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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

DISCRETE DISTRIBUTIONS RELATED TO SUCCESS RUNS OF LENGTH K IN A MULTI-STATE MARKOV CHAIN

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DISCRETE DISTRIBUTIONS RELATED TO SUCCESS RUNS OF LENGTH K IN A MULTI-STATE MARKOV CHAIN*

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Abstract. We consider the time-homogeneous multi-state Markov chain $\{X_n, n \geq 0\}$ with states labaled as "0" (success) and "i" (failure), $i = 1,2,...$ Let k be fixed positive integer and E_0 be the event of a success run of length k in the sequence $X_0, X_1,...$ In this article, joint probability generating function for various statistics, related to the event E_0 is derived. In particular cases the exact distribution of the total number of successes Y, the total number of failures F and the total number of trials N are deduced. The distribution of N is defined as geometric distribution of order k for time-homogeneous $\{0,1,2,...\}$-valued Markov chain.

Key words and phrases: Geometric distribution, discrete distributions, Markov chain, waiting time, geometric distribution of order k, random walk, recursion, joint probability generating function.

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1. Introduction

Let n and k be fixed positive integers such that $n \geq k$. Feller (1968) derived the probability generating function (pgf) of the number of trials N until the occurance of kth consecutive success in independent trials with success probability p, given by

$$g_N(t) = \frac{(1-pt)p^k t^k}{1 - t + (1-p)p^k t^{k+1}} \qquad (1.1)$$

Hahn and Gage (1983) obtained the following recurrence representation for the probability function of N:

$$P\left(N = n\right) = \begin{cases} 0 & \text{if } n < k, \\ p^k & \text{if } n = k, \\ (1-p)p^k & \text{if } n = k+1, \ldots, 2k, \\ (1-p)p^k \left[1 - \sum_{m=1}^{n-2k} P(N = k+m-1)\right] & \text{if } n \geq 2k+1. \end{cases}$$

Philippou *et al.* (1983) obtained the exact probability function corresponding to (1.1)

$$P\left(N = n\right) = \sum_{n_1,\ldots,n_k} \binom{n_1 + \ldots + n_k}{n_1, \ldots, n_k} p^n \left(\frac{q}{p}\right)^{n_1+\ldots+n_k}, \quad n \geq k,$$

where the summation is over all nonnegative integers $n_1, n_2, \ldots, n_k$ such that

$$n_1 + 2n_2 + \ldots + kn_k = n - k \text{ and } q = 1 - p$$

and called it geometric distribution of order k. In the same paper they introduced the negative-binomial distribution of order k and the Poisson distribution of order k. Since then many authors gave their contribution to the exact theory for so called discrete distributions of order k which already can be called classical (cf. Aki *at al.* (1984), Aki (1985), Charalambides (1986), Panaretos and Xekalaki (1986), Philippou (1986, 1988), Ling (1988), Aki and Hirano (1989), Sobel and Ebneshahrashoob (1992) and references therein).

The statistical meaning of considering occurances of k consecutive successes leads to further extention of the distributions based on the independent trials to the dependent trials when the probability of success may vary according to a well defined rule. This enables one to single out some special cases that are in practical importance on the reliability of the consecutive-k-out-of n:F system introduced by Chaing and Niu (1981), on the start-up demonstration tests introduced by Hahn and Gage (1983), on the prevantive statistical process control (cf. Wetherill and Brown (1991)), on the molecular biology, on the typical waiting time problems (cf. Aki (1985), Aki (1992), Ebneshahrashoob and Sobel (1990)).

The study of more general model caries interest in its own right. Rajarshi (1974), Aki and Hirano (1993, 1994), Mohanty (1994) and Hirano and Aki (1993) are derived discrete distributions arising out of success run of length k in a time-homogeneous two-state Markov chain, which is useful model containing independent trials and dependent stationary sequences as special cases. Some distributions are recently obtained by Viveros and Balakrishnan (1993) and Balasubramanian *at. al.* (1994) as a generalization of a problem of practical interest in Hahn and Gage (1983) in a Markovian fashion.

