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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

Densely Two-Valued Metric Projections in Uniformly Convex Banach Spaces

N. V. Zhivkov

Preprint
May 1994
No 1

Department of Operations Research

БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ

1869

BULGARIAN ACADEMY OF SCIENCES

Densely Two-Valued Metric Projections in Uniformly Convex Banach Spaces

N.V. Zhivkov

Abstract

For every uniformly convex Banach space X with $\dim X \geq 2$ there is a residual set $\mathcal{U}$ in the Hausdorff metric space $B(X)$ of bounded and closed sets in X such that a metric projection generated by a set from $\mathcal{U}$ is two-valued and upper semicontinuous on a dense and everywhere continual subset of X.

For any two closed and separated subsets M_1 and M_2 of X the points on the equidistant hypersurface which have best approximations both in M_1 and M_2 form a dense G_δ set in the induced topology.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. A non-empty set $M \subset X$ generates a distance function $d(x, M) = \inf\{\|x-z\| : z \in M\}$ and a set-valued mapping

$$P(x, M) = \{y \in M : \|x-y\| = d(x, M)\}$$

called metric projection or nearest point mapping.

In [St] Stechkin began the study of generic properties of metric projections, i.e. properties satisfied for the points of residual sets. Among the other results he proved that in a uniformly convex Banach space X any metric projection generated by a closed and non-empty subset is single-valued and upper semicontinuous (u.s.c.) at the points of a dense and G_δ subset of X. Later a lot of papers investigating generic properties of metric projections, such as existence of a best approximation, uniqueness and well-posedness appeared [La], [Ko], [Bo], [FZ], [BP], [BF]. It has to be mentioned a characterization result due to Lau and Konyagin: A Banach space X is reflexive

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1991 *Mathematics Subject Classification.* Primary 41A65, 54E52. Secondary 46B20.

Key words and phrases: metric projection, dense G_δ set, uniformly convex, Hausdorff space of sets.

Research partially supported by the National Foundation for Scientific Research at the Bulgarian Ministry of Science and Education.

and satisfies the Kadec-Klee property if and only if arbitrary metric projection generated by a closed subset of X has non-empty images on a dense G_δ subset of X.

In [Za] Zamfirescu showed that a generically single-valued metric projection might be densely multivalued, as well, and that this is typical for compacta in the n-dimensional Euclidean space $\mathbb{R}^n$ whenever $n \geq 2$: In the Hausdorff metric space of compacta $\mathcal{K}(\mathbb{R}^n)$ most compacta (in sense of Baire category) generate metric projections which are multivalued on everywhere continual sets. A set is called everywhere continual in X if its intersection with any non-empty open subset of X contains continuum many elements.

The result of Zamfirescu initiated another line of research. For separable strictly convex Banach space X De Blasi and Myjak proved several analogous theorems: In the Hausdorff space $\mathcal{B}(X)$ of bounded and closed sets most sets generate densely non-well-posed metric projections [BM1], in the spaces $\mathcal{K}(X)$, $\mathcal{C}(X)$, and $\mathcal{S}(X)$ of compacta, continua and starshaped continua respectively most projections are densely multivalued [BM2], [BKM] (with Kenderov).

A subsequent extension of a result of this type concerning $\mathcal{K}(X)$ for arbitrary strictly convex Banach space X is obtained in [Zh].

In this paper we consider generic properties of metric projections generated by closed subsets of uniformly convex Banach spaces and prove the following results:

Theorem 4.1 Let M_1 and M_2 be two closed and separated sets in the uniformly convex Banach space X. Then there is a dense G_δ subset Γ of the equidistant hypersurface such that the metric projections $P(\cdot, M_i)$ for $i = 1, 2$ are single-valued and u.s.c. in X at all points of Γ.

Theorem 5.1 For every uniformly convex Banach space X there is a residual set $\mathcal{U}$ in $\mathcal{B}(X)$ such that a metric projection generated by $M \in \mathcal{U}$ is two-valued and upper semicontinuous on a dense and everywhere continual subset of X.

The latter result gives a partial extension of De Blasi–Myjak's theorem from [BM1] in two directions: First, it shows that the separability assumption can be relaxed and second, the most metric projections associated with sets from $\mathcal{B}(X)$ densely have non-empty multivalued images. In corollary 5.2 we make Zamfirescu's theorem more precise by showing that the images of the metric projections generated by the most compacta in a finite-dimensional strictly convex spaces contain exactly two different elements.

The paper consists of five sections including the present one. In the next section some preliminaries are given. Section 3 contains all auxiliary results. Section 4 deals with equidistant approximations and the final section contains the main result.

2. Preliminaries

Let X be a uniformly convex Banach space of dimension $\dim X \geq 2$, i.e. there exists a continuous non-decreasing function (modulus of convexity) $\delta : [0,2] \longrightarrow [0,1]$ for the closed unit ball B of X such that $0 = \delta(0) < \delta(\varepsilon)$ whenever $\varepsilon > 0$ and $2^{-1}(x+y) + z \in B$ whenever $x, y \in B$ and $\|z\| \leq \delta(\|x-y\|)$. We note that this definition of modulus of convexity differs from the original (see [LT]), but it is more suitable for our purposes. For reasons of purely technical nature a modulus of convexity $\delta(\cdot)$ will be referred as a reduced convexity modulus if there is a modulus of convexity $\delta' : [0,2] \longrightarrow [0,1]$ such that $\delta(2\varepsilon) = \delta'(\varepsilon)$. A convexity modulus δ satisfies the trivial inequality $\delta(\varepsilon) \leq \varepsilon/2$, and if δ is a reduced convexity modulus then $\delta(\varepsilon) \leq \varepsilon/4$.

