that f is regular at x if and only if $\partial f(x) = \partial_p f(x)$.

Following Mifflin [20] we will say that f is semismooth at $x \in E$ if for every $e \in S$ $f'(x; e)$ exists and conditions $x_\alpha \to x$, $y_\alpha \in \partial f(x_\alpha)$ imply $< e, y_\alpha > \to f'(x; e)$.

The following result is established by Spingarn [27] for $E = \mathbf{R}^n$. For an arbitrary Banach space the proof is the same (in one direction it follows by Proposition 1.5 and Theorem 1.6*b*).

Theorem 4.1 *For a locally Lipschitz function $f : E \to R$ ∂f is submonotone at x iff f is semismooth and regular at x.*

The following fact is well known, but it is included for completness.

Proposition 4.2 *For a locally Lipschitz function $f : E \to \mathbf{R}$, which is quasidifferentiable in sense of Pshenichniy, the following are equivalent:*
a) $f'(.;h)$ *is u.s.c. at* x_0 *for every* $h \in E$,
b) $\partial_p f$ *is* $w^* -$ *u.s.c. at* x_0,
c) f *is regular at* x_0.

Proof. *a)* $\Leftrightarrow$ *b)*. First we shall prove that
1) $\partial_p f$ has w^*-closed graph,
2) $\partial_p f$ is locally bounded.

Let $(x_\alpha, x_\alpha^*) \in G := \{(x, x^*) \in (E, E^*) : \ x^* \in \partial f(x)\}$ be a generalized sequence, which converges to (x_0, x_0^*) (E^* considered with w^*-topology). This means that $x_\alpha \to x_0$ and $x_\alpha \xrightarrow{w^*} x_0^*$. Then $< h, x_\alpha^* >\leq f'(x_\alpha; h)$ for every $h \in E$ whence

$< h, x_0^* >\leq \limsup_\alpha f'(x_\alpha; h) \ \leq \ f'(x_0; h)$, since $f'(.;h)$ is u.s.c. This means that $x_0 \in \partial_p f(x_0)$. 2) follows easy by the fact that f is locally Lipschitz. Now b) follows by assuming the contrary.

b) $\Leftrightarrow$ *c)*. By the locally boundedness of $\partial_p f$ it follows that $\partial_p f$ is w^* compact. The mean value theorem in $\mathbf{R}$ states that for every two points $a, b \in E$ there exists $s \in (0, 1)$ such that

$$f(a) - f(b) \ = \ f'(a + s(b-a); b-a).$$

Since $f'(x, .)$ is the support function of $\partial_p f(x)$(which is w^*-compact), there exists $x^* \in \partial_p f(a + s(b-a))$ for which

$f'(a + s(b-a); b-a) =< b-a, x^* >$, i.e. the mean value theorem for Pshenichniy's subdifferentials holds. Now the assertion follows by a result of Lebourg ([18], Theorem 1.8).

c) $\Leftrightarrow$ *a)*. This implication is proved by Clarke [2].∎

The following theorem is established by Spingarn ([27], Proposition 3.5) for $\mathbf{R}^n$ and $T = \partial f$. Here the proof in one direction is the same as the proof of Spingarn and the proof in the other direction is more simple.

Theorem 4.3 *Let $f : E \to \mathbf{R}$ be a locally Lipschitz function, $T : E \to E^*$ be a locally bounded multivalued mapping for which the mean value theorem is valid: for every $a, b \in E$, $a \neq b$, there exists $c \in (a,b)$ and $x^* \in T(c)$ such that $f(a) - f(b) = < a - b, x^* >$. Then T is strictly submonotone at x_0 iff*

$$(4.1) \qquad \liminf_{x \to x_0} \left(\frac{f(x+te) - f(x)}{t} - \sigma_{T(x)}(e)\right) \geq 0 \ for\ every\ e \in S,$$

where $\sigma_K(.)$ is the support function of the set $K \subset E^* : \sigma_K(e) = \sup < e, K >$.

Proof. Let T be strictly submonotone at x_0. This means that for every $\epsilon > 0$ and $e \in S$ there exists $\delta > 0$ such that $\dfrac{< x_1 - x_2, y_1 - y_2 >}{\| x_1 - x_2 \|} \geq -\epsilon$ whenever $x_1 \neq x_2$, $\| x_i - x_0 \| < \delta$, $y_i \in T(x_i)$, $i = 1, 2$ and $\left\| \dfrac{x_1 - x_2}{\| x_1 - x_2 \|} - e \right\| < \delta$. Let $x \in B(x_0; \delta/2)$, $t \in (0, \delta/2)$. Then $\|x + te - x_0\| < \delta$. By the mean value theorem there exists $x_1 \in (x + te, x)$, $x_1 = x + se$, $s \in (0,t)$, $x_1^* \in T(x_1)$ such that $(f(x+te) - f(x))/t = < e, x_1^* >$. Then for $x^* \in T(x_0)_e$ we have

$$\frac{f(x+te) - f(x)}{t} - \sigma_{T(x)}(e) = < e, x_1^* - x^* > = \frac{< x + se - x, x_1^* - x^* >}{\|x + se - x\|} \geq -\epsilon.$$

Let now (4.1) be fulfilled. Then for every $e \in S$ and $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$\frac{f(x+te) - f(x)}{t} - \sigma_{T(x)}(e) > -\epsilon/4 \ and$$

$$\frac{f(x-te) - f(x)}{t} - \sigma_{T(x)}(-e) > -\epsilon/4$$

whenever $\|x - x_0\| < \delta_1$ and $t \in (0, \delta_1)$. There exist $M_1 > 0$ and $\delta_2 > 0$ such that f is Lipschitz in $B[x_0; \delta_2]$ with a Lipschitz constant M_1. Since T is locally bounded at x_0, there exist $M_2 > 0$ and $\delta_3 > 0$ such that $T(x) \subset B[0; M_2]$ for every $x \in B[x_0; \delta_3]$. Let $M = \max\{M_1, M_2\}, \delta = \min\{\delta_1/2,\ \delta_2/4,\ \delta_3,\ \epsilon/(8M)\}$, $x_1 \neq x_2$, $\|x_i - x_0\| < \delta$, $y_i \in T(x_i)$, $i = 1, 2$, $t = \|x_1 - x_2\|$, $u = (x_1 - x_2)/t$, $\|u - e\| < \delta$. Then $t < \delta_1$, $t < \delta_2/2$, $\|x_i + te - x_0\| < \delta_2$, $\|x_i + tu - x_0\| < \delta_2$, $i = 1, 2$ and we can write

