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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

On Certain Duffin and Schaeffer Type Inequalities

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On Certain Duffin and Schaeffer Type Inequalities *

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Abstract

Duffin and Schaeffer type inequalities related to some ultraspherical polynomials are established. One of the results obtained reads as follows:

THEOREM. Let $\{t_j\}_{j=1}^{n-1}$ be the zeros of the ultraspherical polynomial $P_{n-1}^{(\lambda)}$, $t_0 := -1$, $t_n := 1$. Let f be a real algebraic polynomial of degree not exceeding n and satisfying the inequalities

$$|f(t_j| \leq |P_n^{(\lambda)}(t_j)|,\ j = 0, \ldots, n.$$

Then the uniform norms of $f^{(k)}$ and $\frac{d^k}{dx^k} P_n^{(\lambda)}$ satisfy

$$\|f^{(k)}\| \leq \|\frac{d^k}{dx^k} P_n^{(\lambda)}\|$$

for each $k \in \{1, \ldots, n\}$, if $\lambda \geq 1$, and for each $k \in \{2, \ldots, n\}$, if $\lambda \in [1/2, 1)$.

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1. INTRODUCTION

Answering a question of the prominent russian chemist D. Mendeleev, in 1890 A. A. Markov proved that if $f(x) = \sum_{i=0}^{n} a_i x^i$ is a real algebraic polynomial of degree at most n such that $|f(x)| \leq 1$ in [-1,1], then in the same interval

$$|f'(x)| \leq n^2.$$

Two years later, in 1892, A. Markov's younger brother V. A. Markov (being that time a student at St Petersburg University) extended this result proving the following

THEOREM A. *If $f(x) = \sum_{i=0}^{n} a_i x^i$ is a real algebraic polynomial of degree not exceeding n and $|f(x)| \leq 1$ in [-1,1], then*

$$\max_{-1\leq x\leq 1} |f^{(k)}(x)| \leq \frac{n^2(n^2-1^2)\dots(n^2-(k-1)^2)}{1.3.\dots(2k-1)} = T_n^{(k)}(1) \qquad (1.1)$$

for $k = 1, 2, \dots, n$. Equality holds only for $f(x) = \pm T_n(x) = \pm\cos(n.\arccos x)$.

The inequalities of the brothers Markov type has been a challenge for many mathematicians. In 1941 Duffin and Schaeffer [DS] strengthened Theorem A proving that the inequality (1.1) remains true if the requirement $|f(x)| \leq 1$ in [-1,1] is replaced by

$$|f(\eta_j^n)| \leq 1, \quad j = 0, 1, \dots, n, \qquad (1.2)$$

where $\eta_j^n = \cos\frac{j\pi}{n}$ are the points of local extrema of $T_n(x)$ in [-1,1]. In addition, Duffin and Schaeffer showed that (1.1) fails to hold if the conditions (1.2) are replaced by $f|_E \leq 1$, where E is any closed set of points in [-1,1] which does not contain all the points $\{\eta_j^n\}$. In fact Duffin and Schaeffer proved a more general result including inequality for polynomials in a strip in the complex plane, but it does not fall in the frame of this paper. We only mention that their proof involves complex arguments, in particular the Rouche theorem.

Denote by π_n the class of all real algebraic polynomials of degree not exceeding n, and by $\mathbb{P}_n$ the subset of π_n containing only polynomials with n distinct real zeros, located in (-1,1). In our notations Q_n will mean a given algebraic polynomial of exact degree n, and $\|f\| := \sup_{x\in[-1,1]}|f(x)|$. We now formulate our definition for Duffin and Schaeffer type inequality (DS-inequality).

Duffin and Schaeffer Type Inequality The polynomial Q_n and the mesh $\Delta = \{t_j\}_{j=0}^{n}$ $(-1 = t_0 < t_1 < \cdots < t_n = 1)$ are said to admit DS-inequality if for an arbitrary $f \in \pi_n$ the assumptions $|f(t_j)| \leq |Q_n(t_j)|$ $(j = 0, 1, \ldots, n)$ imply the inequalities $\|f^{(k)}\| \leq \|Q_n^{(k)}\|$ for $k = 1, 2, \ldots, n$.

Note that the inequalities DS-type do not hold unconditionally. The validity of such inequalities depends on the choice of the majorant O_n and on the mesh Δ. Actually, to the best of our knowledge, only a few DS-inequalities of the above mentioned type are hitherto known. In [RS] and [RW] the original idea of Duffin and Schaeffer was modified for derivation of some new DS-inequalities.