We consider the time-homogeneous multi-state Markov chain $\{X_n,\ n \geq 0\}$ with states labaled as "0" (success) and "i" (failure), $i = 1,2,\ldots$ We adopt the following classical way of counting of a "success run of length k" (cf. Feller (1968)): a sequence of n outcomes of successes and failures contains as many runs of length k as there are non-overlapping, uninterrupted successions of exactly k zeros.

Let E_0 be the event of a success run of length k in the sequence $X_0, X_1, \ldots$ defined by distribution of initial states of the homogeneous Markov chain

$$P(X_0 = i) = p_i, \quad 0 < p_0 < 1, \quad p_i \geq 0, \quad i = 1,2,\ldots, \quad \sum_{i=0}^{\infty} p_i = 1,$$

and transition probabilities

$$P\left(X_{n+1} = j \,\middle|\, X_n = i\right) = p_{ij}, \quad n = 0,1,\ldots ,$$

where

$$\sum_{j=0}^{\infty} p_{ij} = 1, \quad i = 0,1,\ldots \quad \text{and} \quad p_{ij} \geq 0.$$

Let us define the random variables

$$S_i = \begin{cases} 1 & \text{if } X_0 = i, \\ 0 & \text{if } X_0 \neq i, \end{cases} \quad \text{for } i = 0,1,\ldots$$

and

$$S_{ij} = \left\{\text{number of transitions of type } i \longrightarrow j \text{ in the Markov chain}\right\},$$
$$i\,,\, j = 0\,,1\,,\ldots$$

In this paper we deal with the distribution of random vector

$$S = \left(S_0,\, S_1,\, \ldots\,,\, S_{00},\, S_{01},\, \ldots\right).$$

In Section 2 the joint pgf $g_S\left(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots\right)$ is obtained, and an example in terms of a random walk model is given. In Section 3 are deduced the exact distributions of the following random variables in E_0

Y = the total number of successes,

F = the total number of failures

and

$N = Y + F$ = the total number of trials

until a success run of lenght k in the defined Markov chain occurs.

The distribution of N is defined as geometric distribution of order k for time-homogeneous $\{0,1,2,\dots\}$-valued Markov chain. An interpretation in terms of an urn model is given. At the end the joint pgf $g_{Y,F}(t,s)$ is obtained.

For the two-state homogeneous Markov chain the distribution of N was outlined by Aki and Hirano (1993), and the distributions of Y and F was obtained stylish in various way by Aki and Hirano (1994).

2. The joint pgf of S

Under the given notations the following lemma is true.

LEMMA 2.1. *The joint pgf of* $S = \left(S_0, S_1, \dots, S_{00}, S_{01}, \dots\right)$ *is given by*

$$g_S\left(t_0, t_1, \dots, t_{00}, t_{01}, \dots\right) = \mathbb{E}\left(t_0^{S_0} t_1^{S_1} \dots t_{00}^{S_{00}} t_{01}^{S_{01}} \dots\right) =$$

$$= (p_{00}t_{00})^{k-1}\left\{p_0 t_0\left[1 + \frac{1-(p_{00}t_{00})^{k-1}}{1-p_{00}t_{00}} \sum_{i=1}^{\infty} p_{0i}t_{0i}\mathbb{A}_i\right] + \sum_{i=1}^{\infty} p_i t_i \mathbb{A}_i\right\} \tag{2.1}$$

where $|t_i| \le 1$ and $|t_{ij}| \le 1$, $i,j = 0,1,\dots$ and

$$\mathbb{A}_i = p_{i0}t_{i0}\left[1 - \sum_{j=1}^{\infty} p_{ij}t_{ij} - \frac{1-(p_{00}t_{00})^{k-1}}{1-p_{00}t_{00}} p_{i0}t_{i0} \sum_{j=1}^{\infty} p_{0j}t_{0j}\right]^{-1}, \tag{2.2}$$

$i = 1,2,\dots$.

PROOF. Let $X_0 = 0$ and the first success is followed by k-1 consecutive successes. Then the contribution to the joint pgf $g_S(\bullet)$ is $\mathbb{Y}_0 = p_0 t_0 (p_{00}t_{00})^{k-1}$.