$$\frac{<x_1-x_2,y_1-y_2>}{\|x_1-x_2\|} =< u, y_1-y_2 > = < u-e, y_1-y_2 > + < e, y_1-y_2 >>$$

$$> -\delta\|y_1-y_2\| - \sigma_{T(x_1)}(-e) - \sigma_{T(x_2)}(e) \geq$$

$$\geq -2\delta M_2 + \frac{f(x_1+t(-u))-f(x_1)}{t} - \sigma_{T(x_1)}(-e) + \frac{f(x_2+tu)-f(x_2)}{t}$$

$$\sigma_{T(x_2)}(e) \geq -2\delta M + \frac{f(x_1+t(-e))-f(x_1)}{t} - \sigma_{T(x_1)}(-e) +$$

$$+\frac{f(x_2+te)-f(x_2)}{t} - \sigma_{T(x_2)}(e) + \frac{f(x_1+t(-u))-f(x_1+t(-e))}{t} -$$

$$-\frac{f(x_2+tu)-f(x_2+te)}{t} >$$

$$> -\epsilon/4 - \epsilon/4 - \epsilon/4 - 2M_1\|u-e\| > -3\epsilon/4 - 2M\delta \geq -\epsilon. \blacksquare$$

The cases $T = \partial f$ and $T = \partial_p f$ are important special cases of the above theorem.

Proposition 4.4 *Let $f : E \to \mathbf{R}$ be a locally Lipschitz function. Then we have:*

a) if ∂f is submonotone at $x_0 \in E$ then

$$(4.2) \qquad \liminf_{x \to_e x_0,\ t\downarrow 0} \left(\frac{f(x+te)-f(x)}{t} - f^0(x;e)\right) \geq 0 \qquad \forall e \in S$$

b) if f is quasidifferentiable in the sense of Pshenichniy, then the condition

$$(4.3) \qquad \liminf_{x \to_{-e} x_0,\ t\downarrow 0} \left(\frac{f(x+te)-f(x)}{t} - f'(x;e)\right) \geq 0 \qquad \forall e \in S$$

implies that $\partial_p f$ is submonotone at x_0.

Proof For every $\epsilon > 0$ and $e \in S$ there exists $\delta_1 > 0$ such that

$$\frac{< x - x_0, y - y_0 >}{\|x - x_0\|} \geq -\epsilon/3$$

whenever $x \in U(x_0; e; \delta_1)$, $y \in T(x)$ *and* $y_0 \in T(x_0)$. By the the fact that f is locally Lipschitz, it follows that ∂f is locally bounded at x_1: there exist $\delta_2 > 0$ and $M > 0$ such that $\partial f(x) \subset B[0; M]$ for every $x \in B(x_0; \delta_2)$. The function $f^0(., e)$ is upper semicontinuous (Clarke [3]), therefore there exists $\delta_3 > 0$ such that $f^0(x; e) - f^0(x_0; e) < \epsilon/3$ for every $x \in B(x_0; \delta_3)$. Let $\delta = \min\{\delta_1/2,\ \delta_2/2,\ \delta_3,\ \epsilon/(6M),\ 1\}$, $x \in U(x_0; e; \delta)$, $t \in (0, \delta_3)$. By the mean value theorem (Lebourg [18]) there exist $x_1 \in (x + te, x)$, $x_1 = x + se$, $s \in (0, t)$, $x_1^* \in \partial f(x_1)$ for which $(f(x + te) - f(x))/t =< e, x_1^* >$. It is easy to see that

$$\left\| \frac{x + se - x_0}{\| x + se - x_0 \|} - e \right\| < \delta.$$

Choose $x_0^* \in \partial f(x_0)_e$ and we can write:

$$f(x + te) - f(x))/t - f^0(x; e) =< e, x_1^* > -f^0(x; e) >$$

$$> < e, x_1^* > -f^0(x_0; e) - \epsilon/3 =< e, x_1^* - x_0^* > -\epsilon/3 =$$

$$= \frac{< x_1 - x_0, x_1^* - x_0^* >}{\| x_1 - x_0 \|} + < e - \frac{x_1 - x_0}{\| x_1 - x_0 \|}, x_1^* - x_0^* > > -\epsilon/3 >$$

$$-\epsilon/3 > -2\delta M - \epsilon/3 \geq -\epsilon,$$

and (4.2) is proved.

b) Let (4.3) be fulfilled. Then for every $\epsilon > 0$ and $e \in S$ there exists $\delta_1 > 0$ such that $(f(x - te) - f(x))/t - f'(x, -e) > -\epsilon/4$ whenever $x \in U(x_0; e; \delta_1)$ and $t \in (0, \delta_1)$. There exist $M > 0$ and $\delta_2 > 0$ such that f is Lipschitz in $B(x_0; \delta_2)$ with a Lipschitz constant M. It is easy to see that $\partial f(x) \subset B[0; M]$ for every $x \in B(x_0; \delta/2)$. There exists $\delta_3 > 0$ such that $| (f(x_0 + te) - f(x_0))/t - f'(x_0; e) | < \epsilon/4$ for every $t \in (0, \delta_3)$. Let $\delta = \min\{\delta_1,\ \delta_2/2,\ \epsilon/(8M)\}$, $x \in U(x_0; e; \delta)$, $y \in T(x)$, $y_0 \in T(x_0)$, $t =\| x - x_0 \|$, $u = (x - x_0)//t$. Then $\| x + te - x_0 \| < \delta_2$, $\| x + tu - x_0 \| < \delta_2$ *and*

$$\frac{< x - x_0, y - y_0 >}{\|x - x_0\|} =< u, y - y_0 >=< u - e, y - y_0 > + < e, y - y_0 > >$$

$$> -\delta\|y - y_0\| - f'(x;-e) - f'(x_0;e) \geq -2\delta M + \frac{f(x+t(-u)) - f(x)}{t} -$$

$$- f'(x;-e) + \frac{f(x_0+tu) - f(x_0)}{t} - f'(x_0;e) =$$

$$= -2\delta M + \frac{f(x+t(-e)) - f(x)}{t} - f'(x;-e) + \frac{f(x_0+te) - f(x_0)}{t} -$$

$$-f'(x_0;e) + \frac{f(x+t(-u)) - f(x+t(-e))}{t} +$$

$$+ \frac{f(x_0+tu) - f(x_0+te)}{t} > -\epsilon/4 - \epsilon/4 - \epsilon/4 - 2\delta M > -\epsilon. \blacksquare$$

We shall say that the multivalued mapping $T : E \to E^*$ is *almost monotone* (resp. *almost strictly monotone*) if Definition 1.1 (resp. Definition 1.2) is fulfilled for every e from a dense subset of S.