In a recent paper A. Shadrin [AS] turned back to the original V. Markov's idea - Lagrange interpolation. He presented a simple proof of Theorem A under conditions (1.2). The crucial part for his proof is

Theorem B. *Let $q \in \mathbb{P}_n$, and let $t_j = t_j(q)$ $(j = 0, \ldots, n)$ be the points of all local extrema of q in $[-1, 1]$. Suppose that $f \in \pi_n$ and*

$$|f(t_j)| \leq |q(t_j)|, \quad j = 0, \ldots, n.$$

Then, for every $x \in [-1, 1]$ and for k=1,…,n,

$$|f^{(k)}(x)| \leq \max\{|q^{(k)}(x)|, |\frac{1}{k}(x^2 - 1)q^{(k+1)}(x) + xq^{(k)}(x)|\}.$$

Moreover, Shadrin has conjectured that DS-inequality holds for every $Q_n \in \mathbb{P}_n$ provided the mesh Δ is taken to contain the points of local extrema of Q_n in [-1,1], i.e., if $\Delta = \{-1\} \cup \{t : Q_n'(t) = 0\} \cup \{1\}$. Unfortunately, as some simple examples show, this conjecture is not true in general. Nevertheless, using Theorem B, Bojanov and Nikolov [BN] proved that DS-type inequality holds for such a choice of Δ with majorant $Q_n = P_n^{(\lambda)}$ - the ultraspherical polynomial (the polynomial, orthogonal in [-1,1] with respect to the weight $(1 - x^2)^{\lambda - 1/2}$).

THEOREM C. *Let* $t_j := t_j(P_n^{(\lambda)})$, $(j = 0, \ldots, n)$ *be the extremal points of* $P_n^{(\lambda)}$ *in* $[-1, 1]$. *Let* $f \in \pi_n$ *satisfy*

$$|f(t_j)| \leq |P_n^{(\lambda)}(t_j)|, \; j = 0, \ldots, n.$$

Then the inequality

$$\|f^{(k)}\| \leq \|\frac{d^k}{dx^k} P_n^{(\lambda)}\|$$

holds for all $k \in \{1, \ldots, n\}$, *if* $\lambda \geq 0$, *and for all* $k \in \{2, \ldots, n\}$, *if* $-1/2 \leq \lambda < 0$.

A very interesting result (though not exactly of DS-type) is established in [BR]. There, inequalities for the norms of the derivatives of polynomials are found on the basis of comparison of their corresponding local extrema.

The contents of this paper is organized as follows. In Section 2 we give some preliminary results, including the V. Markov's lemmas about interlacing property of zeros of polynomials, Chebyshev and Zolotarev intervals associated with a pointwise estimate of polynomial derivatives, as well as some properties of the ultraspherical polynomials. In Section 3 we extend the pointwise inequality given by Theorem B (this is the content of Theorems 3.1 and 3.2). Precisely, starting from a fixed mesh Δ we obtain a family of polynomials which may serve as majorants in DS-type inequalities related to Δ. In Section 4 we apply this extention to obtain DS-type inequalities for $Q_n = P_n^{(\lambda)}$ with $\Delta = \{t_j\}_{j=0}^n$, $t_0 = -1$, $t_n = 1$ and $\{t_j\}_1^{n-1}$ being the zeros of $P_{n-1}^{(\lambda)}$ (Theorem 4.1). In section 5 we establish DS-type inequalities for similar choice of Δ but for majorants that vanish at the points -1 and 1 (Theorem 5.1). Section 6 contains some comments and remarks.

2. AUXILIARY RESULTS

The following two lemmas belong to V.A. Markov and reveal the very interesting fact, that if two polynomials have only real simple zeros, which interlace, then the interlacing property remains valid also for their derivatives.

LEMMA 2.1. *Let* $b_1 > b_2 > \cdots > b_{s+1}$; $c_1 > c_2 > \cdots > c_s$, *and let* $b_1 \geq c_1 \geq b_2 \cdots \geq c_s \geq b_{s+1}$. *Let* $p(t) = \prod_{i=1}^{s}(t - c_i)$ *and* $q(t) = \prod_{i=1}^{s+1}(t - b_i)$.

Then for $1 \le k \le s-1$ *the zeros of* $p^{(k)}(t) : \gamma_1 > \gamma_2 > \cdots > \gamma_{s-k}$ *and the zeros of* $q^{(k)}(t) : \beta_1 > \beta_2 > \cdots > \beta_{s+1-k}$ *interlace, i.e., satisfy the inequalities*

$$\beta_1 > \gamma_1 > \beta_2 > \cdots > \beta_{s-k} > \gamma_{s-k} > \beta_{s+1-k}.$$

LEMMA 2.2. *Let* $b_1 > b_2 > \cdots > b_s$*;* $c_1 > c_2 > \cdots > c_s$*, and let* $b_1 \ge c_1 \ge b_2 \cdots \ge c_s$ *with* $b_j \ne c_j$ *for at least one* j*. Let* $p(t) = \prod_{i=1}^{s}(t - c_i)$ *and* $q(t) = \prod_{i=1}^{s}(t - b_i)$.

