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Conditions for Local Integrability of Systems of m Smooth Complex Vector Fields on m+1 Dimensional Real Cartesian Space

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CONDITIONS FOR LOCAL INTEGRABILITY OF SYSTEMS OF m SMOOTH COMPLEX VECTOR FIELDS ON $m+1$ DIMENSIONAL REAL CARTESIAN SPACE

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Abstract

Conditions for local integrability of a system of m smooth complex vector fields on $\mathbf{R}^{m+1}$ *are discussed. One sufficient condition is the real analiticity of the coefficients of the vector fields on the "last" variable. This follows from a theorem of A.Andreotti and C.D.Hill. A direct proof is provided. An example shows that this condition is not a necessary one. A partially converse of the statement is proved. Namely, each locally integrable system of m smooth complex vector fields on* $\mathbf{R}^{m+1}$ *is proved to have coefficients which are boundary values of functions, holomorphic on the "last" complexified variable. Some results for the used in the proofs abstract CR structures of hypersurface type are also given.*

1. PRELIMINARIES.

In this paper we shall consider m smooth complex-valued linearly independent vector fields

$$L_1, L_2, \ldots, L_m, \tag{1}$$

defined on a domain U in the $m+1$-dimensional real Cartesian space $\mathbf{R}^{m+1}$. As we shall consider local properties, we may assume that the considered vector fields are defined on a neighborhood U of the origin 0 in $\mathbf{R}^{m+1}$.

Such systems are investigated for example in [CH], [Dj], [MT] (see also the given there references), but in general for their solvability. Here we propose some more about their local integrability.

The system (1) is called *formally integrable* if the commutator of each cuple of vector fields $[L_j, L_k]$ is a linear combination of the whole vector fields in the system on any open subset of the domain they are defined.

The system (1) is called *locally integrable (in the origin)*, if there exists a neighborhood U of the origin 0 and a smooth function Z on U with a non-zero differential dZ at 0 satisfying the system of equations

$$L_1 u = 0, L_2 u = 0, \ldots, L_m u = 0. \tag{2}$$

The vector fields of the system (1) span a vector subbundle V of the complex tangent bundle $\mathbf{C}TU$ with a fiber dimension equal to m. Note that the formal integrability as well as the local integrability of the system (1) are properties common for each system of vector fields, forming a base for the space of the sections of the vector bundle V. The first one means that the commutator of any two sections of the vector bundle V over an open set where they are defined is a section of V, i.e. the sections of the bundle V forms a Lie algebra. The second one means that the annihilating for the bundle V one-dimensional vector subbundle T of the complex cotangent bundle $\mathbf{C}T^*U$ admits as local generator exact one-form.

It is eazy to see that the bundle V corresponding to the system (1) have a base of the kind

$$L_j = \partial/\partial t^j + a^j(t,x)\partial/\partial x \text{ for } j = 1, 2, \ldots, m, \tag{3}$$

where $a^j(t,x)$ are smooth complex-valued functions on the neighborhood U of the origin 0 in $\mathbf{R}^{m+1}$ with coordinates $t^1, \ldots, t^m, x$. When the system is locally integrable the functions a^j could be choosen so that $a^j(0) = 0$.

It can be directly checked that the system (1) is formally integrable iff it is fulfilled

$$L_j a^k = L_k a^j \text{ for } j, k = 1, \ldots, m. \tag{4}$$

2. ABSTRACT CR STRUCTURES OF HYPERSURFACE TYPE.

The systems of smooth complex vector fields under consideration are closely related with one important notion in the complex analysis of several variables - the abstract CR structures of hypersurface type.

