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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
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ON A THEOREM OF BLATT

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On a Theorem of Blatt

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Abstract: In the present paper, a new sufficient condition for the approximation of the reciprocal of an entire function by reciprocals of polynomials on $[0,\infty)$ with a speed of a geometric progression is provided. This condition sharpens a result of H. P. Blatt.

Introduction:

Let $f(x)$ be a function, real valued and continuous on $R^+, R^+ := [0,\infty)$. We shall deal with the uniform norm, e.g., if B is a compact set in $\mathbf{C}$ and the function g is defined on B, then $\|g\|_B := sup\{|g(x)|, x \in B\}$. For the positive integer $n, (n \in \mathbf{N})$, set

$$\rho_n(f) := inf_{p\in\pi_n} \| \frac{1}{f(x)} - \frac{1}{p(x)} \|_{R^+},$$

where the infimum is taken over all the polynomials p with real coefficients of degree not exceeding n ($p \in \pi_n$).

In the present paper, we come back to the area of the approximation on R^+ of reciprocal of functions of the prescribed kind by reciprocal of polynomials with the speed of a geometric progression, e.g.

$$\limsup_{n\to\infty} \rho_n(f)^{1/n} < 1$$

The background of these investigations are the results of W.J.Cody, G. Meinardus and R.S.Varga concerning the approximation of $\exp(-x)$ (see [CoMeVar]). Later, G. Meinardus and R.S.Varga extended the results in direction of entire functions of completely regular growth (see [MeVar]).

The paper [MeReTayVar] gave rise to investigations devoted to enlarging the class of the functions f those admit geometrical approximatation by polynomials on R^+ in the above sense, as well as to characterizing such functions.

Before presenting the results of [MeReTayVar] we introduce the common notations.

For the positive number r, set $\|f\|_r := \|f\|_{[0,r]}$. For the number $s, s > 1$, let $\mathcal{E}(r,s)$ be the ellipse with foci at $x = 0$ and $x = r$ and major and minor axes $r(s \pm 1)^2/4s$, respectively; that is:

$$\mathcal{E}(r,s) := \{z = x + iy, \frac{(x - r/2)^2}{(r(s+1)^2/4s)^2} + \frac{y^2}{(r(s-1)^2/4s)^2} = 1\}$$

Set $\mathcal{M}_f(r,s) := \|f\|_{\mathcal{E}(r,s)}$.

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The main result of [MeReTayVar] is

Theorem 1: (see [MeReTayVar]): Let the function f be real valued and continuous on R^+. Assume that the inequality

$$\limsup_{n\to\infty} \rho_n(f)^{1/n} = 1/q \tag{1}$$

holds with a number $q, q > 1$.

Then either $f(x) = Const$ or $f(x) \to \infty$ as $x \to \infty$. There exists an entire function $F(x)$ of finite order such that $F(x) = f(x)$ for every $x \in R^+$. For every $s, 1 < s < q$ there exist positive constants $K = K(s,q), \Theta = \Theta(s,q)$ and $r_0 = r_0(s,q)$ such that for every $r > r_0$ we have

$$\mathcal{M}_f(r,s) \le K\|f\|_r{}^{\Theta}. \tag{2}$$

In the same paper the authors established that for a special class of entire functions condition (2) ensures the existence of polynomials the reciprocals of which approximate geometrically on R^+ the reciprocal of f. They proved, namely

Theorem 2: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function with $a_n \ge 0$ for $n > 0$ and $a_0 > 0$. If there exist real numbers K, Θ and r_0 such that condition (2) holds, then condition (1) is valid.

After the appearance of this paper,weaker sufficient conditions for the geometrical approximation in the sense of (1) were shown. We refer to the papers [Blatt1], [Blatt2], [RoTay], [Meinardus] and [ReShi]. It is a matter of concern to close the sufficient and necessary conditions. In [HeRo], the authors called attention to the fact that condition (2) in Theorem 1 does not admit a converse. They showed in particular that if an entire function f, even satisfying condition (2), ocsillates strongly in some sense then condition (1) does not hold. In the same paper, they raised the hypothesis that the complete characterization of functions f with geometrical convergence should involve the rate of growth of $\frac{\|f\|_{[0,r]}}{(min f(x), x\ge r)}$, as $r \to \infty$. H. P .Blatt showed that a "nonstrong ocsillation" in the sense that the ratio $\|f\|/min f(x)$ behaves in an appropriate way yields together with condition (2) a geometrical convergence (see [Blatt3]).