Let $X_0 = 0$ and the first success is followed by no more than k-2 consecutive successes before the first failure. The second failure in the event E_0 (if it exists) may be precided by no more than k-1 consecutive successes followed the first failure. This process will be repeated till the last failure in the event E_0 which is followed by exactly k consecutive successes. The number of failures r in the event E_0 may vary from 1 to ∞. From the Markov property of the sequences we have the following contribution to the joint pgf $g_S(\bullet)$:

$$\mathbb{Y} = \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} \mathbb{Y}_{ir},$$

where

$$\mathbb{Y}_{ir} = p_0 t_0 \left[\sum_{j=1}^{\infty} p_{0j} t_{0j} + p_{00} t_{00} \sum_{j=1}^{\infty} p_{0j} t_{0j} + \ldots + (p_{00} t_{00})^{k-2} \sum_{j=1}^{\infty} p_{0j} t_{0j} \right] \bullet$$

$$\bullet \left\{ \sum_{j=1}^{\infty} p_{ij} t_{ij} + p_{i0} t_{i0} \sum_{j=1}^{\infty} p_{0j} t_{0j} + p_{i0} t_{i0} p_{00} t_{00} \sum_{j=1}^{\infty} p_{0j} t_{0j} + \ldots + \right.$$

$$\left. + \; p_{i0} t_{i0} (p_{00} t_{00})^{k-2} \sum_{j=1}^{\infty} p_{0j} t_{0j} \right\}^{r-1} \bullet \; p_{i0} t_{i0} (p_{00} t_{00})^{k-1}.$$

After some algebra, we obtain

$$\mathbb{Y} = p_0 t_0 (p_{00} t_{00})^{k-1} \frac{1 - (p_{00} t_{00})^{k-1}}{1 - p_{00} t_{00}} \sum_{i=1}^{\infty} p_{0i} t_{0i} \mathbb{A}_i,$$

where $\mathbb{A}_i$ is given by (2.2).

Let $X_0 = i$ for $i = 1,2,\ldots$ Then the distribution of the sequence just after the first failure is the same as the distribution after the first failure in the above case, i.e. the contribution to the joint pgf $g_S(\bullet)$ in this case is

$$\mathbb{F} = (p_{00}t_{00})^{k-1} \sum_{i=1}^{\infty} p_i t_i \mathbb{A}_i.$$

At the end the joint pgf $g_S(\bullet)$ is obtained by

$$g_S\left(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots\right) = \mathbb{Y}_0 + \mathbb{Y} + \mathbb{F}.$$

This completes the proof. ■

Example 2.1 (Random walk model). Let us consider the motion of a particle which moves in discrete jumps with certain probabilities from point to point. We imagine the particle starting at the point $x = i$ (with probability p_i) on the x-axis at time $t = 0$, and for each subsecuent time $t = 1,2,\ldots$ it moves one unit to the right, one unit to the left or remais where it is with probabilities $p_{i,i+1}$, $p_{i,i-1}$ or p_{ii}, respectively where

$$p_{i,i+1} + p_{i,i-1} + p_{ii} = 1, \quad i = 1,2,\ldots$$

If the particle starts at the point $x = 0$ (with probability p_0), it moves at the time $t = 1$ only one unit to the right or remains where it is with probabilities p_{01} or $p_{00} > 0$, respectively where

$$p_{00} + p_{01} = 1.$$

The considered random walk model can be described by homogeneous Markov chain with the following transition probability matrix

$$\begin{pmatrix} p_{00} & p_{01} & 0 & 0 & 0, & \ldots \\ p_{10} & p_{11} & p_{12} & 0 & 0 & \ldots \\ 0 & p_{21} & p_{22} & p_{23} & 0 & \ldots \\ 0 & 0 & p_{32} & p_{33} & p_{34} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