For $x \in X$ and $\varepsilon > 0$, the open (respectively closed) ball with center x and radius ε is denoted by $B(x,\varepsilon)$ (resp. $B[x,\varepsilon]$) and $S[x,\varepsilon]$ is the sphere with same center and radius. The closed ball $B[\theta, r]$ centered at the origin is denoted for convenience by rB or $B[r]$ and the sphere $S[\theta, r]$ by rS or $S[r]$. Also, the symbol $B(M,\varepsilon)$, where M is a set, stands for $\cup_{x\in M} B(x,\varepsilon)$. The Hausdorff metric we denote by H and the balls in $\mathcal{B}(X)$ are denoted by the letter $\mathcal{O}$ instead of B. For arbitrary $x, y \in X$ and $M, N \subset X$ the inequality holds

$$|d(x,M) - d(y,N)| \leq \|x-y\| + H(M,N). \tag{1}$$

A line segment with end-points $x, y \in X$ is denoted by $[x,y]$ and (x,y) means $[x,y] \setminus \{x,y\}$. For a subset M of X we denote by $\mathrm{int}M$, $\mathrm{bd}M$, $\mathrm{co}M$, $\mathrm{diam}M$ and $\mathrm{card}M$ its interior, boundary, convex hull, diameter and cardinality respectively. If M_i, $i = 1,2$, are sets and λ is a real number then $M_1 + \lambda M_2$ substitutes the set $\{x + \lambda y : x \in M_1, y \in M_2\}$.

Given a non-empty set $M \subset X$. A set $N \subset M$ is a ε-net for M if

(i) $\quad \forall x \in M \setminus N \;\; \exists y \in N \;\; \|x-y\| < \varepsilon,$

(ii) $\quad \forall y \in N \;\; \forall z \in N \;\; y \neq z \Longrightarrow \|y-z\| \geq \varepsilon.$

It is a consequence of Zorn's lemma that for every nonvoid set M and every $\varepsilon > 0$ there exists a ε-net N of M. To verify it consider the family $\mathcal{Z}$ of all subsets of M, ordered by inclusion and satisfying condition (ii). A maximal element N in $\mathcal{Z}$ satisfies (i) too, and N is a ε-net.

3. Lemmas

Throughout this section X is supposed to be uniformly convex with $\dim X \geq 2$ and δ stands for a modulus of convexity.

Lemma 1. For every $\varepsilon \in (0,2)$ and $x \in X \setminus B$ it follows

$$\mathrm{diam}\, B[x, d(x,B) + \delta(\varepsilon)] \cap B < \varepsilon.$$

Proof: Assume the contrary: There are $y_n, z_n \in B[x, d(x,B)+\delta(\varepsilon)] \cap B$ such that $(\|y_n - z_n\|)$ is an increasing sequence and $\lim \|y_n - z_n\| \geq \varepsilon$. Obviously, $B[(y_n + z_n)/2, \delta(\|y_n - z_n\|)] \subset B$, whence

$$\|(y_n + z_n)/2 - x\| \geq d(x, B) + \delta(\|y_n - z_n\|).$$

On the other hand, the uniform convexity of $B[x, d(x,B) + \delta(\varepsilon)]$ entails existence of a non-decreasing convergent sequence (α_n) such that

$$\|(y_n + z_n)/2 - x\| \leq d(x, B) + \delta(\varepsilon) + \alpha_n.$$

Now, having in mind continuity of δ [Gu] we conclude $\delta(\varepsilon) \leq \delta(\varepsilon) - \alpha$ where $\alpha = \lim \alpha_n$, but the last inequality contradicts $\alpha > 0$.

Lemma 2. Suppose $B[x_1, r_1]$ and $B[x_2, r_2]$ are two balls (in a strictly convex space) such that $\max\{r_1, r_2\} \leq \|x_1 - x_2\| < r_1 + r_2$. Suppose π is a two-dimensional plane through x_1 and x_2 and denote

$$\eta = r_1 + r_2 - \|x_1 - x_2\|, \quad \{y_1, y_2\} = \pi \cap S[x_1, r_1] \cap S[x_2, r_2].$$

Then $\|y_1 - y_2\| > \eta$.

Proof: Denote

$$\{z\} = [x_1, x_2] \cap [y_1, y_2], \ \{u_1\} = [x_1, x_2] \cap S[x_2, r_2], \ \{u_2\} = [x_1, x_2] \cap S[x_1, r_1].$$

It is seen that $\|u_1 - u_2\| = \eta$ and $z \in [u_1, u_2]$. Suppose $\|u_1 - z\| \geq \|u_2 - z\|$ (the other case is treated analogously). Then $\|u_1 - z\| \geq \eta/2$. Let l be the line passing through x_2 which is parallel to $[y_1, y_2]$. Denote by w_1 and w_2 its intersection points with $S[x_2, r_2]$ such that with respect to the line through x_1 and x_2 the point w_1 is in a halfplane with y_1, and w_2 is in a halfplane with y_2. Denote also

$$\{v_1\} = [u_1, w_1] \cap [y_1, z], \quad \{v_2\} = [u_1, w_2] \cap [z, y_2].$$

It follows from the similarity of the triangles $u_1 z v_1$ and $u_1 x_2 w_1$ that $\|v_1 - z\| = \|u_1 - z\| \geq \eta/2$. Then $\|y_1 - z\| > \eta/2$. Analogously, $\|z - y_2\| > \|z - v_2\| \geq \eta/2$. Therefore $\|y_1 - y_2\| > \eta$.

The next lemma summarizes the previous two and its proof is omitted.

Lemma 3. Suppose $\varepsilon > 0, \tau > 0$ and $x \in X$ are given and $\|x\| > \varepsilon$. Then for $\eta = \varepsilon\delta(\tau/\varepsilon)$ there are points $y_1, y_2 \in S[x, d(x, \varepsilon B) + \eta] \cap \varepsilon S$ such that x, y_1, y_2 and the origin θ belong to a two-dimensional plane, $\eta < \|y_1 - y_2\|$ and $\operatorname{diam} B[x, d(x, \varepsilon B) + \eta] \cap \varepsilon B < \tau$.

Lemma 4. Let $\varepsilon \in (0,2)$ and $x, y \in S$ be such that $\|x-y\| \geq \varepsilon$. Then for any $t \in (2^{-1}\delta(\varepsilon/2), 1]$ the balls $B[tx, 1-t]$ and $B[y, \delta^2(\varepsilon/2)]$ have empty intersection.