Then for $1 \le k \le s-1$ *the zeros of* $p^{(k)}(t) : \gamma_1 > \gamma_2 > \cdots > \gamma_{s-k}$ *and the zeros of* $q^{(k)}(t) : \beta_1 > \beta_2 > \cdots > \beta_{s-k}$ *satisfy the inequalities*

$$\beta_1 > \gamma_1 > \beta_2 > \cdots > \beta_{s-k} > \gamma_{s-k}.$$

As it is pointed out by Prof. Bojanov in ([BB], p.39), the assertion of Lemma 2.2 could be regarded also as monotone dependence of the zeros of the derivative with respect to the zeros of the polynomial. For the proof of lemmas 2.1-2.2 the reader may refer [AS] or Rivlin's book ([TR], Lemma 2.7.1).

The next lemma can be found in ([AS], Lemma 2)). It summarizes some V.A. Markov's observations concerning the pointwise estimates for derivatives of a polynomial. Its proof is based on Lemmas 2.1-2.2 and the Lagrange interpolation formula.

LEMMA 2.3. *Let* $\omega \in \mathbb{P}_{n+1}$ *have zeros* $\{t_j\}_{j=0}^{n}$*. Let* $Q_n \in \mathbb{P}_n$ *be a fixed polynomial, such that* $Q_n(t_{j-1})Q_n(t_j) < 0$ *for* $j = 1, \ldots, n$*. If* $f \in \pi_n$ *satisfies the inequalities*

$$|f(t_j)| \le |Q_n(t_j)| \quad for\ j = 0, \ldots, n,$$

then for every $k \in \{1, \ldots, n\}$ *there exists a set* $I_{n,k} = I_{n,k}(\omega)$*, such that*

$$|f^{(k)}(x)| \le |Q_n^{(k)}(x)| \quad for\ all\ x \in I_{n,k}.$$

The set $I_{n,k}$ *is given by*

$$I_{n,k} = [-1, \alpha_1^k] \cup [\beta_1^k, \alpha_2^k] \cup \ldots \cup [\beta_{n-k-1}^k, \alpha_{n-k}^k] \cup [\beta_{n-k}^k, 1], \tag{2.1}$$

where $\{\alpha_j^k\}_1^{n-k}$ *and* $\{\beta_j^k\}_1^{n-k}$ *are the ordered zeros of* $\omega_0^{(k)}$ *and* $\omega_n^{(k)}$*, respectively, and* $\omega_j(x) = \omega(x)/(x - t_j)$.

REMARK 1. The conditions $Q_n(t_{j-1})Q_n(t_j) < 0$ for $j = 1, \ldots, n$ can be replaced with the weaker requirement the zeros $\{\theta_j\}_1^n$ of Q_n to interlace with the zeros of ω, i.e., to satisfy the inequalities $t_0 \le \theta_1 \le t_1 \le \cdots \le \theta_n \le t_n$. Moreover, Q_n can be allowed to have a zero at ± 1, then ω and f must vanish at this point, too.

The $n-k+1$ intervals forming $I_{n,k}$ are known as the Chebyshev intervals. In fact, Lemma 2.3 remains true with the first and the last intervals in (2.1) replaced by $(-\infty, \alpha_1^k]$ and $[\beta_{n-k}^k, \infty)$. The intervals (α_j, β_j) $(j = 1, \ldots, n-k)$ are called the Zolotarev intervals (see e.g. [VG]). We denote the complementery set by $J_{n,k} := J_{n,k}(\omega)$,

$$J_{n,k} = [-1, 1] \setminus I_{n,k} = \cup_{j=1}^{n-k} (\alpha_j^k, \beta_j^k).$$

Next, we list some properties of the ultraspherical polynomials $P_n^{(\lambda)}$ which will be needed for the proofs of Theorems 4.1 and 5.1.

Properties:

(i) $y = P_n^{(\lambda)}$ satisfies the differential equation

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0 \; ;$$

(ii) For $\lambda > 0$, $\|P_n^{(\lambda)}\| = |P_n^{(\lambda)}(\pm 1)|$;

(iii) $\dfrac{d}{dx} P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x)$;

(iv) $\dfrac{d}{dx} P_{n+1}^{(\lambda)}(x) = x \dfrac{d}{dx} P_n^{(\lambda)}(x) + (n + 2\lambda) P_n^{(\lambda)}(x)$;

(v) For $\lambda > 0$ the ultraspherical polynomials obey the representation

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{n} a_{n,m}(\lambda) T_m(x)$$

with positive coefficients $a_{n,m}(\lambda)$.

The proof of these properties can be found in the book of Szegö [GS] (concerning property (v), the reader can find a more general statement in ([TR], p.158, Remark 1)).

We conclude this section with a lemma, based on the property (v).

LEMMA 2.4. *Let $q = P_n^{(\lambda)}$, $\lambda \ge 0$. Then for $k = 1, 2, \ldots, n$ and for every $s \ge k$*

$$\left\| \frac{x^2 - 1}{s} q^{(k+1)}(x) + x q^{(k)}(x) \right\| = |q^{(k)}(\pm 1)|. \tag{2.2}$$

For $\lambda \in [-1/2, 0)$ the equality (2.2) holds for $k = 2, \ldots, n$.