Remark 4.5 *It is easy to see that all results so far in paragraphs 2, 3 and 4 are also valid if we replace the notion submonotone (resp. strictly submonotone) by almost submonotone (resp. almost strictly submonotone).*

Theorem 4.6 *Let E be a separable Banach space and $f : E \to \mathbf{R}$ be continuous and quasidifferentiable in sense of Phenichniy function. Then $\partial_p f$ is almost strictly submonotone mapping on a dense G_δ subset of E.*

Proof We shall use the following assertion, which is a particular case of Proposition 2.4 of Lebourg [18]:

For every open set $A \subset E$, every $\epsilon > 0$, $e \in S$ there exists open set

(4.4)

$C \subset A$ and $r > 0$ such that

$$\left|\frac{f(x+te) - f(x)}{t} - f'(x;e)\right| \leq \epsilon \ \text{for every} \ x \in C \ \text{and} \ t \in (0,r).$$

Let $\{e_n\}_{n\geq 1}$ be a dense subset of S. By (4.4) it follows that the set

$$\Gamma_{nm} := \{x_0 \in E : \exists \delta > 0 : \| x - x_0 \| < \delta, \ t \in (0,\delta) \Rightarrow$$

$$\left|\frac{f(x+te_n)-f(x)}{t} - f'(x;e_n)\right| \leq 1/m\}$$

is dense and open in E, therefore the set $\Gamma := \bigcap_{n,m=1}^{\infty} \Gamma_{nm}$ is dense and G_δ subset of E. By the proof of Theorem 4.3 it follows that

$\partial_p f$ is almost strictly submonotone at every point of Γ. ∎

Corollary 4.7 *(Kenderov [11]) Every continuous, quasidifferentiable function defined on a separable Banach space is Gateaux differentiable on a dense G_δ subset of the space.*

Proof. By Corollary 3.8 and Remark 4.5 it follows that $\partial_p f$ is single-valued on a dense G_δ subset of the space. It is easy to see that if $\partial_p f(x)$ is single-valued for some x, then f is Gateaux differentiable at x. ∎

Theorem 4.8 *([8]) Every regular in sense of Clarke locally Lipschitz function $f : E \to \mathbf{R}$ defined on an Asplund space is Frechet differentiable on a dense G_δ subset of the space.*

Proof. We shall use the method of "separable reduction" due to Gregory (see [10], p. 160), stating that if the restriction of a locally Lipschitz function to every separable subspace is Frechet differentiable on a residual subset, then the function is Frechet differentiable on a residual subset of the all space.

Let G be a separable subspace of E and $f_1 = f\,|_G$ (the restriction of f to G). Since $\partial f = \partial_p f$ (by the regularity of f) by Theorem 4.6 we have that ∂f_1 is almost strictly submonotone on a dense G_δ subset Γ_1 of G. Since G is also an Asplund space (see [10], or [22]) and ∂f_1 is w^*−uppersemicontinuous (Clarke [3]) by Theorem 3.3 and Lemma 2.1 it follows that there exists a dense G_δ subset Γ_2 of G such that the restriction of ∂f_1 to $\Gamma := \Gamma_1 \cap \Gamma_2$, $(\partial f_1)\,|_\Gamma$ is single-valued and upper-semicontinuous on Γ. Let $x_0 \in \Gamma$. We shall show that $x_0^* := -\partial f_1(x_0)$ is an almost superdifferential of $-f_1$ at x_0 in sense of Zajicek [28] , i.e. $\limsup_{G \ni h \to 0} \frac{-f_1(x_0+h)+f_1(x_0)}{\|h\|} \leq 0$. Let $\epsilon > 0$ and $e \in S_1 := S \cap G$ be given (S is the unit sphere in E). By the upper-semicontinuity of $\partial f_1\,|_\Gamma$ at x_0 it follows that there exists $\delta_1 > 0$ such that the condition $x \in B(x_0;\delta_1) \bigcap \Gamma$

implies $\partial f_1 \mid_\Gamma (x) \subset B(x_0^*; \epsilon)$. Let $t \in (0, \delta_1/2)$. By the mean value theorem (in $\mathbf{R}$) there exists $s \in (0,t)$ such that $(f(x_0+te) - f(x_0))/t = f'(x_0+se; e)$. Since $f'(.;.) = f^0(.;.)$(by the regularity) and $f^0(.,.)$ is upper semicontinuous (Clarke [3]), there exists $\delta_2 > 0$ such that the condition $\| x - x_0 - se \| < \delta_2$ implies $f'(x; e) - f'(x_0 + se; e) < \epsilon/2$.

Let $\delta = \min\{\delta_1/2, \ \delta_2\}$, $x_1 \in \Gamma \cap B(x_0 + se; \delta)$ and $x_1^* = \partial f_1(x_1)$. Then we can write

$$(-f(x_0 + te) + f(x_0))/t - < e, x_0^* >= -f'(x_0 + se; e) - < e, x_0^* > <$$

$$< \ -f'(x_1; e) - < e, x_0^* > +\epsilon/2 = e, x_1^* - x_0^* > +\epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence

$$\sup_{e \in S}[(-f(x_0 + te) + f(x_0))/t - < e, x_0^* >] \ \leq \ \epsilon$$

which shows that x_0^* is an almost superdifferential of $-f_1$ at x_0. Now we apply Theorem 2 and Proposition 1 of Zajicek [28]. ∎

5 A generalization of an Ekeland-Lebourg theorem and differentiability of the distance function

Let A be an arbitrary non-empty set, $G \subset E$ be open and for every $\alpha \in A$ $f_\alpha : G \to \mathbf{R}$ be a function. Define the function $F : G \to \mathbf{R}$ by $F(x) = \inf_{\alpha \in A} f_\alpha(x)$ and assume that $F(x) > -\infty$ for every $x \in E$.

Zajicek's generalization of a theorem of Ekeland and Lebourg from [5] is the following.

Theorem 5.1 *Let E be an Asplund space, and $\{f_\alpha : \ \alpha \in A\}$ be a system of functions on G satisfying the conditions:*

1) there exists $K > 0$ such that every function f_α is K - Lipschitz;

2) every $f_\alpha, \alpha \in A$ is Frechet differentiable on G and for every $x \in G$ the limit $\lim(f_\alpha(x + te) - f_\alpha(x))/t$ is uniform with respect to $(\alpha, e) \in A \times S$. Then F is Frechet differentiable on a dense G_δ subset of G.