Proof: Assume the contrary, i.e. for some $t \in (2^{-1}\delta(\varepsilon/2), 1)$ the balls have non-empty intersection. Denote $x_0 = tx$, $\{z\} = [x_0, y] \cap S[x_0, 1-t]$ and $r = \|y-z\|$. It follows from the strict convexity of X that $r > 0$, and by the assumption $r \leq \delta^2(\varepsilon/2)$. Besides $\varepsilon \leq \|x-y\| \leq \|x-z\| + r$ and $\|z-x\| \geq \varepsilon - r$. Therefore

$$\|(z-x_0)/(1-t) - x\| = \|z-x\|/(1-t) \geq (\varepsilon - r)/(1-t).$$

Since $\delta(\varepsilon) < \varepsilon$ for each $\varepsilon > 0$ then $r < \varepsilon^2/4 < \varepsilon/2$ whence $(\varepsilon - r)/(1-t) > \varepsilon/2$. The uniform convexity of B implies

$$2^{-1}\|(z-x_0)/(1-t) + x\| \leq 1 - \delta((\varepsilon - r)/(1-t)) \leq 1 - \delta(\varepsilon/2).$$

Consider two cases:

(I) $t \leq 1/2$. Estimate $\|z\|/t =$

$$\|x_0/t + (z-x_0)/t\| \leq \|x_0/t + (z-x_0)/(1-t)\| + \|(z-x_0)/t - (z-x_0)/(1-t)\| =$$

$$\|x + (z-x_0)/(1-t)\| + (1/t - 1/(1-t))\|z - x_0\| \leq$$

$$2(1-\delta(\varepsilon/2)) + (1-t)/t - 1 = 1 - 2\delta(\varepsilon/2) + (1-t)/t,$$

whence $\|z\| \leq 1 - 2\delta(\varepsilon/2)t$.

Now, since $\|y-z\| \geq \|y\| - \|z\| = 2\delta(\varepsilon/2)t$, we get a contradiction in the following way: $\delta^2(\varepsilon/2) < 2\delta(\varepsilon/2)t \leq \|y-z\| = r \leq \delta^2(\varepsilon/2)$.

(II) $1/2 < t < 1$. Estimate $\|z\|/(1-t) =$

$$\|x_0/(1-t) + (z-x_0)/(1-t)\| \leq \|x_0/(1-t) - x_0/t\| + \|x + (z-x_0)/(1-t)\| \leq$$

$$(1/(1-t) - 1/t)\|x_0\| + 2(1-\delta(\varepsilon/2)) = 1 - 2\delta(\varepsilon/2) + t/(1-t),$$

whence $\|z\| \leq 1 - 2\delta(\varepsilon/2)(1-t)$.

For $t < 1 - \delta(\varepsilon/2)/2$ a contradiction is obtained in a similar way as in case (I), while for $1 - t \leq \delta(\varepsilon/2)/2$ we have $\varepsilon \leq \|x-y\| \leq \|x-x_0\| + \|x_0 - y\| = 2(1-t) + r \leq \delta(\varepsilon/2) + \delta^2(\varepsilon/2) < \varepsilon$, a contradiction again.

Lemma 5. If $\delta(\cdot)$ is a reduced convexity modulus then for every $\varepsilon > 0$, every two elements x and y in S, $\|x-y\| \geq \varepsilon$, every $t \in (0, \delta(\varepsilon)]$ and $s \in [t, 1)$ it follows $B[sx, 1-s] \cap B[y, t^2] = \emptyset$.

Proof: It is sufficient to observe that for every $\lambda \in (0,1]$ the function $\lambda\delta'(\cdot)$, where $\delta'(\varepsilon) = \delta(2\varepsilon)$, is a convexity modulus and to apply lemma 4 for $t = \lambda\delta'(\varepsilon)$.

Lemma 6. Suppose $\delta(\cdot)$ is a reduced convexity modulus and $y_1, y_2 \in X$ are two elements such that $\|y_i\| \geq d > 0$, $i = 1,2$, $\|y_1 - y_2\| \geq c > 0$. Then for every $y \in [ty_1, y_1]$, where $t \in (0, \delta(c/2d)]$, it follows

(i) $\quad \|y - y_1\| + dt^2 < \|y - y_2\| \quad$ provided $\|y_1\| = d$,

(ii) $\quad \|y - y_1\| + dt^2/2 < \|y - y_2\| \quad$ provided $\|y_1\| < d(1 + t^2/2)$.

Proof: Denote $\varepsilon = dt^2/2$. Since $t \leq 1/2$ and $2\varepsilon \leq dt\delta(c/2d)$ then $\varepsilon < \min\{d/3, c/9\}$ and there is a number σ such that $3\varepsilon < \sigma < c/3$. Consider two cases:

(I) $\quad \|y_2\| \leq d + \sigma$.

Let $\bar{y}_2$ be the radial projection of y_2 on the sphere ρS, i.e. $\bar{y}_2 = \rho y_2/\|y_2\|$, where $\rho = \|y_1\|$. Having in mind that $\rho < d + \varepsilon$ we estimate

$$\rho^{-1}\|y_1 - \bar{y}_2\| \geq (c - \sigma)/(d + \varepsilon) > c/2d.$$

Then by lemma 5

$$B[ty_1, \rho(1 - t)] \cap B[\bar{y}_2, \rho t^2] = \emptyset.$$

If now $\|y_2\| \geq \|y_1\|$ then $\|y_2\| \geq \|\bar{y}_2\|$ and from the fact that the origin θ belongs to the ball $B[ty_1, \rho(1 - t)]$, this ball being a convex set does not intersect $B[y_2, \rho t^2]$. For arbitrary $y \in [ty_1, y_1]$ we have $B[y, \|y - y_1\|] \subset B[ty_1, \rho(1 - t)]$, whence $B[y, \|y - y_1\|] \cap B[y_2, dt^2] = \emptyset$.

If $\|y_2\| < \|y_1\|$ (part (ii) of the lemma), then $\|y_2 - \bar{y}_2\| < \varepsilon$ and for $y \in [ty_1, y_1]$ we have $B[y, \|y - y_1\|] \cap B[y_2, \varepsilon] = \emptyset$. Thus

$$\|y - y_1\| + dt^2/2 < \|y - y_2\| \text{ whenever } y \in [ty_1, y_1].$$

(II) $\quad \|y_2\| > d + \sigma$.