It is seen that $p_{0i} = p_{i0} = 0$ for $i = 2,3,\ldots$ Then the joint pgf $g_S(\bullet)$ given by (2.1) in the considered case has the following representation:

$$g_S^{RW}\left(t_0,t_1,t_{00},t_{01},t_{10},t_{11},t_{12}\right) =$$

$$= (p_{00}t_{00})^{k-1}\left[p_0t_0 + \frac{p_0t_0\dfrac{1-(p_{00}t_{00})^{k-1}}{1-p_{00}t_{00}}p_{01}t_{01}p_{10}t_{10} + p_1t_1p_{10}t_{10}}{1-p_{11}t_{11}-p_{12}t_{12}-\dfrac{1-(p_{00}t_{00})^{k-1}}{1-p_{00}t_{00}}p_{10}t_{10}p_{01}t_{01}}\right]. \quad (2.3)$$

Let us note that the joint pgf $g_S^{RW}(\bullet)$ from (2.3) is unaffected by the values p_{ij}, $i = 2,3,\ldots$ and $j = 3,4,\ldots$

3. Exact Distribution of Y, F, N and (Y,F) in E_0

3.1. Number of successes

The total number of successes Y may be represented as follows

$$Y = S_0 + \sum_{i=0}^{\infty} S_{i0}.$$

THEOREM 3.1. *The distribution of* Y *is defined by pgf*

$$g_Y(t) = \frac{(1-p_{00}t)p_{00}^{k-1}t^k}{1-t+(1-p_{00})p_{00}^{k-1}t^k}. \quad (3.1)$$

PROOF. The relation (3.1) is obtained from (2.1) by substitu-tution

$$t_0 = t_{i0} = t, \quad i = 0,1,\ldots$$

and

$$t_i' = 1, \quad i = 1,2,\ldots; \quad t_{ij} = 1, \quad i = 0,1,\ldots \quad \text{and} \quad j = 1,2,\ldots \quad \blacksquare$$

The presentation (3.1) is pgf of the shifted geometric distribution of order k-1 so that its support begins with k. The exact distribution of Y is given by

$$P\left(Y = n\right) = \sum_{n_1,\ldots,n_{k-1}} \binom{n_1 + \ldots + n_{k-1}}{n_1, \ldots, n_{k-1}} p_{00}^n \left(\frac{q_{00}}{p_{00}}\right)^{n_1+\ldots+n_{k-1}}, \quad n \geq k,$$

where the summation is over all nonnegative integers $n_1, n_2, \ldots, n_{k-1}$ such that

$$n_1 + 2n_2 + \ldots + (k-1)n_{k-1} = n - k + 1 \quad \text{and} \quad q_{00} = 1 - p_{00}.$$

It is somewhat surprising to note from (3.1) that the distribution of Y depends only on p_{00} and it is unaffected by the values p_i and p_{ij}, $i = 0,1,\ldots$ and $j = 1,2,\ldots$ Hense we can treat homogeneous multi-state Markov chain as the independent and identically distributed sequence. This conclusion was obtained in a different way by Aki and Hirano (1994) for the (0,1)-valued homogeneous Markov chain (cf. their THEOREM 3.2, p. 199).

3.2. *Number of failures*

The total number of failures F may be represented as follows

$$F = \sum_{i=1}^{\infty} S_i + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} S_{ij}.$$

THEOREM 3.2. *The distribution of F is defined by pgf*

$$g_F(t) = p_0 p_{00}^{k-1} + \sum_{i=1}^{\infty}\left[\frac{1-(p_{00})^{k-1}}{1-p_{00}} p_0 p_{0i} + p_i\right] \frac{p_{i0} p_{00}^{k-1} t}{1-\left(1-p_{i0} p_{00}^{k-1}\right) t}. \quad (3.2)$$

PROOF. The relation (3.2) is obtained from (2.1) by substitution

$$t_0 = t_{i0} = 1, \quad i = 0,1,\ldots$$

and

$$t_i = t, \quad i = 1,2,\ldots; \quad t_{ij} = t, \quad i = 0,1,\ldots \quad \text{and} \quad j = 1,2,\ldots$$ ■

COROLLARY 3.1. *The exact distribution of F is given by*

$$P\left(F = 0\right) = p_0 p_{00}^{k-1},$$

$$P\left(F = n\right) = \sum_{i=1}^{\infty}\left[\frac{1-(p_{00})^{k-1}}{1-p_{00}} p_0 p_{0i} + p_i\right] p_{i0} p_{00}^{k-1}\left(1 - p_{i0} p_{00}^{k-1}\right)^{n-1}, \quad n \geq 1.$$

PROOF: From (3.2) it is seen that

$$\frac{p_{i0} p_{00}^{k-1} t}{1-\left(1-p_{i0} p_{00}^{k-1}\right) t} = \sum_{n=1}^{\infty} p_{i0} p_{00}^{k-1}\left(1 - p_{i0} p_{00}^{k-1}\right)^{n-1} t^n.$$ ■

Let us denote by Ge(p) the regular geometric distribution with parameter $p \in (0,1)$.