Here we prove this theorem by the properties of the submonotone mappings and present another generalization of the Ekeland-Lebourg theorem. For a function $g : E \times A \to \mathbf{R}$, where A is an arbitrary set and E is a Banach space deine the function $f(x) = \sup_{\alpha \in A} g(x, \alpha)$. Define

$$M_\epsilon(x) = \{\alpha \in A : g(x,\alpha) \geq f(x) - \epsilon\}.$$

The following theorem is a modification of a result of Pshenichniy ([23], Theorems 3.2 and 3.4 when A is a compact space).

Theorem 5.2 *Let $g(.,\alpha) : E \to \mathbf{R}$ be a quasidifferentiable functions for every $\alpha \in A$ and*

a) for every $x \in E$ and $e \in S$ there exists $\epsilon > 0$ such that the limit $\lim_{t \downarrow 0}(g(x+th,\alpha) - g(x,\alpha))/t$ is uniform with respect to $\alpha \in M_\epsilon(x)$ and the set $D_\epsilon(x) := \{\cup \partial_p g(x,\alpha) : \ \alpha \in M_\epsilon(x)\}$ is norm bounded in E^*;

b) for every $x_0 \in E$ and $\epsilon_2 > 0$ there exist $\epsilon_1 > 0$ and $\delta > 0$ such that $M_{\epsilon_1}(x) \subset M_{\epsilon_2}(x_0)$ whenever $\| x - x_0 \| < \delta$;

c) $f(x) := \sup_{\alpha \in A} g(x,\alpha) \ < \ +\infty$.

Then the function f is quasidifferentiable in sense of Pshenichniy and the following conditions hold:

$$(5.1) \qquad f'(x;h) = \inf_{\epsilon > 0} \sup_{\alpha \in M_\epsilon} g'(x,\alpha;h) \ \text{ for every } \ h \in E$$

$$(5.2) \qquad \partial_p f(x) = \bigcap_{\epsilon > 0} \overline{\text{co}}^* \{ \bigcup_{\alpha \in M_\epsilon(x)} \partial_p g(x,\alpha)\}, \ \text{ where}$$

$g'(x,\alpha;h)$ is the derivative in direction h of the function $g(x,\alpha)$ and $\overline{\text{co}}^* D$ denote the convex and w^*-closed hull of the set $D \subset E^*$.

If $g(.,\alpha)$, $\alpha \in A$ are regular at $x \in E$ for every $e \in S$ and $\{g'(.,\alpha;e), \alpha \in M_\epsilon(x)\}$ are uniform upper-semicontinuous at x for some $\epsilon > 0$, then f is also regular at x.

Proof. Let $x \in E$ be fixed. It is enough to consider only the case when $\| h \| = 1$. Denote $a = \inf_{\epsilon > 0} \sup_{\alpha \in M_\epsilon(x)} g'(x,\alpha;h)$. By a) for every $\epsilon > 0$ there exist $\epsilon_0 > 0$ and $t_0 = t_0(\epsilon)$ such that

$$\left| \frac{(g(x+th),\alpha) - g(x,\alpha)}{t} - g'(x,\alpha;h) \right| \ < \ \epsilon/3$$

for every $\alpha \in M_{\epsilon_0}(x)$ and $t \in (0,t_0)$. Let $t_1 \in (0,t_0)$ and $0 < \epsilon_1 < \min\{t_1\epsilon/3, \epsilon_0\}$. There exists $\alpha_1 \in M_{\epsilon_1}(x)$ such that

$$\sup_{\alpha \in M_{\epsilon_1}(x)} g'(x,\alpha;h) \; < \; g'(x,\alpha;h) + \epsilon/3.$$

Since $\epsilon_1 \; < \; \epsilon_0$ we have $M_{\epsilon_1}(x) \subset M_{\epsilon_0}(x)$ and :

$$(f(x+t_1h) - f(x))/t_1 \;\geq\; (g(x+th,\alpha_1) - g(x,\alpha_1) - \epsilon_1)/t_1 \;>$$

$$> g'(x,\alpha_1;h) - \epsilon/3 - \epsilon_1/t_1 \;>$$

$$> \sup_{\alpha \in M_{\epsilon_1(x)}} g'(x,\alpha;h) - \epsilon/3 - \epsilon/3 \;\geq\; a - \epsilon.$$

Hence

$$\liminf_{t \downarrow 0}(f(x+th) - f(x))/t \geq \; a - \epsilon$$

and since $\epsilon > 0$ is arbitrary small ,

$$\liminf_{t \downarrow 0}(f(x+th) - f(x))/t \;\geq\; a.$$

Let $\epsilon_3 > 0$ be such that $\sup\limits_{\alpha \in M_{\epsilon_3(x)}} g'(x,\alpha';h) - a \; < \; \epsilon/3.$

By $b)$ there exist $\delta > 0$ and $\epsilon_4 > 0$ such that $M_{\epsilon_4}(x') \subset M_{\epsilon_3}(x)$ whenever $\| x' - x \| < \delta$. Let $0 < t_2 < \min\{t_0, \delta\}$, $0 < \epsilon_2 < \min\{t_2\epsilon/3, \epsilon_0, \epsilon_4\}$ and $\alpha_2 \in M_{\epsilon_2}(x+t_2h)$. Then $M_{\epsilon_2}(x+t_2h) \subset M_{\epsilon_4}(x+t_2h) \subset M_{\epsilon_3}(x)$ and we can write:

$$(f(x+t_2h) - f(x))/t_2 \;\leq (g(x+t_2h,\alpha_2) + \epsilon_2 - g(x,\alpha_2))/t_2 \;\leq$$

$$\leq\; g'(x,\alpha_2;h) + \epsilon/3 + \epsilon_2/t \;<$$

$$< \;\sup\{g'(x,\alpha;h) : \alpha \in M_{\epsilon_2}(x+t_2h)\} + 2\epsilon/3 \;\leq$$

$$\leq\; 2\epsilon/3 + \sup_{\alpha \in M_{\epsilon_3(x)}} g'(x,\alpha;h) \;<\; a + \epsilon.$$

Hence $\limsup\limits_{t \downarrow 0}(f(x+th) - f(x))/t \;\leq\; a + \epsilon$ and since $\epsilon > 0$ is arbitrary small, $\limsup\limits_{t \downarrow 0}(f(x+th) - f(x))/t \;\leq\; a$ and (5.1) is proved.