Proof. We apply the approach proposed in [BN]. Instead of (2.2) we shall show that for all $x \in [-1,1]$ and for $\lambda \geq 0$

$$|(x^2-1)q^{(k+1)}(x) + sxq^{(k)}(x)| \leq sq^{(k)}(1). \tag{2.3}$$

In the case $s = k$ and $q = T_n$ (i.e., for $\lambda = 0$) (2.3) has already been proved by Shadrin ([AS], Lemma 3). Then, for $\lambda > 0$, we make use of the properties (v) and (ii) to obtain

$$\begin{aligned}
&|(x^2-1)q^{(k+1)}(x) + kxq^{(k)}(x)| \\
&= |(x^2-1)\sum_{m=0}^{n} a_{n,m}(\lambda)T_m^{(k+1)}(x) + kx\sum_{m=0}^{n} a_{n,m}(\lambda)T_m^{(k)}(x)| \\
&\leq \sum_{m=0}^{n} a_{n,m}(\lambda)|(x^2-1)T_m^{(k+1)}(x) + kxT_m^{(k)}(x)| \\
&\leq \sum_{m=0}^{n} a_{n,m}(\lambda)kT_m^{(k)}(1) = kq^{(k)}(1),
\end{aligned}$$

proving in such a way (2.3) for $s = k$. For $s > k$ we have

$$\begin{aligned}
&|(x^2-1)q^{(k+1)}(x) + sxq^{(k)}(x)| \\
&\leq |(x^2-1)q^{(k+1)}(x) + kxq^{(k)}(x)| + |(s-k)xq^{(k)}(x)| \\
&\leq kq^{(k)}(1) + (s-k)q^{(k)}(1) = sq^{(k)}(1).
\end{aligned}$$

In the last step we have taken into account that, according to (iii) $q^{(k)}$ is ultraspherical polynomial, too, and therefore in view of (ii) for $x \in [-1,1]$ $|xq^{(k)}(x)| \leq q^{(k)}(1)$.

Finally, for the case $\lambda \in (-1/2, 0)$ one can apply the above arguments to $q'(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x)$. The proof of lemma is completed. ∎

REMARK 2. In [BN] the same reasonings are applied for the proof of Theorem C, the case $\lambda \geq 0$, while the proof of the case $\lambda \in (-1/2, 0)$ relies on different arguments. Lemma 2.4 furnishes a short proof of Theorem C for both cases.

3. POINTWISE INEQUALITIES

For the proof of our DS-inequalities we will need the following extension of Theorem B.

THEOREM 3.1. *Let* $q \in \mathbb{P}_n$ *and let* $\{t_j\}_{j=1}^{n-1}$ *be the zeros of* q', $t_0 := -1$, $t_n := 1$. *Let* $Q_n(x) = mxq'(x) + q(x)$, *where* m *is a real parameter such that*

$$m \geq \max\{\frac{q(-1)}{q'(-1)}, -\frac{q(1)}{q'(1)}\}. \tag{3.1}$$

If $f \in \pi_n$ *satisfies the inequalities*

$$|f(t_j)| \leq |Q_n(t_j)| \ \text{for } j = 0, \dots, n, \tag{3.2}$$

then for all $k \in \{1, \dots, n\}$ *and for every* $x \in [-1, 1]$

$$|f^{(k)}(x)| \leq \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}, \tag{3.3}$$

where

$$Z_{n,k}(x) = (\frac{x^2 - 1}{k} - m)q^{(k+1)}(x) + xq^{(k)}(x). \tag{3.4}$$

Proof. Without any restriction we may regard the leading coefficient of q positive. It is easily seen that the polynomial Q_n has exactly n real zeros, which interlace with the zeros of $\omega(x) = (x^2 - 1)q'(x)$. Indeed, we have

$$\operatorname{sign} Q_n(t_j) = \operatorname{sign} q(t_j) = (-1)^{n-j} \text{ for } j = 0, \dots, n-1, \tag{3.5}$$

hence each of the intervals (t_{j-1}, t_j), $(j = 2, \dots, n-1)$ contains a zero of Q_n. The requirement (3.1) implies $Q_n(1) = mq'(1) + q(1) \geq 0$, while $Q_n(t_{n-1}) < 0$ in view of (3.5), therefore Q_n has a zero in the interval $(t_{n-1}, t_n]$. Analogously, we obtain the existence of one additional zero of Q_n located in $[t_0, t_1)$. Thus we established the desired interlacing property. Moreover, it follows from (3.5) that Q_n has a positive leading coefficient. By the above reasons we conclude that the zeros of $\omega_0(x) = (x+1)q'(x)$ and $Q_n(x)$ as well as the zeros of $Q_n(x)$ and $\omega_n(x) = (x-1)q'(x)$ interlace. Then Lemma 2.2 asserts the same property for the zeros of their k-th derivatives, i.e., for $j = 1, \dots, n-k$ the j-th zero of $Q_n^{(k)}$ is surrounded by the j-th zero α_j^k of $\omega_0^{(k)}$ and the j-th zero β_j^k of $\omega_n^{(k)}$, and consequently

$$\operatorname{sign} Q_n^{(k)}(\alpha_j^k) = -\operatorname{sign} Q_n^{(k)}(\beta_j^k) = (-1)^{n-k-j-1} \ (j = 1, \dots, n-k). \tag{3.6}$$