Let f be a function of the prescribed kind and $r > 0$ be arbitrary. In order to present precisely Blatt's theorem, we introduce the number $\mu(r)(= \mu_f(r))$, that is $\mu(r) := min_{x\ge r} f(x)$.

Theorem 3 [Blatt3]: Let $f(x)$ be an entire transcendental function, real valued on R^+ and having not more than a finite number of zeros there; $f(x) \to \infty$ as $x \to \infty$. . Assume that condition (2) holds . Further assume that there exist real numbers $s > 1, \gamma > 0$ and $r' > 0$ such that

$$\|f\|_r{}^{\gamma} \le \mu\left(\frac{r}{s+2}\right)$$

for all $r \ge r'$.

Then condition (1) holds.

In the present paper, we prove

Theorem 4: In the conditions of Theorem 3, assume that for every $r > r_0$ the inequality

$$\|f\|_r{}^{\gamma} \le \mu(r)$$

is valid with γ positive.

Then condition (1) is true.

Auxilliary Lemmas

In what follows, the function g is continuous and real valued on R^+.

The base of our forthcoming considerations is the following lemma, established by H.P.Blatt (see [BLATT1]).

Lemma 1 [Blatt 1]: Denote by β the total number of zeros of g on R^+. If $\beta < \infty$ then for every positive integer n either $\rho_n(g) = 0$ or there exists a unique polynomial $p_n, p_n \in \pi_n, p_n = p_n(g)$, with $\|\frac{1}{g} - \frac{1}{p_n}\|_R^+ = \rho_n(g)$.

If $deg p_n = n$, then the difference $(1/g - 1/p_n)(x)$ attains alternatively the value $\rho_n(g)$ at $n+2-2\beta$ (different) points $y_{n,i}$, e.g., there are $y_{n,i}$ (different) points with $i \geq n+2-2\beta$ for those

$$\frac{1}{g(y_{n,i})} - \frac{1}{p_n(y_{n,i})} = a(-1)^i \rho_n(g)$$

with $a = \pm 1$.

If $deg p_n < n$, then the number of the alternation points $y_{n,i}$ is at least $n + 1 - 2\beta$ and the last alternation point is a $(+)$-point of the difference $(1/g - 1/p_n)(x)$. (e.g. $(1/g - 1/p_n)(y) > 0$.)

For a set $E \in R^+$ and an integer $n, n \in \mathrm{N}$, we introduce the notations $\rho_n(g, E)$, that is $\rho_n(g, E) = inf\|\frac{1}{g(x)} - \frac{1}{p(x)}\|_E$. Further, let $E_n(g, E)$ be the error of the best uniform approximation of g on E in the class π_n. In [MeReTayVar], it was shown that

Lemma 2: If the function g is in addition nowhere zero on R^+, then there exists a positive constant $C, C = C(g)$ such that for every $n \in \mathrm{N}$

$$\rho_n(g, E) \leq C.E_n(g, E). \tag{3}$$

In what follows we will use a modification of this lemma. Namely, let y and a be real numbers, $y \notin E$. Set $\rho_n(g, E, y, a) := \inf_{p \in \pi_{n-1}} \|1/g(x) - 1/(a - (x - y)p(x))\|_E$. Similarly, we define $E_n(g, E, y, a)$ as the error of the best approximation of g on E by polynomials $p \in \pi_n$ satisfying the additional condition $p(y) = a$. We have the analogue to Lemma 2, namely

Lemma 2': For any $y, a \in R^+, y \notin E$, there exists a positive constant C' such that for every $n \in \mathrm{N}$ it is true that

$$\rho_n(g, E, y, a) \leq C'.E_n(g, E, y, a). \tag{4}$$

It is easy to verify that the constants C, C' appearing in both lemmas satisfy the inequality $C, C' < 4/m^2$ with $m := min_{R^+} g(x)$. Set $max(C, C') := C_1$.

In what follows, we will denote by $C_i, i \in \mathrm{N}$ positive constants which do not depend on the integer i.

At several stages of the proof of Theorem 4, we will use the following classical result due to S. Bernstein.