Let $p \in \mathbf{R}^{2m+1}$ with $m \geq 1$. Let $L_1, L_2, \ldots, L_m$ be smooth complex vector fields defined near p. The system L_j, $j = 1, 2, \ldots, m$ is called *an abstract CR structure of hypersurface type near p* if it is fulfilled

$$L_1, L_2, \ldots, L_m, \bar{L}_1, \bar{L}_2, \ldots, \bar{L}_m \text{ are linearly independent at } p \text{ and} \tag{5}$$

$$L_1, L_2, \ldots, L_m \text{ is formally integrable.} \qquad (6)$$

A function Z annihilated by the vector fields of an abstract CR structure of hypersurface type is called *a CR function*.

The abstract CR structure is called *realizable*, if it is locally integrable, i.e. there exist smooth functions $Z^1, Z^2, \ldots, Z^{m+1}$ such that they are CR functions near p and $dZ^1, dZ^2, \ldots, dZ^{m+1}$ are linearly independent.

Lemma 1. Let is given a system of the kind (3), satisfying the conditions (4). Then the system of smooth complex-valued vector fields

$$L'_j = 2\partial/\partial \bar{z}^j + a^j(t,x)\partial/\partial x \text{ for } j = 1, \ldots, m \qquad (7)$$

forms an abstract CR structure of hypersurface type near the origin.

Here are used the complex variables $z^j = t^j + is^j, s = (s^1, \ldots, s^m) \in \mathbf{R}^m$ and the complex derivatives $\partial/\partial \bar{z}^j = 1/2(\partial/\partial t^j + i\partial/\partial s^j)$. Let us recall also that $\partial/\partial z^j = 1/2(\partial/\partial t^j - i\partial/\partial s^j)$.

Proof. Indeed, the vector fields $L'_1, L'_2, \ldots, L'_m, \bar{L}'_1, \bar{L}'_2, \ldots, \bar{L}'_m$ are linearly independent near the origin because such are the complex derivatives $\partial/\partial z^1, \partial/\partial z^2, \ldots, \partial/\partial z^m, \partial/\partial \bar{z}^1, \partial/\partial \bar{z}^2, \ldots, \partial/\partial \bar{z}^m$ and the vector field $\partial/\partial x$. The formal integrability of the system follows by the equalities (4). Indeed, from (4) and the definition of the vector fields L'_j $j = 1, 2, \ldots, m$ follows

$$L'_j a^k = L'_k a^j \text{ for } j, k = 1, 2, \ldots, m \qquad (8)$$

which enshure the formal integrability of the system (7). The lemma is proved.

Lemma 2. The system (3) is locally integrable if and only if the abstract CR structure of hypersurface type (7) is realizable.

Proof. Let the system (3) be locally integrable and the function $Z(t,x)$ be a solution of the system of equations (2) on a neighborhood of the origin in $\mathbf{R}^{m+1}$ with non-zero differential at the origin. Then the functions $Z^1 = t^1 + is^1, Z^2 = t^2 + is^2, \ldots, Z = t^m + is^m, Z^{m+1} = Z(t,x)$ will annihilate the vector fields (7) on a neighborhood of the origin in $\mathbf{R}^{2m+1}$ as it can be checked.

Also it can be verified directly that their differentials are linearly independent on some neighborhood of this origin. This prove the realizability of the system (7) in this case.

Let now the abstract CR structure of hypersurface type is realizable and $Z^1(t,s,x), Z^2(t,s,x), \ldots, Z^{m+1}(t,s,x)$ are CR functions with linearly independent differentials near the origin.
We will find a function $Z(t,x)$ with non-zero differential at the origin which annihilate the vector fields (3) of the kind

$$Z = F(Z^1, Z^2, \ldots, Z^{m+1})$$

where F is a holomorphic function of $m+1$ variables. As we ask Z to satisfy (2), we consider the system of equations

$$\partial F(Z^1, Z^2, \ldots, Z^{m+1})/\partial s^k =$$

$$= \sum_{j=1}^{m+1} \partial Z^j/\partial s^k \partial F/\partial Z^j = 0 \text{ for } k = 1, 2, \ldots, m. \qquad (9)$$