Let $D(x) = \bigcap_{\epsilon>0} \overline{co^*} \bigcup_{\alpha\in M_\epsilon(x)} \partial_p g(x,\alpha)\}$. By $b)$ and by the Alaoglou-Bourbaki theorem the set $\overline{co^*} D_\epsilon(x)$(for D_ϵ from a) is w^*-compact, therefore $D(x)$ is non-empty (and w^*-compact). Let $x^* \in D(x)$ and $\epsilon > 0$. Then $x^* = w^* - \lim x^*_\gamma$, where $x^*_\gamma = \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} x^*_{\gamma,i}$, $\lambda_{\gamma,i} \in [0,1]$, $i = 1 - n(\gamma)$, $\sum_{i=1}^{n(\gamma)} \lambda_{\gamma i} = 1$, $x^*_{\gamma,i} \in \partial_p g(x, \alpha_{\gamma,i})$, $\alpha_{\gamma,i} \in M_\epsilon(x)$, $\gamma \in \Gamma$. We can write:

$$< h, x^*_\gamma >= \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} < h, x^*_{\gamma,i} >\leq \max_{1\leq i\leq n(\gamma)} < h, x^*_{\gamma,i} >\leq$$

$$\leq \sup\{< h, z^* >: z^* \in \partial_p g(x,\alpha), \alpha \in M_\epsilon(x)\} =$$

$$= \sup_{\alpha\in M_\epsilon(x)} \sup\{< h, z^* >: z^* \in \partial_p g(x,\alpha)\} = \sup_{\alpha\in M_\epsilon(x)} g'(x,\alpha;h),$$

whence $< h, x^* >\leq \sup_{\alpha\in M_\epsilon(x)} g'(x,\alpha;h)$. Since the last inequality is fulfilled for every $\epsilon > 0$, we have

$$< h, x^* >\leq \inf_{\epsilon>0} \sup_{\alpha\in M_\epsilon(x)} g'(x,\alpha;h) = f'(x;h).$$

Hence f is quasidifferentiable in sense of Pshenichniy at x and

$$D(x) \subset \partial_p f(x).$$

By the other hand for every $\epsilon > 0$ and $h \in S$ we have:

$$\sup\{< h, z^* >: z^* \in \overline{co^*}\{\cup\{\partial_p g(x,\alpha) : \alpha \in M_\epsilon(x)\}\} \geq$$

$$\geq \sup\{< h, z^* >: z^* \in \cup\{\partial_p g(x,\alpha) : \alpha \in M_\epsilon(x)\}\} =$$

$$= \sup_{\alpha\in M_\epsilon(x)} \sup\{< h, z^* >: z^* \in \partial_p g(x,\alpha)\} = \sup_{\alpha\in M_\epsilon(x)} g'(x,\alpha;h) \geq f'(x;h) =$$

$$= \max\{< h, x^* >: z^* \in \partial_p f(x)\}.$$

Hence by the separation theorem we obtain

$$\partial_p f(x) \subset \overline{co^*}\{\cup\{\partial_p g(x,\alpha) : \alpha \in M_\epsilon(x)\}\}$$

and since this is true for every $\epsilon > 0$, we have $\partial_p f(x) \subset D(x)$ and (5.2) is proved.

Let $d)$ be fulfilled for $x_0 \in E$ and let $\epsilon > 0$, $e \in S$. Then there exist $\delta_1 > 0$ and $\epsilon_0 > 0$ for which $g(x, \alpha; e) - g(x_0, \alpha; e) < \epsilon/3$ whenever $x \in B(x_0; \delta_1)$. By (5.1) it follows that there exists $\epsilon' \in (0, \epsilon_0)$ such that $g'(x_0, \alpha; e) - f'(x_0; e) < \epsilon/3$ for every $\alpha \in M_{\epsilon'}(x_0)$. By b) there exists $\epsilon_1' \in (0, \epsilon']$ and $\delta > 0$ such that $M_{\epsilon_1'}(x) \subset M_{\epsilon'}$ whenever $\|x - x_0\| < \delta$. Let $x \in B(x_0; \min\{\delta, \delta_1\})$. By (5.1) there exist

$\alpha' \in M_{\epsilon_1'}(x)$ for which $f'(x; e) - g'(x, \alpha; \epsilon_1') < \epsilon/3$. Now we can write

$$f'(x; e) - f'(x_0; e) \ < \ g'(x, \alpha'; e) - g'(x_0; \alpha'; e) + 2\epsilon/3 \ < \ \epsilon,$$

which shows that $f'(.; e)$ is upper-semicontinuous and by Proposition 4.3 we obtain that f is regular. ∎

In the following lemma we use some ideas of Ekeland and Lebourg from the proofs of Propositions 2.3 and 2.4 from [5].

Lemma 5.3 *Let U be an open set of the Banach space E, A be an arbitrary set and $g(., \alpha) : U \to \mathbf{R}, \alpha \in A$ be continuous, quasidifferentiable in sense of Pshenichniy functions, $f(x) := \sup_{\alpha \in A} g(x, \alpha)$ for every $x \in U$ and let the following condition hold:*

(5.3) for every $x_0 \in U$ there exists $\delta > 0$ such that the set $\cup\{\partial_p g(x, \alpha) : \| x - x_0 \| < \delta,\ g(x, \alpha) \geq f(x_0) - \delta\}$ is norm bounded in E^*. Then the function f is locally Lipschitz and the following condition holds:

(5.4) for every $x_0 \in U$ and every $\epsilon_0 > 0$ there exist $\delta > 0$ and $\epsilon_1 > 0$ such that $M_\epsilon(x) \subset M_{\epsilon_0}(x_0)$ whenever $\| x - x_0 \| < \delta$.

Proof. Let $x_0 \in U$ and $\epsilon_0 > 0$. By (5.3) there exist $\delta_1 > 0$ and $c > 0$ such that

(5.5) the conditions $\| x - x_0 \| < \delta_1$, $g(x, \alpha) \geq f(x_0) - \delta_1$, $z^* \in \partial_p g(x, \alpha)$ imply $\| z^* \| < C$.