Lemma 3: [Bernstein] Let r, s be positive numbers, $s > 1$ and let g be analytic in the ellipse $\mathcal{E}(r,s)$ and continuous on $\overline{\mathcal{E}}(r,s)$. Then, for any $n, n \in \mathbb{N}$ we have

$$E_n(g,r) \leq 2.\frac{\mathcal{M}_g(r,s)}{s^n(s-1)}$$

Proof of Theorem 4

For the sake of clearance and simplicity, we will assume that the function f under consideration is nowhere zero. Later on we will give the sketch of the proof for the general case.

Before presenting the new sufficient condition, we introduce for the positive integer r the set I_r, that is

$$I_r := \{x, x \in R^+, f(x) \leq r\}.$$

In the conditions of Theorem 4, the set I_r consists apparently of a finite number of subintervals $I_{r,k}, 1 \leq k \leq k_r$. Further, we introduce into considerations the number $m(r)$ as follows:

$$m(r) := min\{f(x), x \in \bigcup_{k\geq 2} I_{r,k}\}$$

Fix now a positive number $s', 1 < s' < s$ and set $s_1; = s'^{\frac{1}{2\theta}}$. For the number $n, n \in \mathbb{N}$ denote $I_n := I_{s_1^n}, k_n := k(s_1^n)$ and $I_n := \bigcup_{k=1}^{k_n} I_{n,k}$ with $I_{n,k} := [a_{n,k}, b_{n,k}]$ for $k \geq 2$. Let the number r_n be determined by the condition $f(r_n) = \|f\|_{r_n} = s_1^n$. It will be assumed throughout that $0 \in I_{n,1}$. Set, finally, $m(r_n) := m_n$, if $k_n \geq 2$ and $m_n = f(r_n)$, otherwise.

We have obviously

$$m_n = \mu(r_n).$$

Set $\delta_n := \frac{\ln m_n}{n \ln s_1}$. With respect to the conditions of Theorem 4, we may write

$$\alpha := \inf_{n > n_0} \delta_n > 0 \tag{5}$$

for an appropriate integer $n_o \in \mathbb{N}$.

Preserving these notations, we will prove the following statement:

Theorem 4': Let $f(x)$ be an entire transcendental function, real valued and positive on $R^+, f(x) \to \infty$ as $x \to \infty$. Assume that there exist positive numbers s, r_0 and $\theta, s, \theta > 1$ such that condition (2) holds . Assume, further that for all $n > n_0$ inequality (5) is valid.

Then condition (1) holds.

Proof of Theorem 4'

Denote by r_{n,s_1} the crossing point of the ellipse $\mathcal{E}(r_n, s_1)$ with R^+; the calculation implies $r_{n,s_1} := (r_n/2).(1 + 1/2(s_1 + 1/s_1))$. Set further $I_{n,s_1} := \{x, 0 \leq x \leq r_{n,s_1}\}$. In accordance to the situation of the point r_{n,s_1} we have to consider three cases: a) If

$r_{n,s_1} \in [b_{n,k_1}, a_{n,k_1+1}]$, for some $k_1, 1 \leq k_1 < K_n$, then let l be the segment with endpoints r_{n,s_1} and a_{n,k_1+1},, that is $\frac{y(x)-f(r_{n,s_1})}{r-f(r_{n,s_1})} = \frac{x-r_{n,s_1}}{a_{n,k_1+1}-r_{n,s_1}}$.

We now introduce into considerations the function $f_{n,s_1}(x)$ constructed in the following way:

$$f_{n,s_1}(x) := f(x), x \in I_{n,s_1},$$

$$f_{n,s_1}(x) := y(x), r_{n,s_1} < x < a_{n,k_1+1},$$

$$f_{n,s_1}(x) := f(x), x \in I_{n,k}, k \geq k_1 + 1$$

$$f_{n,s_1}(x) := s_1^n, x \in R^+ - I_n \bigcup [r_{n,s_1}, a_{n,k_1+1}] \bigcup I_{n,s_1},$$

b) If there is a number $k_2, 1 < k_2 \leq k_n$, such that $r_{n,s_1} \in I_{n,k_2}$, then we define f_{n,s_1} in the following way:

$$f_{n,s_1}(x) := f(x), 0 \leq x \leq b_{n,k_2}$$

$$f_{n,s_1}(x) := f(x), x \in I_{n,k}, k > k_2,$$

$$f_{n,s_1}(x) := s_1^n, x \in R^+ - I_n \bigcup I_{n,s_1},$$

c) Finally, if $r_{n,s_1} > b_{n,k_n}$, then set

$$f_{n,s_1} := f(x), x \in I_{n,s_1}$$

and

$$f_{n,s_1}(x) := f(r_{n,s_1}), x \in R^+ - I_{n,s_1}.$$

For any $n, n > n_0$, the next inequality could be easily verified

$$\rho_n(f) \leq \rho_n(f_{n,s_1}) + 1/s_1^n \tag{6}$$

As we see from here, it suffieces to show that $\limsup \rho_n(f_{n,s_1})^{1/n} < 1$.

Further, we may apparently assume that (for any $n, n \in \mathbf{N}, n \geq n_0$ the inequality

$$\rho_n(f_{n,s_1}) \geq 1/s_1^n \tag{7}$$

is fulfilled.

Indeed, denoting by p_{n,s_1} the polynomial for which the infimum $\rho_n(f_{n,s_1})$ is attained and assuming that (7) is false, we conclude that in cases a) and b) there holds that $1/p_{n,s_1}(x) \geq \rho_n(f_{n,s_1}) - 1/s_1^n > 0$ for every positive real number x sufficiently ($x > b_{n,k_n}$), which is obviously impossible. Consequently, (7) holds in the cases a) and b). As for case c), we have in general $\rho_n(f_{n,s_1}) \geq 1/f(r_{n,s_1})$ so that (7) may be destroyed. For such a case we may exlude of the considerations the possibility $\rho_n(f_{n,s_1}) \leq 1/s_1^n$, since otherwise we get, by (6), the requiered geometrical estimation of $\rho_n(f)$.

Coming back to (7), we see that all the (+)-alternation points of the difference $\frac{1}{f_{n,s_1}} - \frac{1}{p_{n,s_1}}$, say y^+, lie on I_n.

Assume, for the first that there are (+)-alternation points those lie on $\bigcup_{k>2} I_{n,k}$. Then, using (5), we obtain

$$\rho_n(f_{n,s_1}) < 1/s_1^{n\delta_n}.$$

Hence, for $n > n_0$ we get

$$\rho_n(f_{n,s_1}) \leq 1/s_1^{n\alpha}. \tag{8}$$

Assume now that there is no (+)-alternation point that lies on $\bigcup_{k\geq 2} I_{n,k}$. By the preceding argumentation, we get that all they lie in $[0, r_n)$. We concern ourselves with the case when

$$\rho_n(f_{n,s_1}) > 1/s_1^{n\delta_n}. \tag{9}$$

In fact, if $\rho_n(f_{n,s_1}) \leq 1/s_1^{n\delta_n}$, then we come accordingly to (6) again to inequality (8).

Under the assumption (9), we see that the (+)- alternation points have to be be located on $[0, r_n)\bigcap I_{s_1^{n\delta_n}}$. (Notice the obvious relation $[0, r_n)\bigcap I_{s_1^{n\delta_n}} \in [0, r_n)$).

Consider first the case when $\deg p_{n,s-1} < n$. In view to Lemma 1 and to the previous considerations, the set of all the alternation points belongs to $[0, r_n)$. Hence for $\rho_n(f_{n,s_1})$ we obtain

$$\rho_n(f_{n,s_1}) = \rho_{n-1}(f, r_n).$$

Applying now successively Lemma 2, Lemma 3 and condition (2) of Theorem 4', we get

$$\rho_{n-1}(f_{n,s_1}) \leq C_1.E_{n-1}(f, r_n) < C_1.K.\frac{\mathcal{M}_f(r_n, s)}{s^{n-1}(s-1)} < C_2(s).\frac{\|f\|^{\theta}_{r_n}}{s^n}$$

Taking account of the choice of the number r_n, we finally obtain

$$\rho_n(f_{n,s_1}) \leq C_2(s)\frac{s_1^{n\theta}}{s^n} \tag{10}$$

Consider now the case when $\deg p_{n,s-1} = n$. There are again two subcases: the set of all the alternation points belongs to the segment $[0, r_{n,s_1}]$ and there are alternation points those lie outside this segment. In the first case we have

$$\rho_n(f_{n,s_1}) = \rho_n(f, r_{n,s_1})$$

so that analogously to the previous case

$$\rho_n(f_{n,s_1})) \leq C_1.E_n(f, r_{n,s_1}) < C_1.K.\frac{\mathcal{M}_f(r_{n,s_1}, s)}{s^n(s-1)} < C_2(s).\frac{\|f\|^{\theta}_{r_{n,s_1}}}{s^n}$$