The coefficients $\partial Z^j/\partial s^k$ are CR functions, as the differential operators $\partial/\partial s^k$ commute with the complex vector fields in (7). Hence they can be represented as a composition of holomorphic functions of $m+1$ variables with the given CR functions $Z^1, Z^2, \ldots, Z^{m+1}$, i.e. there exist holomorphic functions G_{jk} such that

$$\partial Z^j/\partial s^k = G_{jk}(Z^1, Z^2, \ldots, Z^{m+1}).$$

The system (9) becomes

$$\sum_{j=1}^{m+1} G_{jk}(Z^1, Z^2, \ldots, Z^{m+1}) \partial F/\partial Z^j = 0, k = 1, 2, \ldots, m.$$

This is a system of m first order partial differential equations with holomorphic coefficients for a holomorphic function F of $m+1$ variables. Then we can apply Cauchy - Kowalewskaya theorem. It follows that there exists a holomorphic solution F_o of the system (9) with non-zero differential.

The asked for the local integrability function $Z(t,x)$ may be choosen to be the function $F_o(Z^1, Z^2, \ldots, Z^{m+1})$. Indeed, this function doesn't depend on the variables $s^1, s^2, \ldots, s^{m+1}$ and annihilate the vector fields (3). Moreover, its differential is non-zero at the origin, as this of F_o is so. This prove the local integrability of the system (3).

3. SUFFICIENT CONDITION FOR LOCAL INTEGRABILITY.

A.Andreotti and C.D.Hill have proved in [AH] a theorem for local integrability of formally integrable systems of l smooth complex vector fields in normal form like in (3) on n dimensional real Cartesian space. In the particular case when $l = m$, $n = m + 1$ this theorem is the following one

Theorem 1. Each formally integrable system of the kind (3) with smooth complex valued functions $a^j(t, x)$ for $j = 1, \ldots, m$ which are really analytic on the variable x is locally integrable.

We will state a new more simple proof of this theorem, based on the result for realizability of the considered abstract CR structures of hypersurface type.

Lemma 3. If the complex valued functions $a^j(t, x)$ in the system (3) satisfying the condition (4) are really analytic on the variable x, then the corresponding abstract CR structure of hypersurface type (7) is realizable.

Proof. Let us prolonged the complex valued real-analytic functions $a^j(t, x)$ as holomorphic functions $A^j(t, z)$ of the complex variable $z = x + iy$ for $(t, x) \in U, -\varepsilon < y < \varepsilon$ for some $\varepsilon > 0$, i.e. we construct holomorphic functions $A^j(t, z)$ of z such that

$$A^j(t, x) = a^j(t, x) \text{ for } (t, x) \in \mathrm{U}. \tag{10}$$

We shall consider the system of smooth complex vector fields

$$L'_j = 2\partial/\partial \bar{z}^j + A^j(t, z)\partial/\partial x \text{ for } j = 1, \ldots, m, \tag{11}$$

$$L' = \partial/\partial \bar{z}$$

on the domain $U' = \{(t, s, x + iy) : (t, x) \in U, s \in \mathbf{R}^m, y \in \mathbf{R}, -\varepsilon < y < \varepsilon\}$in $\mathbf{R}^{2m+2}$. This system consist of $m + 1$ linearly independent smooth complex vector fields. They commute each other because of the holomorphicity of the functions $A_j(t, z)$ of the variable z and as the condition (8) holds also for the holomorphic prolongations $A_j(t, z)$ of the functions $a_j(t, x)$. Such a system of vector fields forms an integrable almost complex structure on the domain U'. The important theorem of integrability of Newlander-Nirenberg [NN] for integrable almost complex structures assure that on the set U' there exist complex coordinates $Z^1, Z^2, \ldots, Z^{m+1}$. These are smooth complex valued functions which annihilate the vector fields (11) and have linearly independent differentials. Then the functions $Z^1(t, s, x), Z^2(t, s, x), \ldots, Z^{m+1}(t, s, x)$

provide the needed functions for the realizability of the abstract CR structure of hypersurface type (7). Indeed, they annihilate the vector fields (7) and their differentials are alzo linearly independent as for them holds $\partial Z^j/\partial x = -\partial Z^j/\partial y$. The lemma is proved.