Since $y_1 \in B[d + \varepsilon]$ and $y_2 \notin B[d + \sigma]$ then for $y \in [ty_1, y_1]$ it follows $\|y - y_2\| - \|y - y_1\| > \sigma - \varepsilon > 2\varepsilon$. As is seen

$$\|y - y_1\| + dt^2 < \|y - y_2\| \text{ whenever } \|y_1\| = d \text{ and } y \in [ty_1, y_1].$$

Lemma 7. Let $\delta(\cdot)$ be a reduced convexity modulus, $\eta, \varepsilon, r \in \mathbb{R}$ and $x, y_1, y_2 \in X$ be given such that $0 < \eta < \varepsilon < r/2$, $\|x\| = r$, $y_1, y_2 \in S[x, d(x, \varepsilon B) + \eta] \cap \varepsilon S$ and x, y_1, y_2 and θ belong to a two-dimensional plane. Let for σ, $0 < \sigma < 4^{-1}(r - \varepsilon)\delta^2(\eta/r)$ and for $i = 1,2$ the sets M_i satisfy $\emptyset \neq M_i \subset B(y_i, \sigma)$. Then for every $\rho \in [\varepsilon, r - \sqrt{4\sigma r}]$ there is a point $x_\rho \in \rho S \cap \mathrm{co}\{y_1, x, y_2\}$ such that $d(x_\rho, M_1) = d(x_\rho, M_2)$.

Proof: Denote $d = r - \varepsilon + \eta$. It follows from lemma 2 that $\|y_1 - y_2\| > \eta$ and then by lemma 5

$$B[s(y_j - x)/d, (1 - s)] \cap B[(y_k - x)/d, t^2] = \emptyset$$

whenever $t \in (0, \delta(\eta/d)]$, $s \in [t,1)$ and $j,k \in \{1,2\}$, $j \neq k$. Therefore

$$B[sy_j + (1-s)x, d(1-s)] \cap B[y_k, dt^2] = \emptyset.$$

Let $t_0 = \sqrt{4\sigma/r}$ and $s \geq t_0$. Denote $x_i(s) = sy_i + (1-s)x$ for $i = 1,2$. Since $t_0 < \delta(\eta/d)$ and $d > r/2$ then the previous formula implies

$$B[x_j(s), d(1-s)] \cap B[y_k, 2\sigma] = \emptyset,$$

whence $d(x_j(s), M_j) < d(x_j(s), M_k)$ for $s \geq t_0$ and $j,k = 1,2$. The estimations $\|x_i(t_0) - x\| = t_0 d < \sqrt{4\sigma r}$ for $i = 1,2$ imply $\|x_i(t_0)\| > r - \sqrt{4\sigma r}$ and then for $\rho \in [\varepsilon, r - \sqrt{4\sigma r}]$ we have $d(x_j(\rho), M_j) < d(x_k(\rho), M_k)$, where $x_i(\rho) = [x, y_i] \cap \rho S$. Now, the intermediate value theorem guarantees existence of a point x_ρ on the arc of ρS via $x_1(\rho)$ and $x_2(\rho)$ such that $d(x_\rho, M_1) = d(x_\rho, M_2)$.

Lemma 8. Suppose $\delta(\cdot)$ is a reduced convexity modulus and $\varepsilon > 0, \tau > 0$, $x' \in X$, $\|x'\| > 2\varepsilon$ and $y_1, y_2 \in X$, $y_1 \neq y_2$ are given such that $y_1, y_2 \in S[x', d(x', \varepsilon B) + \eta] \cap \varepsilon S$ for $\eta = \varepsilon\delta(\delta(\tau/4\varepsilon))$. Suppose also $0 < \sigma \leq \eta/4$ and M_i are two non-empty sets, $M_i \subset B(y_i, \sigma)$, $i = 1,2$. Denote $r_1 = \|x'\| - (\varepsilon + \sigma)(1 - \delta(\tau/2(\varepsilon+\sigma))) - \eta/2$ and $r_2 = r_1 + \eta/2$. Then for every x from the relative interior of $\mathrm{co}\{y_1, x', y_2\}$ such that $d(x, M_1) = d(x, M_2)$ it follows

(i) $$B[x, d(x, M_1)] \subset B[x', r_1],$$

(ii) $$\mathrm{diam}(\varepsilon + \sigma)B \cap B[x', r_2] < \tau/2$$

(iii) $$B(y_i, \sigma) \subset (\varepsilon + \sigma)B \cap B[x', r_1] \ \text{ for } i=1,2.$$

Proof: Lemma 1 implies $\mathrm{diam}\, B[x', d(x', \varepsilon B) + \eta] \cap \varepsilon B < \varepsilon\delta(\tau/4\varepsilon)$. Suppose $x \in \mathrm{reintco}\{y_1, x', y_2\}$ and $d(x, M_1) = d(x, M_2)$. The ray $\{x' + t(x - x') : t > 1\}$ meets (y_1, y_2) at a point z. Let $\|z - y_1\| = \min\{\|z - y_i\| : i = 1,2\}$ $\leq 2^{-1}\varepsilon\delta(\tau/4\varepsilon)$. Then

$$\|x - y_1\| \leq \|x - z\| + \|z - y_1\| \leq \|x' - z\| - \|x' - x\| + 2^{-1}\varepsilon\delta(\tau/4\varepsilon) <$$
$$d(x', \varepsilon B) + \eta + 2^{-1}\varepsilon\delta(\tau/4\varepsilon) - \|x' - x\|.$$

Since $d(x, M_1) \leq \|x - y_1\| + \sigma$ and obviously $\sigma < \varepsilon$ then

$$d(x', M_1) \leq d(x, M_1) + \|x' - x\| < d(x', \varepsilon B) + \eta + 2^{-1}\varepsilon\delta(\tau/4\varepsilon) + \sigma =$$
$$d(x', (\varepsilon + \sigma)B) + \eta + 2\sigma + 2^{-1}\varepsilon\delta(\tau/4\varepsilon) \leq$$
$$d(x', (\varepsilon + \sigma)B) + 2\eta + 2^{-1}\varepsilon\delta(\tau/4\varepsilon) - \eta/2 \leq$$
$$d(x', (\varepsilon + \sigma)B) + \varepsilon\delta(\tau/4\varepsilon) - \eta/2 <$$
$$d(x, (\varepsilon + \sigma)B) + (\varepsilon + \sigma)\delta(\tau/2(\varepsilon + \sigma)) - \eta/2 = r_1,$$

which implies the inclusion (i).