By direct calculations we obtain that $Q_n^{(k)}$ and $Z_{n,k}(x)$ satisfy the relationships

$$Z_{n,k}(x) - Q_n^{(k)}(x) = \frac{1}{k}(x - 1 - km)\omega_0^{(k)}(x), \tag{3.7}$$

$$Z_{n,k}(x)+Q_n^{(k)}(x)=\frac{1}{k}(x+1+km)\omega_n^{(k)}(x). \tag{3.8}$$

We therefore have for $j=1,\ldots,n-k$

$$Z_{n,k}(x)=\begin{cases} Q_n^{(k)}(x) & \text{for } x=\alpha_j^k, \\ -Q_n^{(k)}(x) & \text{for } x=\beta_j^k. \end{cases} \tag{3.9}$$

The last relation yields, by virtue of (3.6),

$$\operatorname{sign} Z_{n,k}(\beta_j^k)=-\operatorname{sign} Z_{n,k}(\alpha_{j+1}^k)=(-1)^{n-k-j-1} \tag{3.10}$$

for $j=1,\ldots,n-k-1$. Moreover, since

$$Z_{n,k}(\beta_{n-k}^k)-Q_n^{(k)}(\beta_{n-k}^k)=-2Q_n^{(k)}(\beta_{n-k}^k)<0,$$

while $Z_{n,k}-Q_n^{(k)}>0$ for x large enough, it follows that $x_0=x_0(k):=1+km$ is the last zero of $Z_{n,k}-Q_n^{(k)}$, i.e., $x_0>\beta_{n-k}^k$. By similar reasonings, $-x_0<\alpha_1^k$.

Now let $f\in\pi_n$ be an arbitrary polynomial satisfying (3.2), then according to Lemma 2.3 the k-th derivatives of f and Q_n satisfy the inequalities

$$|f^{(k)}(x)|\le|Q_n^{(k)}(x)| \text{ for all } x\in I_{n,k}. \tag{3.11}$$

The theorem will be proved if we show that

$$|f^{(k)}(x)|\le|Z_{n,k}(x)| \text{ for all } x\in J_{n,k}. \tag{3.12}$$

From (3.9) and (3.11), for $j=1,\ldots,n-k$ we get

$$|f^{(k)}(\alpha_j^k)|\le|Z_{n,k}(\alpha_j^k)|, \tag{3.13}$$

$$|f^{(k)}(\beta_j^k)|\le|Z_{n,k}(\beta_j^k)|. \tag{3.14}$$

This coupled with (3.10) yields

$$(Z_{n,k}\pm f^{(k)})(\beta_j^k).(Z_{n,k}\pm f^{(k)})(\alpha_{j+1}^k)\le 0,$$

therefore $Z_{n,k}\pm f^{(k)}$ has at least one zero in $[\beta_j^k,\alpha_{j+1}^k]$ for $(j=1,\ldots,n-k-1)$. The same observation applies to the intervals $[-x_0,\alpha_1^k)$ and $(\beta_{n-k}^k,x_0]$. We prove this for the interval $(\beta_{n-k}^k,x_0]$, the proof of the second case is analogous. Since $Z_{n,k}$ is a polynomial of exact degree $n-k+1$ with a positive leading coefficient and x_0 is the last zero of $Z_{n,k} - Q_n^{(k)}$, we have $Z_{n,k}(x) \geq Q_n^{(k)}(x) > 0$ for $x \geq x_0$ (we recall that the zeros of $Q_n^{(k)}$ are located in the interval $(\alpha_1^k, \beta_{n-k}^k)$). Then, from $|f^{(k)}(x)| \leq |Q_n^{(k)}(x)|$ for $x \geq \beta_{n-k}^k$, we obtain

$$\text{sign}\,\{(Z_{n,k} \pm f^{(k)})(x_0)\} \geq 0.$$

On the other hand, according to (3.9), $Z_{n,k}(\beta_{n-k}^k) = -Q_n^k(\beta_{n-k}^k) < 0$, therefore

$$\text{sign}\,\{(Z_{n,k} \pm f^{(k)})(\beta_{n-k}^k)\} = -1,$$

whence the desired result holds.

Thus, we showed that each of the polynomials $Z_{n,k} \pm f^{(k)}$ has at least $n-k+1$ distinct zeros, located outside $J_{n,k}$. Since their degree is $n-k+1$, they have no zeros in $J_{n,k}$. The inequality (3.12) then holds by virtue of (3.13)-(3.14). The theorem is proved. ∎

REMARK 3. The requirement (3.1) is fulfilled, e.g., if $m \geq 0$. In the special case $m = 0$ Theorem 3.1 reproduces Shadrin's Theorem B.