Let $0 < \epsilon_1 < \min\{\delta_1, \epsilon_0\}$. As supremum of continuous functions f is lower-semicontinuous. So there exists $\delta_2 > 0$ such that $f(x_0) - f(x) < \epsilon_1/3$ whenever $\| x - x_0 \| < \delta_2$. Let $0 < \delta_3 < \min\{\delta_1/2, \delta_2, \epsilon_1/(3C)\}$, $x \in B(x_0; \delta_3)$, $y \in B(x; \delta_3)$, $y \neq x$ and $\alpha \in M_{\epsilon_1/3}(x)$. Then we have

$$g(x, \alpha) \ \geq \ f(x) - \epsilon_1/3 > f(x_0) - 2\epsilon_1/3. \tag{5.6}$$

Consider the function $\varphi(t) := g(x, \alpha) - g(x + t(y - x), \alpha)$ with derivative $\varphi'(t) = g'(x + t(y - x), \alpha; y - x)$. Let $t_0 > 0$ is the gratests number for which we have $\varphi(t) < \epsilon_1/3$ for $t \in [0, t_0)$. Let us assume that $t_0 \leq 1$. Then for $t \in [0, t_0]$ we have $\varphi(t) \leq \epsilon_1/3$ and by (5.6) obtain

$$g(x + t(y - x), \alpha) > f(x_0) - \epsilon_1 > f(x_0) - \delta_1.$$

Since $\| x + t(y-x) - x_0 \| \leq \| x - x_0 \| + t \| y - x \| \leq t\delta_3 + \delta_3 \leq 2\delta_3 < \delta_1$, by (5.5) we obtain $\| z^* \| < C$ for every $z^* \in \partial_p g(x + t(y-x), \alpha)$. Then

$$\epsilon_1/3 \leq \varphi(t_0) = \int_0^{t_0} \varphi'(t)dt = \int_0^{t_0} g'(x + t(y-x), \alpha; y - x)dt =$$

$$\int_0^{t_0} \max\{< y-x, x^* >: x^* \in \partial_p g(x+t(y-x), \alpha)\}dt \leq \| y-x \| t_0 C < \delta_3 t_0 C,$$

whence $t_0 \geq \epsilon_1/(3C\delta_3) > 1$, which is a contradiction with the assumption that $t_0 \leq 1$. Therefore $t_0 > 1$ and we can write:

$$g(x, \alpha) - g(y, \alpha) = \varphi(1) = \int_0^{t_0} \varphi'(t)dt \leq C \| x - y \| . \tag{5.7}$$

By (5.7) and (5.6) we obtain

$$g(y, \alpha) \geq g(x, \alpha) - C \| x - y \| > f(x_0) - 2\epsilon_1/3 - C\delta_3 >$$

$$> f(x_0) - \epsilon_1.$$

Now we take $y = x_0$ and obtain $\alpha \in M_{\epsilon_1}(x_0)$. So we have $M_{\epsilon_1/3}(x) \subset M_{\epsilon_1}(x_0) \subset M_\epsilon(x_0)$ and (5.4) is proved.

Now let $\epsilon_2 \in (0, \epsilon_1/3)$, $\| x - x_0 \| < \delta_3/2$, $\| y - x_0 \| < \delta_3/2$, $x \neq y$, $\alpha_2 \in M_{\epsilon_2}(x) \subset M_{\epsilon_1/3}(x)$. By (5.7) we have

$$f(x) - f(y) \leq g(x, \alpha_2) - g(y, \alpha_2) + \epsilon_2 \leq C \| x - y \| + \epsilon_2$$

and since this inequality is fulfilled for every $\epsilon_2 \in (0, \epsilon_1/3)$, we obtain $f(x) - f(y) \leq C \| x - y \|$. Exchanging the roles of x and y, in the same way we obtain $f(y) - f(x) \leq C \| x - y \|$ and the lemma is proved. ∎

Now we can prove the Zajicek theorem.

Proof of Theorem 5.1 Let us assume the notions of Theorem 5.1. Put $g(., \alpha) = -f$ and f=-F. Then $f(.) = \sup_{\alpha \in A} g(., \alpha)$. By 1) and 2) of Theorem

5.1 (5.3) in Lemma 5.3 is fulfilled and then (5.4) shows that condition $b)$ of Theorem 5.2 is fulfilled. So f is locally Lipschitz and quasidifferentiable in sense of Pshenichniy.

We shall use the following assertion, which is also analogous to a result of Lebourg ([18], Proposition 2.4) and the proof is the same:

For every open set $B \subset E$ and every $\epsilon > 0$ there exist an open set $C \subset B$ and $r > 0$ such that the conditions $x \in C$ and $t \in (0, r)$ imply

$$(5.8) \quad \left|\frac{g(x+te,\alpha)-g(x,\alpha)}{t} - g'(x,\alpha,e)\right| \leq \epsilon \ \forall e \in S \ \forall \alpha \in A.$$

We shall proof that $\partial_p f$ is a strictly submonotone mapping on a dense G_δ subset of G. By (5.8) it follows that the set

$$G_n := \{x_0 \in G : \exists \delta > 0 : \| x - x_0 \| < \delta,\ t \in (0,\delta) \Rightarrow$$
$$\left|\frac{g(x+te,\alpha)-g(x,\alpha)}{t} - g'(x,\alpha,e)\right| \leq 1/n \ \ \forall e \in S, \ \ \forall \alpha \in A\}$$

is dense and open in G. Let $x_0 \in G_n$. There exists $\delta > 0$ for which the conditions $\| x - x_0 \| < \delta, t \in (0,\delta)$ imply

$$(5.9) \quad \left|\frac{g(x+te,\alpha)-g(x,\alpha)}{t} - g'(x,\alpha,e)\right| \leq 1/n \ \forall e \in S, \ \ \forall \alpha \in A.$$

Let $e \in S$ is fixed and $x \in B(x_0;\delta)$, $t \in (0,\delta)$, $0 < \epsilon_1 < t/(n)$, $x^* \in \partial_p f(x)_e$, (which means that $< e, x^* >= \max_{z^* \in \partial_p f(x)} < e^*, z >$. By (5.2) of Theorem 5.2 we have $x^* = w^* - \lim_{\gamma \in \Gamma} x^*_\gamma$, where $x^*_\gamma = \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} x^*_{\gamma,i}$, $x^*_{\gamma,i} = \partial_p g(x,\alpha_{\gamma,i})$, $\alpha_{\gamma,i} \in M_{\epsilon_1}(x)$, $\lambda_{\gamma,i} \in [0,1]$, $\sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} = 1$ and by (5.9) we can write

$$\frac{f(x+te)-f(x))}{t} \geq \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i}(g(x+te,\alpha_{\gamma,i}) - g(x,\alpha_{\gamma,i})) - \epsilon_1]/t \geq$$

$$\geq \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i}[g'(x,\alpha_{\gamma,i};e) - 1/n] - \epsilon_1/t >$$

$$> \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} < e, x^*_{\gamma,i} > -1/n = < e, x^*_\gamma > -2/n.$$

Hence $\frac{f(x+t_1 e)-f(x)}{t} - < e, x^* > \geq -2/n$. Since $< e, x^* >= \max\{< e, z^* >: z^* \in \partial_p f(x)\} = f'(x; e)$, Theorem 4.4 shows that $\partial_p f$ is strictly submonotone on $\Gamma := \cap_{n=1}^{\infty} G_n$. By Theorem 3.3 $\partial_P f$ has property (*) on a dense G_δ subset G_0 of G.