COROLLARY 3.2. *The conditional distribution of F*

(i) *given that* $X_0 = i$ *is* $\mathrm{Ge}\left(p_{i0} p_{00}^{k-1}\right)$, $i = 1,2,\ldots$,

(ii) *given that* $X_0 = 0$ *is the mixture* $p_{00}^{k-1}\delta_0 + (1 - p_{00}^{k-1})$G, where δ_0 means the distribution function of S_0 and G means the distribution function of the mixture $\sum_{i=1}^{\infty} \frac{p_{0i}}{1 - p_{00}} \mathrm{Ge}\left(p_{i0}p_{00}^{k-1}\right)$.

PROOF: It follows directly from THEOREM 3.2 by considering the conditional pgf $g_{F|X_0=i}(t)$, $i = 1,2,\ldots$ and $g_{F|X_0=0}(t)$, respectively.■

Remark 3.1. The last result was obtained in a different way by Aki and Hirano (1994) for the (0,1)-valued homogeneous Markov chain (cf. their THEOREM 3.1, p. 199).

3.3. Number of trials

The total number of trials N can be represented as follows

$$N = Y + F = \sum_{i=0}^{\infty} S_i + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} S_{ij}.$$

THEOREM 3.3. *The distribution of N is defined by pgf*

$$g_N(t) = p_0 p_{00}^{k-1} t^k +$$

$$+ p_{00}^{k-1} t^k \sum_{i=1}^{\infty} \frac{p_0\left[1-(p_{00}t)^{k-1}\right] p_{0i} p_{i0} t^2 + (1-p_{00}t) p_i p_{i0} t}{1 - (p_{00}+1-p_{i0})t + (p_{00}-p_{i0})t^2 + (1-p_{00}) p_{i0} p_{00}^{k-1} t^{k+1}} \quad (3.3)$$

PROOF. The relation (3.3) is obtained from (2.1) by substitution

$$t_i = t_{ij} = t, \quad i,j = 0,1,\ldots \quad ■$$

COROLLARY 3.3. *The expected number of trials in* E_0 *is given by*

$$\mathbb{E}\left(N\right) = 1 - \frac{1 - p_0}{1 - p_{00}} + \frac{p_0(1 - p_{00}^{k-1})}{(1 - p_{00})p_{00}^{k-1}}\left[\sum_{i=1}^{\infty} \frac{p_{0i}}{p_{i0}} + 1\right].$$

Definition. The disrtibution with pgf given by (3.3) we coin the geometric distribution of order k for homogeneous multi-state Markov chain.

COROLLARY 3.4. *The exact distribution of* N *is given by*

$$P\left(N = k\right) = p_0 p_{00}^{k-1},$$

$$P\left(N = n\right) = \sum_{i=1}^{\infty}\left[p_0 \sum_{m=2}^{k} p_{00}^{m-2} p_{0i}\mathbb{P}_i\left(N = n-m\right) + p_i\mathbb{P}_i\left(N = n-1\right)\right], \qquad n > k$$

where for fixed $a = 1,2,...k$ *and* $i = 1,2,....$

$$\mathbb{P}_i\left(N = n-a\right) = \sum_{n_{1ia},...,n_{kia}} \binom{n_{1ia} + ... + n_{kia}}{n_{1ia}, ..., n_{kia}} \left(1-p_{i0}\right)^{n_{1ia}} \bullet$$

$$\bullet \left(1-p_{00}\right)^{n_{2ia}+...+n_{kia}} \left(p_{i0}\right)^{n_{2ia}+...+n_{kia}+1} \left(p_{00}\right)^{n_{3ia}+...+(k-2)n_{kia}+k-1}$$