Theorem 3.1 treats the symmetric case only, but applying the same arguments as above one can extend it as follows (the proof is identical with that of Theorem 3.1, therefore we omit it here):

THEOREM 3.2. *Let $q \in \mathbb{P}_n$ and let $\{t_j\}_{j=1}^{n-1}$ be the zeros of q', $t_0 := -1$, $t_n := 1$. Let $Q_n(x) = (mx+s)q'(x) + q(x)$, where m and s are real parameters such that*

$$m - s \geq \frac{q(-1)}{q'(-1)}, \quad m + s \geq -\frac{q(1)}{q'(1)}.$$

If $f \in \pi_n$ satisfies the inequalities

$$|f(t_j)| \leq |Q_n(t_j)| \ \text{for } j = 0, \ldots, n,$$

then for all $k \in \{1, \ldots, n\}$ and for every $x \in [-1, 1]$

$$|f^{(k)}(x)| \leq \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) = \left(\frac{x^2-1}{k} - sx - m\right)q^{(k+1)}(x) + (x - ks)q^{(k)}(x).$$

4. A Duffin and Schaeffer Inequality

As an application of Theorem 3.1 we prove in this section another DS-type inequality where the majorant Q_n is the ultraspherical polynomial $P_n^{(\lambda)}$.

THEOREM 4.1. *Let $\{t_j\}_{j=1}^{n-1}$ be the zeros $P_{n-1}^{(\lambda)}$ $(\lambda \geq 1)$, $t_0 := -1$, $t_n := 1$. If $f \in \pi_n$ satisfies the inequalities*

$$|f(t_j| \leq |P_n^{(\lambda)}(t_j)|, \; j = 0, \ldots, n,$$

then

$$\|f^{(k)}\| \leq \|\frac{d^k}{dx^k} P_n^{(\lambda)}\| \tag{4.1}$$

for each $k \in \{1, \ldots, n\}$.

For $\lambda \in [1/2, 1)$ the inequalities (4.1) hold for $k \geq 2$.

Proof. We set $q := P_n^{(\mu)}$, $\lambda = \mu + 1$ in the statement of Theorem 3.1, then property (iii) yields $q' = 2\mu P_{n-1}^{(\lambda)}$, therefore $\{t_j\}_1^{n-1}$ are the zeros of $P_{n-1}^{(\lambda)}$.

We consider firstly the case $\mu > 0$. For $Q_n = P_n^{(\lambda)}$ properties (iii) and (iv) yield

$$Q_n(x) = \frac{1}{2\mu} \cdot \frac{d}{dx}\{P_{n+1}^{(\mu)}(x)\} = \frac{n+2\mu}{2\mu}\left(\frac{1}{n+2\mu} x q'(x) + q(x)\right), \tag{4.2}$$

and since $1/(n+2\mu) > 0$, we can apply Theorem 3.1 to obtain $|f^{(k)}(x)| \leq \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$, where

$$Z_{n,k}(x) = \frac{1}{2\mu}\left[(n+2\mu)[\frac{x^2-1}{k} q^{(k+1)}(x) + x q^{(k)}(x)] - q^{(k+1)}(x)\right]. \tag{4.3}$$

The theorem will be proved for $\lambda > 1$ if we succeed to show that $\|Q_n^{(k)}\| \geq \|Z_{n,k}\|$. In order to prove this we apply Lemma 2.4 with $s = k$. We have

$$\begin{aligned} 2\mu\|Z_{n,k}\| &\leq (n+2\mu)\|\tfrac{x^2-1}{k} q^{(k+1)}(x) + x q^{(k)}(x)\| + \|q^{(k+1)}\| \\ &= (n+2\mu) q^{(k)}(1) + q^{(k+1)}(1). \end{aligned} \tag{4.4}$$

On the other hand, from (4.2) we obtain

$$\begin{aligned} 2\mu\|Q_n^{(k)}\| &= \|x q^{(k+1)}(x) + (n+2\mu+k) q^{(k)}(x)\| \\ &= (n+2\mu+k) q^{(k)}(1) + q^{(k+1)}(1), \end{aligned} \tag{4.5}$$

where we have taken into account property (ii) of ultraspherical polynomials. Now comparison of the right hand sides (4.4) and (4.5) asserts the desired result, hence Theorem 4.1 is proved for $\mu > 0$ (i.e., for $\lambda > 1$). The proof of the case $\lambda = 1$ needs a slight modification because of the different normalization of the Chebyshev polynomials of first and second kind. Namely, we can replace property (iii) with $T_n'(x) = nU_{n-1}(x)$, the identity (4.2) with $U_n(x) = \frac{1}{n}xT_n'(x) + T_n(x)$ and to repeat the above arguments to prove Theorem 4.1 for $\lambda = 1$. Finally, the case $\lambda \in [1/2, 1)$ is treated in a similar way, however, in this case Lemma 2.4 is valid for $k \geq 2$ only. The proof of Theorem 4.1 is completed. ∎

5. A Duffin-Schaeffer-Schur Inequality

A Duffin-Schaeffer-Schur inequality (DSS-inequality) is called any DS-type inequality, for which the majorant Q_n vanishes at the end points $t_0 = -1$ and $t_n = 1$. The reason is I. Shur's paper [IS], where A. Markov's problem has been examined subject to zero boundary conditions.