Let us assume the notation in the proof of Theorem 3.4. In our case $T = \partial_P f$ and $D(T) = G$. We shall prove that T_0 is single-valued and norm to norm upper semicontinuous on $\Gamma_0 := G_0 \cap \Gamma$. Let $\epsilon > 0$ be given, $x_0 \in \Gamma_0$ and $y \in T_0(x_0)$ be the point from the definition of the property (*). Then there exists $\delta_1 > 0$ such that $\Gamma_0(x) \cap (y + \epsilon B^*) \neq \emptyset$ for every $x \in x_0 + \delta_1 B$. For $n > \frac{1}{\epsilon}$ there exists $\delta_2 > 0$ such that $(x_0 + \delta_2 B \subset G_n$. Put $\delta = \min\{\delta_1, \delta_2\}$.

Assume that $T_0(x) \not\subset y + 2\epsilon B^*$ for some $x \in x_0 + \delta B$. Let $e \in S$ strictly separate $x^* + \epsilon B^*$ and $y + \epsilon B^*$, where $x^* \in T_0(x) \setminus (y + 2\epsilon B^*)$. Then by the monotonicity at x we have

$$\sup_{z \in x^* + \epsilon B^*} \langle e, z \rangle < \inf_{z \in X^* + \epsilon B^*} \langle e, z \rangle \leq \langle e, x^* \rangle - \epsilon \leq$$

$$\leq \sup_{z \in T_0(x)} \langle e, z \rangle - \epsilon \leq \inf \langle e, z \rangle \quad \forall z \in U(x, e, \gamma)$$

for some $\gamma > 0$. But $U(x, e; \gamma) \cap (x_0 + \delta B) \neq \emptyset$, a contradiction. Therefore $\partial_P f$ is single-valuedd and u.s.c. on Γ_0.

Let $x_0 \in \Gamma_0$ and $\epsilon > 0$ be fixed. Since f is locally Lipschitz, $\partial_p f$ is w^*-compact and by the mean-value theorem for $n > 1/\epsilon$ and $t \in (0, \delta_n(x_0))$ we can write

$$\left| \frac{f(x_0 + te) - f(x_0)}{t} - f'(x_0; e) \right| = | f'(x_0 + se; e) - f'(x_0; e) | =$$

$$= < e, x_1^* - \partial_p f(x_0) > \leq \| x_1^* - \partial_p f(x_0) \| \leq 1/n < \epsilon,$$

for every $e \in S$, where $s \in (0, t)$, $x_1^* \in \partial_p f(x_0 + se)_e$, recall that $< e, x_1^* >= \max_{z \in \partial_P f(x_0 + se)} < e, z >$.

Therefore f is Frechet differentiable at x_0 and the theorem is proved. ∎

Now we present new generalization of the Ekeland and Lebourg theorem (theorem 5.1).

Theorem 5.4 *Let U be an open subset of the Banach space E and $g(.,): U \to \mathbf{R}$, $\alpha \in A$ be continuous, quasidifferentiable in sense of Pshenichniy functions, A be an arbitrary non-empty set and let $f(.) := \sup_{\alpha\in A} g(.,\alpha) < +\infty$ for every $\alpha \in A$. Let us assume that the condition (5.3) of Lemma 5.3 and the following condition is fulfilled:*

(5.10) for every $x_0 \in U$ and every $e \in S$ there exist $\epsilon > 0$ and $\delta > 0$ such that the limit $\lim_{t\downarrow 0} \frac{g(x+te,\alpha)-g(x,\alpha)}{t}$ is uniform with respect to $x \in B(x_0;\delta)$ and $\alpha \in M_\epsilon(x_0)$.

Then

a) f is locally Lipschitz in U, quasidifferentiable in U and $\partial_P f$ is strictly submonotone,

b) if E is an Asplund space, then f is Frechet differentiable on a dense G_δ subset Γ of U(with Frechet derivative f'). If $g(.,\alpha), \alpha \in A$ are Gateaux (resp. Frechet) differentiable in U with Gateaux (resp. Frechet) derivative $\nabla g(.,\alpha)$(resp. $g'(.,\alpha)$) then for every $x \in \Gamma$ we have // $g(x,\alpha_n) \to f(x) \Rightarrow \nabla g(x,\alpha_n) \to f'(x)$ in the w^*–topology (resp. $g(x,\alpha_n) \to f(x) \Rightarrow g'(x,\alpha_n) \to f'(x)$ in the norm topology),

c) If there exists a w^*–lower semicontinuous strictly convex function in E^* then f if Gateaux differentiable on a dense G_δ subset of E.

Proof. By Lemma 5.3 and theorem 5.2 it follows that f is locally Lipschitz, quasidifferentiable in sense of Pshenichniy in U and the conditions (5.1) and (5.2) of Theorem 5.2 are fulfilled. Let $x_0 \in U$, $e \in S$ and $\epsilon > 0$ be fixed. By (5.10) it follows that there exist $\epsilon_1 > 0$ and $\delta_1 > 0$ such that $|\,(g(x+te,\alpha) - g(x,\alpha))/t - g'(x,\alpha;e)\,| < \epsilon/2$ for every $x \in B(x_0;\delta_1)$, $t \in (0,\delta_1)$, $\alpha \in M_{\epsilon_1}(x_0)$. Let $t_1 \in (0,\delta_1), 0 < \epsilon_2 < \min\{t_1\epsilon/2, \epsilon_1\}$, $x \in B(x_0;\delta_1)$, $x^* \in \partial_P f(x)_e$. By *(5.2)* of Theorem 5.2 we have $x^* = w^* - \lim x^*_\gamma$, where $x^*_\gamma = \sum_{i=1}^{n(\gamma)} \lambda_\gamma x^*_{\gamma,i}$, $x^*_{\gamma,i} \in \partial_P g(x,\alpha_{\gamma,i})$, $\alpha_{\gamma,i} \in M_{\epsilon_2}(x) \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} = 1$. We can write

$$f(x+t_1e) - f(x))/t_1 \geq [\sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i}(g(x+t_1e,\alpha_{\gamma i}) - g(x,\alpha_{\gamma,i}) - \epsilon_2]/t_1 \geq$$

$$\geq \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i}(g'(x,\alpha_{\gamma,i};e) - \epsilon/2) - \epsilon_2/t_1 > \sum_{i=1}^{n(\gamma)} \lambda_{\gamma,i} < e, x^*_{\gamma,i} > -\epsilon/2 - \epsilon/2 =$$

$$= < e, x_\gamma^* > -\epsilon.$$

Hence $\frac{f(x+t_1 e)-f(x)}{t_1} - < e, x^* > \geq -\epsilon$. Since $< e, x^* > = \max\{< e, z^* >: z^* \in \partial_p f(x)\} = f'(x; e)$, Theorem 4.4 shows that $\partial_p f$ is strictly submonotone at x_0.

b) By Theorem 3.4 it follows thsat $\partial_P f$ is single-valued and norm to norm upper semicontinuous on a dense G_δ subset Γ of U. By the proof of Theorem 5.1 one can see that f is Frechet differentiable on Γ.