Proof of theorem 1. First we constrruct for the system (3), satisfying the equalities (4) the correspondind abstract CR structure of hypersurface type (7) according Lemma 1. Then we apply the Lemma 3 to the structure (7) and obtaine its realizability. Finally according the Lemma 2 we obtain the local integrability of the given system (3). The theorem is proved.

Now we will give the following example, which shows that this sufficient condition for local integrability is not a necessary one.

Example. Let us consider the system of m smooth (complex) vector fields

$$M_j(t,x) = \partial/\partial t_j - \qquad (12)$$

$$-\frac{t_j e^{-(t_1^2+t_2^2+\ldots+t_m^2+x^2)^{-1}}}{(t_1^2+t_2^2+\ldots+t_m^2+x^2)x}\partial/\partial x \text{ if } x \neq 0 \text{ and}$$

$$M_j(t,0) = \partial/\partial t_j \text{ for } j = 1,2,\ldots,m$$

This is formally integrable system of smooth complex vector fields with coefficients - functions which anniuhilate with all derivatives in the origin, i.e. flat in the origin functions. Hence for this system can not be found a coordinate system in $\mathbf{R}^{m+1}$ where they would be analytic functions of one of the variables.

In spite of this, the system is locally integrable, as the function

$$Z(t,x) = e^{-(t_1^2+t_2^2+\ldots+t_m^2+x^2)^{-1}} \text{ if } (t,x) \neq 0,\ Z(0) = 0$$

is a smooth function which annihilate the vector fields of this system on each neighborhood of the origin.

4. NECESSARY CONDITION FOR LOCAL INTEGRABILITY.

The example above shows that the condition in Theorem 1 is not a necessary one. Nevertheless the following partially converse statement of the Theorem 1 is true.

Theorem 2. If the formally integrable system of the kind (3) is locally integrable, then there exist smooth complex valued functions $A^j(t,x,y)$ for $j = 1,\ldots,m$ defined on $U \times (-\varepsilon,\varepsilon)$ with properties

$$\partial A^j/\partial \bar{z} = 0 \text{ on } U \times (0,\varepsilon) \tag{13}$$

where $z = x + iy$ and

$$A^j(t,x,0) = a^j(t,x) \text{ for } (t,x) \in \text{U}. \tag{14}$$

i.e. the coefficients $a^j(t,x)$ in the system (3) are boundary values on the boundary $z = 0$ of functions A^j on $U \times (0,\varepsilon)$ holomorphic with respect the variable z.

To prove this we need the following result, proved by N. Hanges and H. Jacobowitz.

Theorem 3. [HJ] Let $m \geq 1$. Assume that $L'_1, L'_2, \ldots, L'_m$ generate an abstract CR structure near $p \in \mathbf{R}^{2m+1}$. If the structure is realizable, then there exists a real change of coordinates such that the transformed structure is generated by vector fields of the form (7) satisfying (8), (13) and (14).

We give the proof of this theorem for completeness and because of its constructive character who illuminate the situation.

Proof of theorem 3. [HJ] From the realizability of the given abstract CR structure of hypersurface type follows that there exist real coordinates

$$(t^1, t^2, \ldots, t^m, s^1, s^2, \ldots, s^m, x)$$

near $0 \in \mathbf{R}^{2m+1}$ such that the structure is generated by

$$L'_j = \partial/\partial \bar{z}^j + b^j(t,s,x)\partial/\partial x \text{ for } j = 1,\ldots,m$$

$z^j = t^j + \text{i}s^j$, $j = 1,2,\ldots,m$ with coefficients b_j smooth complex-valued functions, defined near 0 with $b_j(0) = 0$. It may be choosen for CR functions with linearly independent differentials of this structure the functions

$$z^j = t^j + \text{i}s^j \text{ for } j = 1,2,\ldots,m$$

$$v = x + i\phi(t,s,x)$$

with ϕ real smooth function, $\phi(0) = 0$ and $d\phi(0) = 0$.