It follows from lemma 1 that (ii) is true. In order to prove (iii) take a point $x \in \text{reintco}\{y_1, x', y_2\}$ such that $\|x - y_1\| = \|x - y_2\|$ and for any $y \in B(y_1, \sigma)$ assign $M_1 = \{y\}$ and let M_2 be an arbitrary non-empty closed subset of $S[x, \|x - y\|] \cap B(y_2, \sigma)$. The preceding arguments show that $y \in B[x', r_1]$ and more generally $B(y_1, \sigma) \subset B[x', r_1] \cap (\varepsilon + \sigma)B$. Similarly, it is shown that $B(y_2, \sigma) \subset B[x', r_1] \cap (\varepsilon + \sigma)B$.

At the end of the section a lemma from [St] is recalled.

Lemma 9 [St]. Let $y_0 \in P(x_0, M)$, where $x_0, y_0 \in X$ and $\emptyset \neq M \subset X$. Then for every $x \in (x_0, y_0]$ the metric projection $P(\cdot, M)$ is single-valued and u.s.c. at x.

4. Equidistant Approximations

The problem of generic equidistant approximation in a uniformly convex Banach space X is considered in this section. Suppose two non-empty closed subsets M_1 and M_2 are given and they are separated:

$$\inf\{\|y_1 - y_2\| : y_1 \in M_1, y_2 \in M_2\} > 0$$

The equidistant hypersurface is denoted by

$$\Sigma(M_1, M_2) = \{x \in X : d(x, M_1) = d(x, M_2)\}.$$

Obviously, $\Sigma(M_1, M_2)$ is a closed set and might be viewed as a complete metric space with respect to the induced from X topology. It will be seen meanwhile, by the proof of the next result, that $\Sigma(M_1, M_2)$ has empty interior. What we are concerned is the existence of points on $\Sigma(M_1, M_2)$ which have best approximations both in M_1 and M_2. It follows from Stechkin's theorem that there is a dense G_δ set in X of points with best approximations in M_1 and M_2, but it is not apparent that this set intersects $\Sigma(M_1, M_2)$. The next result gives an affirmative answer.

Theorem 4.1 Let M_1 and M_2 be two closed and separated sets in the uniformly convex Banach space X. Then there is a dense G_δ subset Γ of $\Sigma(M_1, M_2)$ such that the metric projections $P(\cdot, M_i)$ for $i = 1, 2$ are single-valued and upper semicontinuous in X at all points of Γ.

Proof: Denote by Γ_i the sets of continuity of $P(\cdot, M_i)$, i.e. where $P(\cdot, M_i)$ are single-valued and u.s.c., $i = 1, 2$. It is known that Γ_i are G_δ sets. Hence $\Gamma_i \cap \Sigma(M_1, M_2)$ are G_δ sets since $\Sigma(M_1, M_2)$ is a G_δ set being closed. What remains to be proved is that for each $i = 1, 2$ the set Γ_i intersects the hypersurface $\Sigma(M_1, M_2)$ in a proper dense subset. Then the Baire category theorem would imply that $\Gamma = \Gamma_1 \cap \Gamma_2 \cap \Sigma(M_1, M_2)$ is a dense and G_δ set in the induced space.

Suppose x_0 is an arbitrary point from $\Sigma(M_1, M_2)$ and σ is an arbitrary positive number. It has to be shown that Γ_1 intersects $\Sigma(M_1, M_2)$ at a point which is at less than σ distance from x_0. For the sake of convenience we might assume that x_0 coincides with the origin θ. Denote

$$c = \inf\{\|y_1 - y_2\| : y_1 \in M_1, y_2 \in M_2\} > 0,$$

$$d(\theta, M_1) = d = d(\theta, M_2).$$

Obviously $c \le 2d$. Assume additionally $\sigma < c/2$, whence $\sigma < d$. Let $\delta(\cdot)$ be a reduced convexity modulus. Choose a positive number t such that

$$t \le \delta\left(\frac{c}{2d}\right) \quad \text{and} \quad d(t^2/4 + 2t) < \sigma.$$

Put

$$\varepsilon = dt^2/3, \quad r = d + \varepsilon, \quad T_1 = rB \cap M_1,$$

$$U = \bigcup_{y \in T_1} \operatorname{co}(B(\varepsilon/2) \cup \{y\}), \quad D = U \cap B(\sigma).$$

Since $t < 1$ then obviously $\varepsilon < d$ and $\sigma > rt + \varepsilon/2$. The next aim is to prove

$$\Sigma(M_1, M_2) \cap (U \backslash D) = \emptyset. \tag{2}$$

Apply lemma 6 (ii) for every $y_1 \in T_1$, and $y_2 \in M_2$:

$$\|y - y_1\| + dt^2/2 < \|y - y_2\|, \text{ whenever } y \in [ty_1, y_1],$$

whence

$$d(y, M_1) + \varepsilon \le d(y, M_2) \text{ for } y \in [ty_1, y_1] \text{ where } y_1 \in T_1. \tag{3}$$

Suppose $z \in U\backslash D$. There are $y_1 \in T_1$ and $y \in [\theta, y_1]$ such that $y \in B(z, \varepsilon/2)$. In fact $y \in [ty_1, y_1]$ since $B(z, \varepsilon/2) \cap B(rt) = \emptyset$. Therefore

$$d(z, M_1) + \varepsilon/2 \le d(z, M_2) \text{ whenever } z \in U\backslash D,$$

and (2) is satisfied.

It should be mentioned that (3) implies $\operatorname{int}\Sigma(M_1, M_2) = \emptyset$. Indeed, we showed $B(x_0, \sigma) \not\subset \Sigma(M_1, M_2)$ since $\sigma > rt$, but x_0 is an arbitrary point and σ might be arbitrarily small. Thus no open ball is contained in the equidistant hypersurface. Since, in the reasonings, the places of M_1 and M_2 can be changed we conclude that any neighborhood of $x_0 \in \Sigma(M_1, M_2)$

contains points which are closer to M_1 and points which are closer to M_2. This argument will be employed.