and the summation is over all nonnegative integers n_{1ia}, ..., n_{kia} *such that*

$$n_{1ia} + 2n_{2ia} + ... + kn_{kia} = n - k.$$

Example 3.1 (Urn model). Suppose that there are urns numbered by 0,1,2,... Each urn contains the balls numbered by 0,1,2,... The content of each urn remains constant (each sampled ball is returned to the urn). Let p_i denote the probability that the initial drawing will be from the urn with number i, $i = 0,1,2,...$, $\sum_{i=0}^{\infty} p_i = 1$. Let p_{ij} be the probability that a ball bearing the number j is drawn from the ith urn, $\sum_{j=0}^{\infty} p_{ij} = 1$. Then the next ball will be drawn from the urn numbered by j, $j = 0,1,2,...$ This process continues until k consecutive times the drawing is made from the urn numbered by 0.

If N denote the number of drawn balls, then N has geometric distribution of order k for the homogeneous multi-state Markov chain, defined by COROLLARY 3.4.

Remark 3.2. For two-state Markov chain the exact distribution of N and its pgf was outlined and by Aki and Hirano (1993) where it is understood that X_0 is not counted in measuring the length of the run (cf. their COROLLARY on p. 469).

COROLLARY 3.5. *Let us consider the random walk model given in Example 2.1. Then the distribution of N is defined by pgf*

$$g_N^{RW}(t) = \frac{p_{00}^{k-1} t^k \left\{ p_0 + \left[p_{10}(p_0+p_1) - p_0 - p_0 p_{00} \right] t + p_{00} \left[p_0 - p_{10}(p_0+p_1) \right] t^2 \right\}}{1 - (p_{00}+1-p_{10})t + (p_{00}-p_{10})t^2 + (1-p_{00})p_{10}p_{00}^{k-1} t^{k+1}} \tag{3.4}$$

and by following recursion

$$P\left(N = n\right) = 0 \quad \text{for} \quad n < k,$$

$$P\left(N = k\right) = p_0 p_{00}^{k-1},$$

$$P\left(N = k+1\right) = (1-p_0) p_{10} p_{00}^{k-1},$$

$$P\left(N = k+2\right) = \left[p_0 p_{01} + p_1 (1-p_{10}) \right] p_{10} p_{00}^{k-1},$$

$$P\left(N = n\right) = (p_{00}+1-p_{10}) P\left(N = n-1\right) - (p_{00}-p_{10}) P\left(N = n-2\right) -$$

$$- p_{01} p_{10} p_{00}^{k-1} P\left(N = n-k-1\right), \quad n \geq k+3.$$

PROOF. The relation (3.4) is obtained from (2.3) by substitution all arguments equal to t. After equating the coefficients of t^n on both sides in (3.4) yields the recorded recursion for $P\left(N = n\right)$, $n = 0,1,..$ ■

3.4. *The joint pgf* $g_{Y,F}(t,s)$

By substituting

$$t_0 = t_{i0} = t, \quad i = 0,1,\ldots$$

and

$$t_i = s, \quad i = 1,2,\ldots; \qquad t_{ij} = s, \quad i = 0,1,\ldots \text{ and } j = 1,2,\ldots$$ ■

in (2.1) we obtain the following

THEOREM 3.4. *The joint distribution* (Y,F) *is defined by pgf*

$$g_{Y,F}(t,s) = (p_{00}t)^{k-1}\left\{p_0 t\left[1 + \frac{1-(p_{00}t)^{k-1}}{1-p_{00}t}\, ts\sum_{i=1}^{\infty} p_{0i}p_{i0}\mathbb{B}_i\right] + ts\sum_{i=1}^{\infty} p_i p_{i0}\mathbb{B}_i\right\},$$

where

$$\mathbb{B}_i = \left[1 - (1-p_{i0})s - (1-p_{00})\frac{1-(p_{00}t)^{k-1}}{1-p_{00}t}\, p_{i0}ts\right]^{-1}, \qquad i \geq 1.$$

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