In this section we discuss the possibility for derivation of DSS-inequalities on the basis of Theorem 3.1. Our starting point will be the property (i) of the ultraspherical polynomials. With $q = P_n^{(\lambda)}$ we have the representation

$$(x^2 - 1)q''(x) = n(n + 2\lambda)[-\frac{2\lambda + 1}{n(n + 2\lambda)}xq'(x) + q(x)] \qquad (5.1)$$

Clearly, the parameter $m = -\dfrac{2\lambda + 1}{n(n + 2\lambda)}$ satisfies the requirement (3.1) with equality sign, therefore Theorem 3.1 is valid with $Q_n(x) = (x^2 - 1)q''(x)$. In this special case it reads as follows:

THEOREM 3.1′. *Let $q = P_n^{(\lambda)}$, let $\{t_j\}_{j=1}^{n-1}$ be the zeros of q', and $t_0 := -1$, $t_n := 1$. Let $Q_n(x) = (x^2 - 1)q''(x)$. If $f \in \pi_n$ satisfies*

$$|f(t_i)| \leq |Q_n(t_i)|, \quad i = 0, \ldots, n,$$

then for each $k \in \{1, \ldots, n\}$ and for every $x \in [-1, 1]$ we have the pointwise inequality

$$|f^{(k)}(x)| \leq \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) = n(n+2\lambda)[\frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x)] + (2\lambda+1)q^{(k+1)}(x). \quad (5.2)$$

In order to prove DSS-inequality, we have to show that $\|Q_n\| \geq \|Z_{n,k}\|$. It is easily seen on the basis of Lemma 2.4 that for $\lambda \geq 0$ or for $k \geq 2$

$$\|Z_{n,k}\| = |Z_{n,k}(1)| = n(n+2\lambda)|q^{(k)}(1)| + (2\lambda+1)|q^{(k+1)}(1)|.$$

On the other hand, it follows from the proof of Theorem 3.1 that in this case $x_0(k) < 1$ and therefore $|Q_n^{(k)}(1)| < |Z_{n,k}(1)|$. These observations show that we may obtain DSS-inequality on the basis of Theorem 3.1′ only if $\|Q_n^{(k)}\|$ is attained at an interior point for [-1,1]. However, it is seen from (5.1) that this is not the case for λ equal or close to -1/2. In the following DSS-inequality we have chosen $\lambda \in [-1,-1/2]$ in sense of Theorem 3.1′.

THEOREM 5.1. *Let $(t_j)_1^{n-1}$ be the zeros of $q := P_{n-1}^{(\lambda)}$, ($\lambda \in [0,1/2]$), and let $t_0 = -1$, $t_n = 1$. Let $Q_n(x) = (x^2-1)q'(x)$. If $f \in \pi_n$ satisfies the inequalities*

$$|f(t_j)| \leq |Q_n(t_j)|, \quad j = 0, \ldots, n,$$

then for each $k \in \{2, \ldots, n\}$

$$\|f^{(k)}\| \leq \|Q_n^{(k)}\| \quad (5.3)$$

For $\lambda \in (-1/2, 0)$ the inequality (5.3) holds for $k \geq 3$.

Proof. Based on the the property (i) we obtain

$$Q_n^{(k)}(x) = (1-2\lambda)xq^{(k)}(x) + [n(n+2\lambda-2) + k(1-2\lambda)]q^{(k-1)}(x) \quad (5.4)$$

Following the reasonings from the proof of Theorem 3.1, we obtain in the same fashion $|f^{(k)}(x)| \leq \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$ for each $k \in \{1, \ldots, n\}$ and for every $x \in [-1,1]$, where

$$Z_{n,k}(x) = n(n+2\lambda-2)[\frac{x^2-1}{k}q^{(k)}(x) + xq^{(k-1)}(x)] + (2\lambda-1)q^{(k)}(x). \quad (5.5)$$

(both (5.4) and (5.5) can be derived from (5.1) and (5.2) by formal replacement of k by $k-1$ in the derivatives of q and λ by $\lambda - 1$). For $\lambda \in [0, 1/2]$ properties (ii) and (iii) and (5.4) imply

$$\|Q_n^{(k)}\| = (1-2\lambda)q^{(k)}(1) + [n(n+2\lambda-2) + k(1-2\lambda)]q^{(k-1)}(1). \qquad (5.6)$$

From (5.5) we get

$$\|Z_{n,k}\| \le n(n+2\lambda-2)\|\frac{x^2-1}{k}q^{(k)}(x) + xq^{(k-1)}(x)\| + (1-2\lambda)\|q^{(k)}\|.$$

Then application of Lemma 2.4 implies

$$\|\frac{x^2-1}{k}q^{(k)}(x) + xq^{(k-1)}(x)\| = q^{(k-1)}(1) \text{ for } k \ge 2,$$

whence

$$\|Z_{n,k}\| \le n(n+2\lambda-2)q^{(k-1)}(1) + (1-2\lambda)q^{(k)}(1) \text{ for } k \ge 2. \qquad (5.7)$$