There is a point $x_1 \in B(\varepsilon/2)$ such that $x_1 \in \Gamma_1$ and $d(x_1, M_2) < d(x_1, M_1)$. Let $\{y_1\} = P(x_1, M_1)$. According to lemma 9 the entire segment (x_1, y_1) is contained in Γ_1. On the other hand (x_1, y_1) intersects $\Sigma(M_1, M_2)$ at some point x. It is easily checked, by means of the triangle inequality, that $y_1 \in T_1$. Therefore $x \in [x_1, y_1] \subset U$ and (2) implies $x \in B(\sigma)$.

It is proved that $\Gamma_1 \cap \Sigma(M_1, M_2)$ is dense in the hypersurface, what is required. Q.E.D.

Corollary 4.2 Let $M_i \subset X$, i =1,2 be as in theorem 4.1. Then every open set U which has non-empty intersection with the equidistant hypersurface $\Sigma(M_1, M_2)$ contains at least continuum many points at which the metric projection $P(\cdot, M_1 \cup M_2)$ is two-valued and u.s.c.

Proof: According to a classical theorem of Alexandroff and Uryson every non-empty compact set without isolated points has a power greater or equal to the continuum [AU]. Their argument works quite analogously in this case too, since by theorem 4.1 the set of points at which $P(\cdot, M_1 \cup M_2)$ is two-valued and u.s.c. does not have isolated points.

5. Main Result

Theorem 5.1 For every uniformly convex Banach space X, $\dim X \geq 2$, a residual subset $\mathcal{U}$ of the Hausdorff metric space $\mathcal{B}(X)$ exists such that for every $M \in \mathcal{U}$ the metric projection $P(\cdot, M)$ is two-valued and upper semicontinuous on a dense and everywhere continual subset of X.

Proof: Define for any $M \in \mathcal{B}(X)$ and $n \geq 2$

$$T_M = \{x \in X : P(x, M) \text{ is two-valued and u.s.c. at } x\},$$

$$\mathcal{U}_n = \{M \in \mathcal{B}(X) : \forall x \in X \quad B(x, n^{-1}) \subset B(M, n) \Longrightarrow \operatorname{card} T_M \cap B(x, n^{-1}) \geq \mathbf{c}\}$$

and

$$\mathcal{U} = \bigcap_{n=2}^{\infty} \mathcal{U}_n.$$

It will be established that all $\mathcal{U}_n$ contain open and dense in $\mathcal{B}(X)$ sets, which will imply that $\mathcal{U}$ contains a dense G_δ subset of $\mathcal{B}(X)$. Obviously, for any $M \in \mathcal{U}$ the set T_M is everywhere continual in X.

Fix a reduced convexity modulus $\delta : [0,2] \longrightarrow [0,2^{-1}]$, then $\delta(s) \leq s/4$ whenever $s \geq 0$. Let $M_0 \in \mathcal{B}(X)$, $n \geq 2$ be an integer and $\varepsilon \in (0, n^{-1})$ be given. We have to show that there exist $N \in \mathcal{B}(X)$ and $\sigma > 0$ such that

$$\mathcal{O}(N,\sigma) \subset \mathcal{O}(M_0,\varepsilon) \cap \mathcal{U}_n.$$

Assign

$$\tau = \frac{1}{2n}\left(\frac{\varepsilon}{16}\right)^2 \delta^2\left(\frac{\varepsilon}{4n}\right), \ \eta = \frac{\varepsilon}{16}\delta\left(\delta\left(\frac{4\tau}{\varepsilon}\right)\right), \ \sigma = \frac{n-1}{4}\delta^2\left(\frac{\eta}{n}\right)$$

The following comparative inequalities are a simple consequence of the property of δ: $\tau \leq 2^{-17}n^{-3}\varepsilon^4$, $\eta \leq 2^{-6}\tau$, $\sigma \leq 2^{-6}n^{-1}\eta^2$.

Let N_0 be a $\varepsilon/2$-net for M_0 and N_1 be a τ-net for the sphere $\varepsilon' S$, where $\varepsilon' = \varepsilon/16$. Define the maps $f_1, f_2 : N_1 \longrightarrow \varepsilon' S$ such that for $e \in N_1$ the images $f_j(e)$ are chosen from $S[ne/\varepsilon', n - \varepsilon' + \eta] \cap \varepsilon' S$ according to lemma 3 and the points $\theta, f_1(e), ne/\varepsilon'$ and $f_2(e)$ belong to a two-dimensional plane and

$$\eta < \|f_1(e) - f_2(e)\|, \ \operatorname{diam} B[ne/\varepsilon', n - \varepsilon' + \eta] \cap \varepsilon' B < \varepsilon' \delta(\tau/4\varepsilon'). \qquad (4)$$

Let

$$N = \{z + f_j(e) : z \in N_0, e \in N_1, j = 1,2\}.$$

For every $M \in \mathcal{O}(N,\sigma)$ and all $v \in N$ the sets $M(v) = M \cap B(v,\sigma)$ are closed and non-empty. The claim is proved by showing that the balls $B(v,\sigma)$ are uniformly separated one from another. Suppose $w_i \in B(v_i,\sigma)$, where $v_i = z_i + f_{j_i}(e_i)$, $z_i \in N_0$, $e_i \in N_1$, $j_i \in \{1,2\}$ for i=1,2 and distinguish between the following cases:

(I) $z_1 \neq z_2$. Since $\|w_i - v_i\| < \sigma$ and $\|v_i - z_i\| = \varepsilon/16$ then

$$\varepsilon/2 \leq \|z_1 - z_2\| \leq \varepsilon/8 + 2\sigma + \|w_1 - w_2\|,$$

whence $\|w_1 - w_2\| > 3\varepsilon/8 - 2\sigma$.

(II) $z_1 = z_2$ and $e_1 \neq e_2$. Having in mind (4) we estimate

$$\tau \leq \|e_1 - e_2\| < 2\varepsilon'\delta(\tau/4\varepsilon') + \|f_{j_1}(e_1) - f_{j_2}(e_2)\| \leq \tau/8 + \|v_1 - v_2\|.$$

Then

$$7\tau/8 < \|v_1 - v_2\| \leq 2\sigma + \|w_1 - w_2\|,$$

whence $\|w_1 - w_2\| > 7\tau/8 - 2\sigma$.

(III) $z_1 = z_2$ and $e_1 = e_2$. Since

$$\eta < \|f_1(e_1) - f_2(e_1)\| = \|v_1 - v_2\| \leq 2\sigma + \|w_1 - w_2\|$$

then $\|w_1 - w_2\| > \eta - 2\sigma$.