Let $g(., \alpha)$ be Gateaux differentiable in U and let $x \in \Gamma, g(x, \alpha_n) \to f(x)$. Then withowt loss of generality we can assume that there exists a sequence $\{\epsilon_n\}_{n\geq 1}$ of positive numbers for which $\alpha_n \in M_{\epsilon_n}(x)$ and $\epsilon_n \downarrow 0$. Let $\epsilon > 0$, and $e \in S$. By (5.10) it follows that that there exist $\delta_1(e) > 0$ and $\nu(e) \in N$ such that $| g'(x, \alpha_n; \pm e) - (g(x \pm te, \alpha_n) - g(x, \alpha_n))/t | < \epsilon/3$ for every $t \in (0, \delta_1(e))$ and every $n > \nu(e)$. There exists $\delta_2 > 0$ for which

$$| f'(x; \pm e) - (f(x \pm te) - f(x))/t | < \epsilon/3 \quad \forall t \in (0, \delta_2) \quad \forall e \in S.$$

Let $0 < t_0 < \min\{\delta_1(e), \delta_2\}$, n_1 be such that $\epsilon_{n_1} < t_0\epsilon/3$ and let $n_0 > \max\{\nu(e), n_1\}$. Then for every $n \geq n_0$ we have

$$g'(x, \alpha_n; e) - f'(x; e) <$$

$$< [g(x + t_0 e, \alpha_n) - g(x, \alpha_n) - f(x + t_0 e) + f(x)]/t_0 + 2\epsilon/3 \leq$$

$$\leq 2\epsilon/3 + \epsilon_n/t_0 < \epsilon$$

and

$$f'(x; e) - g'(x, \alpha_n; e) =$$

$$= g'(x, \alpha_n; -e) - f'(x; -e) <$$

$$< [g(x - t_0 e, \alpha_n) - g(x_0, \alpha_n) - f(x - t_0 e) + f(x)]/t_0 + 2\epsilon/3 \leq$$

$$\leq \epsilon_n/t_0 + 2\epsilon/3 < \epsilon.$$

In such a way we have shown that for every $h \in E$

$$(5.11) \qquad |< h, \nabla g(x, \alpha_n - f'(x) >| = | g'(x, \alpha_n; h) - f'(x; h) | \to 0,$$

which show that $\nabla g(x, \alpha_n) \to f'(x)$ in the w^*-topology. If $g(., \alpha)$ are Frechet differentiable in U, then $\delta_1(e)$ and $\nu(e)$ do not depend on $e \in S$, therefore the convergence in (5.11) will be uniform with respect to $e \in S$, i.e.

$$\| \nabla g(x, \alpha_n - f'(x) \| = \sup_{e \in S} | < e, \nabla g(x, \alpha_n - f'(x) > | \to 0.$$

c) follows directly by a) and by Corollary 3.7. ∎

When A consists of only one element, we obtain the following.

Corollary 5.5 *Let E be an Asplund space, $f : E \to \mathbf{R}$ be a quasidifferentiable in sense of Pshenichniy continuous function function, which is locally directionally uniformly differentiable at every point, which means that for every $x_0 \in E$ and $h \in E$ there exists $\delta > 0$ for which the limit $\lim_{t \downarrow 0}(f(x+th) - f(x))/t$ is uniform with respect to $x \in B(x_0; \delta)$. Then f is Frechet differentiable on a dense G_δ subset of E.*

Every convex continuous function satisfy the condition of Corollary 5.5. This follows by the fact that its subdifferential is a strictly submonotone mapping and by Theorem 4.4.

Let A be an arbitrary nonempty subset of the Banach space E. If we denote $g(x, \alpha) = - \| x - \alpha \|$ for $x \in E$ and $\alpha \in A$, then for the function "distance to a set" $d(x, A) := \inf_{\alpha \in A} \| x - \alpha \|$ we have $-d(x, A) = \sup_{\alpha \in A} g(x, \alpha)$. The functions $g(., \alpha)$, $\alpha \in A$ are Lipschitz with Lipschitz constant 1 and by Theorem 5.4 we obtain the following.

Corollary 5.6 *([9]) Let E be an Asplund space with a uniformly Gateaux differentiable norm (which means that the norm is Gateaux differentiable and for every $h \in E$ the limit $\lim_{t \downarrow 0} \dfrac{\| x + th \| - \| x \|}{t}$ is uniform with respect to $x \in S$) and let $A \subset E$ be an arbitrary non-empty set. Then the distance function $d(., A)$ is Frechet differentiable on a dense G_δ subset of E.*

Corollary 5.7 *(Zajicek [28, Corollary 2). Let the Banach space E has an uniformly Frechet differentiable norm (i.e. the norm is Frechet differentiable on $E \backslash \{0\}$ and the limit $\lim_{t \downarrow 0} \dfrac{\| x + th \| - \| x \|}{t}$ is uniform with respect to $(x, h) \in S \times S$.*

Then every distance function $d(., A)$, where A is an arbitrary non empty subset of E is Frechet differentiable on a dense G_δ subset of E.

Proof. If E has a Frechet differentiable norm, then E is an Asplund space (see for instance [10], [22]). Now we apply Corollary 5.6. ∎

Corollary 5.8 *Let the Banach space E has an uniformly Gateaux differentiable norm. Then every distance function $d(.,A)$, where A is an arbitrary nonempty subset of E is Gateaux differentiable on a dense G_δ subset of E.*

Proof. By a result of Smulian [26] it follows that the dual norm in E^* is w^*-locally uniformly convex, which means that for every two sequences $\{x_n^*\}_{n\geq 1}, \{y_n^*\}_{n\geq 1}$ from the unit ball of E^* for which $\| x_n^* + y_n^* \| \to 2$ it is fulfilled $x_n^* - y_n^* \to 0$ in the w^*-topology. It is easy to see that if the dual norm is w^*-locally uniformly convex, then it is strictly convex. Now we apply Corollary 3.7. ∎

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