Then this could be extended to an almost complex structure.

Let (t, s, x, y) be coordinates near $0 \in \mathbf{R}^{2m+2}$, and introduce the vector field

$$L' = \partial/\partial x + (i - \phi_x)\partial/\partial y.$$

Then the vector fields $L'_1, L'_2, \ldots, L'_m, L'$ give an almost complex structure. The functions $z^1, z^2, \ldots, z^m$ and $w = x + i(\phi(t, s, x) + y)$ are holomorphic functions for this almost complex structure. This means that the almost complex structure is integrable, moreover it is a complex structure with complex coordinates $z^1, z^2, \ldots, z^m, w$.

It may be assumed that this almost complex structure is defined on a small ball B containing $0 \in \mathbf{R}^{2m+2}$. Let us define

$$B(t, s)^+ = \{(x, y) \in \mathbf{R}^2 : (t, s, x, y) \in B \text{ and } y > 0\}$$

for each(s, t). Let $O(t, s)^+ \subset \mathbf{C}$ be the image of $B(t, s)^+$ under the map $w(t, s, ., .)$. Since $O(t, s)^+$ is simply connected there exists a map $z = z(t, s, .)$ mapping $O(t, s)^+$ into the complex plane such that:
1. z is conformal for each (t, s)
2. $\mathrm{Im} z > 0$ on $O(t, s)^+$
3. $\mathrm{Im} z(t, s, , w(t, s, x, 0) = 0$ for all (t, s, x)
4. z is smooth in all arguments
5. $\partial z/\partial w(0, 0, 0) \neq 0$.

In particular the map $(t, s, x) \to (t, s, z(t, s, x, 0)$ is a real diffeomorphism. After the change of coordinates

$$(t, s, x, y) \to (t, s, \mathrm{Re} z, \mathrm{Im} z)$$

the almost complex structure becomes

$$L'_j = \partial/\partial z^j + L'_j z\partial/\partial z + L'_j z\partial/\partial z \text{ for } j = 1, 2, \ldots, m,$$
$$L' = 2\bar{z}_{\bar{w}}\partial/\partial \bar{z}.$$

It follows from the integrability conditions for the almost complex structure that $L'_j z$ is holomorphic in z for $\mathrm{Im} z > 0$ and $j = 1, 2, \ldots$,m. The fact that L'_j, $j = 1, 2, \ldots, m$ are tangent to $\mathrm{Im} z$=0 shows that for $j = 1, 2, \ldots, m$ we have $L'_j z = L'_j \mathrm{Re} z$ when $\mathrm{Im} z = 0$. Hence the CR structure induced on $\mathrm{Im} z = 0$ is generated by

$$L'_j = \partial/\partial z^j + L'_j \mathrm{Re} z\partial/\partial \mathrm{Re} z \text{ for } j = 1, 2, \ldots, m$$

and (13) and (14) follow with $A_j = L'_j z$. This completes the proof.
Proof of theorem 2. First we need to coinstruct the corresponding system (7) for the given system of smooth complex vector fields (3). As the system (3) is locally integrable from Lemma 2 follows that the system (7) is realizable. Then we could to apply the Theorem 3 and to construct the functions $A_j(t, s, x, y), j = 1, 2, \dots, m$. The rest is to restrict the functions $A_j(t, s, x, y), j = 1, 2, \dots, m$ to the intersection of the domain they are defined with the plane in $\mathbf{R}^{2m+2} : s^1 = 0, s^2 = 0, \dots, s^m = 0$.

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