It is a routine calculation to verify that $\eta = \min\{\eta, 7\tau/8, 3\varepsilon/8\}$ and $\eta - 2\sigma > 0$. Hence $M(v)$ are closed and non-empty sets for all $v \in N$.

Suppose $M \in \mathcal{O}(N,\sigma)$ is a fixed set and x_0 satisfies $B(x_0, n^{-1}) \subset B(M,n)$. We are going to find consecutively points x_1, x_2, x_3, as indicated in the sketch below, all in the ball $B(x_0, n^{-1})$ such that the metric projection $P(\cdot, M)$ is two-valued and u.s.c. at continuum many points from a neighborhood of the last point x_3.

If $d(x_0, N_0) \leq 3\varepsilon'$, then there is x_1 such that $\|x_0 - x_1\| < 1/4n$ and $d(x_1, N_0) > 3\varepsilon'$. Indeed, let $\{z_1\} = P(x_0, N_0)$ and take arbitrary x_1 from the set
$\{z_1 + s(x_0 - z_1) : s > 1\} \cap B(z_1, 4\varepsilon') \backslash B[z_1, 3\varepsilon']$.

In the other case: $d(x_0, N_0) > 3\varepsilon'$, we make use of Stechkin's theorem to ensure existence of a point x_1 such that for some $z_1 \in N_0$

$$P(x_1, N_0) = \{z_1\}, \quad d(x_1, N_0 + \varepsilon' B) > 2\varepsilon',$$

$$\|x_0 - x_1\| < 1/4n. \tag{5}$$

It is not difficult to observe that

$$3\varepsilon' < \|x_1 - z_1\| < n. \tag{6}$$

To prove the right-hand side inequality make use of (1)

$$\|x_1 - z_1\| = d(x_1, N_0) \leq d(x_0, M) + \|x_0 - x_1\| + H(M, N_0) <$$

$$n - 3/4n + \varepsilon' + \sigma < n,$$

since $d(x_0, M) \leq n - n^{-1}$.

Find now a point $x_2 \in [x_1, z_1]$ such that $x_2 = (1-t)x_1 + tz_1$ where $t = \delta(\varepsilon/4n)$ and for arbitrary $z \in N_0 \backslash \{z_1\}$ make use of lemma 6 (i) with respect to $z_1 - x_1$, $z - x_1$ and $d = \|x_1 - z_1\|$. Then $\|x_2 - z_1\| + dt^2 < \|x_2 - z\|$, whence

$$\|x_2 - z_1\| + 3\varepsilon' t^2 < \|x_2 - z\| \text{ whenever } z \in N_0 \backslash \{z_1\}. \tag{7}$$

and

$$\|x_1 - x_2\| = t\|x_1 - z_1\| < n\delta(\varepsilon/4n) \leq \varepsilon' < 1/16n. \tag{8}$$

Let $[x_2, z_1] \cap S[z_1, \varepsilon'] = \{u\}$. Since N_1 is a τ-net in $\varepsilon' S$ there is $e_1 \in N_1$ such that $\|z_1 + e_1 - u\| < \tau$. Denote

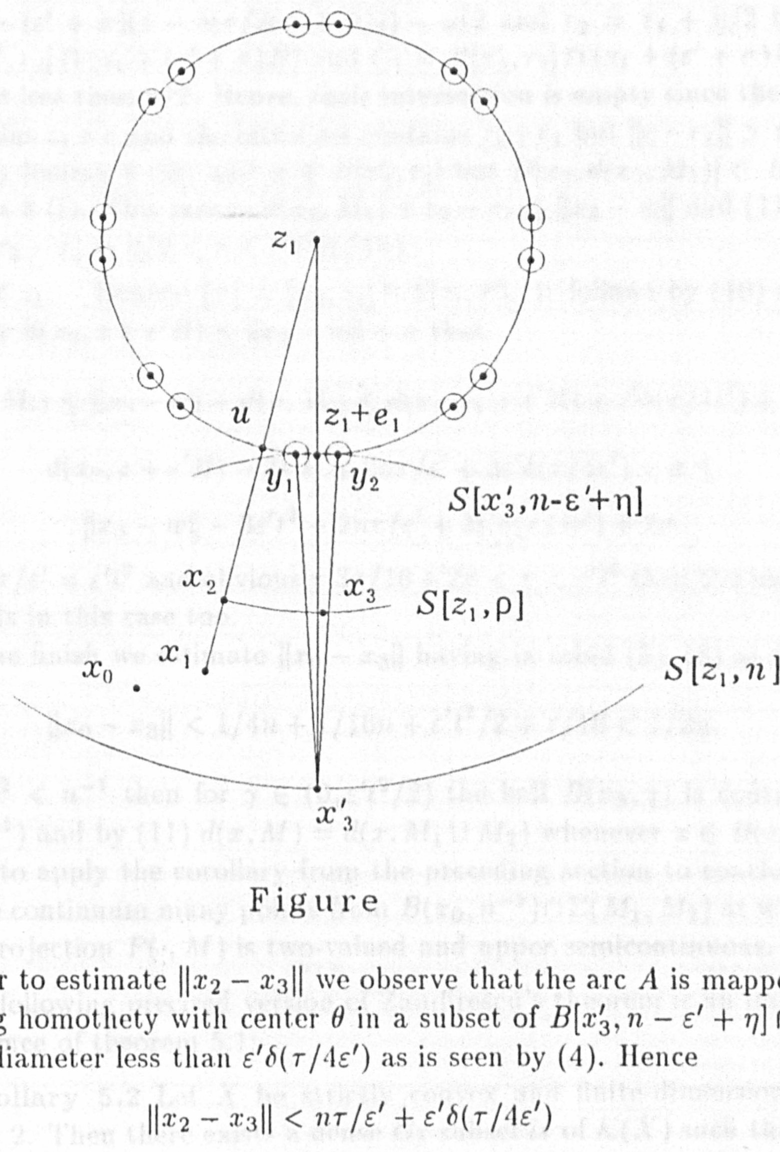
$$y_i = z_1 + f_i(e_1), \quad M_i = M \cap B(y_i, \sigma), \quad i = 1, 2$$

and $x_3' = z_1 + ne_1/\varepsilon'$. It is possible to apply lemma 7 for x_3' and the sets M_i, because $\sigma < 4^{-1}(n - \varepsilon')\delta^2(\eta/n)$. Then for arbitrary $\rho \in [\varepsilon', n - \sqrt{4\sigma n}]$

there exists $x_\rho \in S[z_1, \rho] \cap \operatorname{co}\{y_1, x_3', y_2\}$ such that $d(x_\rho, M_1) = d(x_\rho, M_2)$. However, (6) and (8) imply

$$2\varepsilon' < \|x_2 - z_1\| = (1-t)\|x_1 - z_1\| < n - n\delta(\varepsilon/4n).$$