Taking account of the obvious fact that

$$\|f\|_{r_{n,s_1}} \leq \mathcal{M}_f(r_n, s),$$

and having im mind condition (2), we finally get

$$\rho_n(f_{n,s_1}) < C_2(s)\frac{s_1^{2n\theta}}{s^n} \tag{11}$$

As for the second case, we notice that there is only one alternation point y with

$$y > r_{n,s_1} \tag{12}$$

and it is necessarily a (-)-alternation point of the difference $\frac{1}{f_{n,s_1}} - \frac{1}{p_{n,s_1}}$. Recall that $r_{n,s_1} = \frac{r_n}{2}.(1+\frac{1}{2}(s_1+\frac{1}{s_1}))$ Let τ be the biggest (+)- alternation point; it suffices necessarily the inequality

$$\tau < r_n \tag{13}$$

In the case under consideration we have

$$\rho_n(f_{n,s_1}) = \rho_n(f(x), \tau, y, f_{n,s_1}(y))$$

so that, by mean of Lemma 2', we may write

$$\rho_n(f_{n,s_1}) \leq C_1.E_n(f(x), \tau, y, f_{n,s_1}(y)) < C_1.y.E_{n-1}(\frac{f(x) - f_{n,s_1}(y)}{x-y}, \tau)$$

The application of Lemma 3 leads to the estimation

$$E_{n-1}(\frac{f(x) - f_{n,s_1}(y)}{x-y}, \tau) \leq 2.C_1.\frac{\overline{\mathcal{M}}_{\frac{f(x)-f_{n,s_1}(y)}{x-y}}(\tau, s)}{s^{n-1}(s-1)} \leq \frac{C_3(s)}{y-\tau}.\frac{\mathcal{M}_{f(x)-f_{n,s_1}(y)}(\tau, s)}{s^n}$$

By mean of (2), (12) and (13), we have

$$\rho_n(f_{n,s_1}) \leq C_3(s).\frac{r_{n,s_1}}{r_{n,s_1} - r_n}.\frac{(2.K).s_1^{n\theta}}{s^n}.$$

which implies, finally

$$\rho_n(f_{n,s_1}) \leq C_4(s)\frac{s_1^{n\theta}}{s^n}.$$

Consequently, by (8), (10), (11) and the last inequality

$$\rho_n(f_{n,s_1}) \leq \max(1/s_1^n, 1/s_1^{n\alpha}, C_2(s)\frac{s_1 n\theta}{s^n}, C_2(s)\frac{s_1^{2n\theta}}{s^n}, \frac{(2.K).s_1^{n\theta}}{s^n}).$$

This inequality combined with (6) proves Theorem 4 for the special case when the function f is nowhere zero on R^+.

The proof in the general case does not contain essentially new elements. Assume $f(x) = w(x).\phi(x)$, where $w(x) = \prod_{i=1}^{l}(x - x_i)^{\beta_i}$, $\sum_{i=1}^{l} \beta_i = \beta$ and $\phi(x)$ is positive on R^+. It was shown in [Blatt3] that for every $n > \beta$ the polynomial p_n for which $\rho_n(f)$ is achieved, is given by $p_n = p + w^2(x).p_n^*$, where $p_n^* \in \pi_{n-2\beta}$, and the polynomial p is determined by the conditions $p^{(k)}(x_i) = f^{(k)}(x_i)$ for $k = 0, ..., 2\beta_i - 1, i = 1, ..., l$. There exists a polynomial P of degree $\leq 3\beta - 1$ such that $f(x) - P(x) = w^3(x).F(x)$; in the conditions of Theorem 4, the function $F(x)$ is of the kind presented in Theorem 4'. In [Blatt3], the validity of Lemma

2 was established with $E_n(f, E)$ replaced in (3) by $E_n(F, E)$. Analogously, $E_n(f, E, y, a)$ in (4'), Lemma 2', should be substituted by $E_n(F, E, y, a)$.

Taking account of these results, the proof of Theorem 4 repeats the previous ideas.

Theorem 4 is completely proved.

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