The comparison of the right hand sides of (5.6) and (5.7) proves the assertion of the theorem for the case $\lambda \in [0, 1/2]$ (by $P_m^{(0)}$ we mean, as in Theorem 4.1, the Chebyshev polynomial T_m). Finally, the case $\lambda \in (-1/2, 0)$ is treated similarly, but in this case the inequality holds for $k \ge 3$ only. The theorem is proved. ∎

6. Concluding Remarks

1. Concerning DS-inequalities, some questions arise in a natural way. Such a question is, for instance, for a fixed majorant Q_n, what is the set of all meshes Δ admitting DS-type inequality ? As we already mentioned, the original DS-inequality fails to hold if in (1.2) some of the points η_j^n is omitted. However, is it not true that $\Delta = \{\eta_j^n\}_{j=0}^n$ is the unique mesh allowing DS-type inequality with $Q_n = T_n$. A trivial alternative choice is any $n+1$-tuple, containing the zeros of T_n. The converse question is, for a given mesh Δ (i.e., a set of $n+1$ distinct points, located in [-1,1]), what is the class of all majorants Q_n at these points, admitting Duffin and Schaeffer type inequality? Theorems 3.1 and 3.2 give some possible candidates for such majorants. As regards the first question, Theorem C and Theorem 4.1 assert that, for $Q_n = P_n^{(\lambda)}$ $(\lambda \ge 1)$

DS-inequality holds for two choices of a mesh $\Delta = \{t_j\}$, namely for $\{t_j\}_1^{n-1}$ being the zeros of $P_{n-1}^{(\lambda+1)}$ and $P_{n-1}^{(\lambda)}$. We conjecture that DS-inequality holds with $Q_n = P_n^{(\lambda)}$ for any choice of $\{t_j\}_1^{n-1}$ - zeros of $P_{n-1}^{(\mu)}$ with $\lambda \le \mu \le \lambda + 1$.

2. The special case $\lambda = 1/2$ in Theorem 5.1 corresponds to Theorem C ($\lambda = -1/2$) (see also ([BN], Theorem 3.2)), while Theorem 5.1 ($\lambda = 0$) reproduces a result of Rahman and Schmeisser ([RS], Theorem 2).

3. Theorem 3.2 may be applied for derivation of certain DS-inequalities for non-symmetric majorants, e.g., for $Q_n = P_n^{(\alpha,\beta)}$ - the Jacobi orthogonal polynomials.

References

[BB] B.D. Bojanov, An Inequality of Duffin and Schaeffer Type, East J. Approx., **1**(1995), 37-46.

[BN] B.D. Bojanov and G.P. Nikolov, Duffin and Schaeffer Type Inequality for Ultraspherical Polynomials, to appear in J. Approx. Theory

[BR] B.D. Bojanov and Q.I. Rahman, On Certain Extremal Problems for Polynomials, J. Math. Anal. Apl., 189(1995), 781-800.

[DS] R.J. Duffin and A.C. Schaeffer, A refinement of an inequality of the brothers Markoff, Trans. Amer. Math. Soc. **50** (1941), 517-528.

[VG] V. Gusev, Functionals of derivatives of an algebraic polynomial and V.A. Markov's theorem, Izv. Akad. Nauk SSSR Ser. Mat. **25**(1961), 367-384 (in Russian); English translation: Appendix, *"The Functional Method and Its Application"* (E.V. Voronovskaja,Ed.), Transl. of Math. Monographs, (**28**)1970, AMS, Providence, RI.

[AM] A.A. Markov, On a question of D.I. Mendeleev, Zap. Peterb. Akad. Nauk **62**(1890), 1-24 (in Russian).

[VM] V.A. Markov, On functions least deviated from zero in given interval, St. Petersburg, 1892 (in Russian). German translation: Über Polynome, die in einem gegeben Intervalle möglichst wenig von Null abweichen, Math. Ann. **77**(1916), 213-258.

[TR] T.J. Rivlin, *The Chebyshev Polynomials*, John Wiley & Sons, Inc., 1974.

[RS] Q.I. Rahman and G. Schmeisser, Markov-Duffin-Schaeffer inequality for polynomials with a circular majorant, Trans. Amer. Math Soc. **310**(1988), 693-702.

[RW] Q.I. Rahman and A.Q. Watt, Polynomials with a Parabolic Majorant and the Duffin-Schaeffer Inequality, J. Approx. Theory **69**(1992), 338-354.

[AS] A.Yu. Shadrin, Interpolation with Lagrange polynomials. A simple proof of Markov inequality and some of its generalizations, Approx. Theory and Appl. **8**(1992), 51-61.

[IS] I. Schur, Über das Maximum des absoluten Betrages eines Polinoms in einem gegebenen Intervall, Math. Z. **4**(1919), 271-287.

[GS] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, New York, 1959.