On the other hand $\sqrt{4\sigma n} < n\delta(\varepsilon/4n)$, since obviously $\eta < \varepsilon/4$ and $\delta(\eta/n) \leq \delta(\varepsilon/4n)$. Thus $\|x_2 - z_1\| \in [\varepsilon', n - \sqrt{4\sigma n}]$ and then there is a point which we denote by x_3 such that $x_3 \in A = S[z_1, \|x_2 - z_1\|] \cap \operatorname{co}\{y_1, x_3', y_2\}$ and $d(x_3, M_1) = d(x_3, M_2)$; see the sketch.



Figure

In order to estimate $\|x_2 - x_3\|$ we observe that the arc A is mapped by a retracting homothety with center θ in a subset of $B[x_3', n - \varepsilon' + \eta] \cap \varepsilon' B$ and has a diameter less than $\varepsilon'\delta(\tau/4\varepsilon')$ as is seen by (4). Hence

$$\|x_2 - x_3\| < n\tau/\varepsilon' + \varepsilon'\delta(\tau/4\varepsilon') \tag{9}$$

and by (7) and (9)

$$d(x_3, z_1 + \varepsilon' B) < d(x_3, z + \varepsilon' B) - 3\varepsilon' t^2 + 2n\tau/\varepsilon' + 2\varepsilon'\delta(\tau/4\varepsilon'), \tag{10}$$

where $t = \delta(\varepsilon/4n)$.

Our next aim is to prove that $d(x_3, M_i) = d(x_3, M)$ and more precisely:

$$d(x_3, M_i) + \varepsilon' t^2 \leq d(x_3, M \setminus \bigcup_{i=1,2} M_i). \tag{11}$$

Take arbitrary $w \in M \setminus \bigcup_{i=1,2} B(y_i, \sigma)$. There are $z \in N_0, e \in N_1$ and $y \in N$ such that $w \in B(y, \sigma)$ and $y = z + f_j(e)$ for $j \in \{1, 2\}$. Consider two cases:

(I) $z = z_1$. Denote $x' = z_1 + ne/\varepsilon'$. According to lemma 8 (ii) for $r_1 = n - (\varepsilon' + \sigma)(1 - \delta(\tau/2(\varepsilon' + \sigma))) - \eta/2$ and $r_2 = r_1 + \eta/2$ the sets $C = B[x', r_2] \cap (z_1 + (\varepsilon' + \sigma)B)$ and $C_1 = B[x'_3, r_2] \cap (z_1 + (\varepsilon' + \sigma)B)$ have diameters less than $\tau/2$. Hence, their intersection is empty since the former set contains $z_1 + e$ and the latter set contains $z_1 + e_1$ but $\|e - e_1\| > \tau$. Thus $w \in C$ by lemma 8 (iii) and $w \notin B[x'_3, r_2]$ but $B[x_3, d(x_3, M_1)] \subset B[x'_3, r_1]$ by lemma 8 (i). This means $d(x_3, M_1) + r_2 - r_1 \leq \|x_3 - w\|$ and (11) holds because $r_2 - r_1 = \eta/2 < \tau < \varepsilon'\delta^2(\varepsilon/4n)$.

(II) $z \neq z_1$. Denote $\{v\} = [x_3, z_1] \cap S[z_1, \varepsilon']$. It follows by (10) and the inequality $d(x_3, z + \varepsilon' B) \leq \|x_3 - w\| + \sigma$ that

$$d(x_3, M_i) \leq \|x_3 - v\| + d(v, M_i) \leq d(x_3, z_1 + \varepsilon' B) + \varepsilon'\delta(\tau/4\varepsilon') + \sigma <$$
$$d(x_3, z + \varepsilon' B) - 3\varepsilon' t^2 + 2n\tau/\varepsilon' + 3\varepsilon'\delta(\tau/4\varepsilon') + \sigma \leq$$
$$\|x_3 - w\| - 3\varepsilon' t^2 + 2n\tau/\varepsilon' + 3\varepsilon'\delta(\tau/4\varepsilon') + 2\sigma.$$

Since $2n\tau/\varepsilon' = \varepsilon' t^2$ and obviously $3\tau/16 + 2\sigma < \tau < \varepsilon' t^2$ then the inequality (11) holds in this case too.

At the finish we estimate $\|x_0 - x_3\|$ having in mind (5), (8) and (9):

$$\|x_0 - x_3\| < 1/4n + 1/16n + \varepsilon' t^2/2 + \tau/16 < 1/2n.$$

Since $\varepsilon' t^2 < n^{-1}$ then for $\gamma \in (0, \varepsilon' t^2/2)$ the ball $B(x_3, \gamma)$ is contained in $B(x_0, n^{-1})$ and by (11) $d(x, M) = d(x, M_1 \cup M_2)$ whenever $x \in B(x_3, \gamma)$. It remains to apply the corollary from the preceding section to conclude that there are continuum many points from $B(x_0, n^{-1}) \cap \Sigma(M_1, M_2)$ at which the metric projection $P(\cdot, M)$ is two-valued and upper semicontinuous. Q.E.D.

The following precised version of Zamfirescu's theorem is an immediate consequence of theorem 5.1:

Corollary 5.2 Let X be strictly convex and finite-dimensional with $\dim X \geq 2$. Then there exists a dense G_δ subset $\mathcal{U}$ of $\mathcal{K}(X)$ such that every compact from $\mathcal{U}$ generates two-valued on an everywhere continual subset of X metric projection.

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Institute of Mathematics, Bulgarian Academy of Sciences
Sofia 1113, Bulgaria
e-mail address: nickor@bgearn.bitnet