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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
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AN ANALOGUE OF MONTEL'S THEOREM TO SOME RATIONAL APPROXIMATING FUNCTIONS

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AN ANALOGUE OF MONTEL'S THEOREM TO SOME RATIONAL APPROXIMATING FUNCTIONS

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Abstract: In the present paper, we consider families of rational approximants with an unbounded number of the finite poles, analytic in a domain and not taking a given complex value in this domain. Under additional conditions, we establish the normality of these sequences.

Introduction

Let B be a domain in the extended complex plane $\overline{\mathbf{C}}$ and F - a family of functions, holomorphic (analytic and single valued) in the domain B. The set F is equipped with the sup-norm on compact subsets. In what follows, we shall deal with local uniform convergence inside B, that is an uniform convergence in the sup-norm on compact subsets there.

The family F is said to be normal on the compact subset $K, K \in B$, if each sequence $F', F \supset F'$ subsets is either uniformly bounded or converges uniformly to infinity on K. The family F is normal in the domain B, if it is normal on compact subsets in B. For normal families, we shall use the notation $F \in \mathcal{N}(B)$.

The classical result of Montel provides a sufficient condition for the normality of a family; that is

Theorem 1 (Montel): *Let F be a family of functions holomorphic in the domain B. If there are two points a and b, $a, b \in \mathbf{C}, a \neq b$ such that $f(z) \neq a, b$ for every $f \in F$ and $z \in B$ then F is a normal family in B.*

For details in the theory of the normal families, and for the proof of Montel's theorem, in particular, the reader is referred to [1], for instance.

We remind of the important generalization of Montel's theorem; that is

Theorem 1' (Montel): *If there are two points a and b, $a, b \in \mathbf{C}, a \neq b$ and a positive integer m such that each $f \in F$ takes the values a and b in B not more than m times, then*

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the family F is normal in the domain B.

If, in addition to Montel's conditions, the family F converges uniformly on some continuum K in B to a function f, then it necessarily converges locally uniformly inside the domain B itself; thus there is a holomorphic function F in $B, (F \in \mathcal{H}(B))$ with $f(z) = F(z)$ for every $z, z \in B$.

In the present paper, we show that for some sequences of rational approximating functions, it suffices only one finite point for the sequence under considerations to be normal. This means that the function, being approximated, admits a holomorphic continuation into a wider domain (provided the approximating sequences converge to f on a continuum belonging to the domain).

For simplicity, we shall deal with the zero-point. It is clear that the generality of the considerations will not be lost.

Let Δ be a real segment $[-1,1]$ and $\phi(z) = z + \sqrt{z^2 - 1}$ with $\phi(\infty) = \infty$. For each $\rho, \rho > 1$, we denote by E_ρ the ellipse $\mathcal{E}_\rho := \{z = x + iy, x^2/((\rho + 1/\rho)/2)^2 + y^2/((\rho - 1/\rho)/2)^2 \leq 1\}$; $\Gamma_\rho := \partial\mathcal{E}_\rho$. As it is known, $|\phi(z)|_{\Gamma_\rho} = \rho$.

Suppose, the function f is real-valued and continuous on $\Delta (f \in C_R(\Delta))$. For a nonnegative integer $n, (n \in \mathrm{N})$, we set

$$e_n = e_n(f, \Delta) = \inf \|f - p\|_\Delta,$$

where the infimum is taken over all the polynomials p of degree not exceeding n.

Let P_n be a polynomial of best uniform approximation to f on the segment Δ, that is:

$$\|f - P_n\|_\Delta = e_n$$

In the conditions for f, the polynomial P_n, $n \in \mathrm{N}$ always exists and is uniquely determined by the alternation - theorem of Chebyshev (see [2]).

We say that the function f is holomorphic on Δ, $(f \in \mathcal{H}(\Delta))$, if it admits a holomorphic continuation in some region E_ρ, $\rho > 1$. We define the radius $\rho_0 = \rho_0(f, \Delta)$ of holomorphy as follows: $\rho_0 = \sup\{\rho, \rho > 1, f \in \mathcal{H}(E_\rho)\}$. In the case when $f \notin \mathcal{H}(\Delta)$, we set $\rho_0 = 1$. S. N. Bernstein pointed at the relation between $\limsup_{n\to\infty} e_n^{1/n}$ and ρ_0. He proved, namely (see [1])

Theorem 2: *If $f \in C_R(\Delta)$, then*

$$\limsup_{n\to\infty} \|f - P_n\|_\Delta^{1/n} = 1/\rho_0(f).$$

Hence, $\{P_n\} \in \mathcal{N}(E_{\rho_0})$.

Bernstein also studied the connection between the distribution of the zeros of the sequence $\{P_n\}$ and the holomorphic continuability. The following theorem (see [3]) is valid.

Theorem 3: *Let $f \in C_R(\Delta)$. Assume that there is a domain $U, U \supset \Delta$, such that $P_n(z) \neq 0$ for each $z \in U$ and every $n \in N$ starting with a number n_o.*

Then $P_n(z) \to f$ locally uniformly inside U.

Herewith, $f \in \mathcal{H}(\mathcal{U})$ and $P_n(z) \in \mathcal{N}(U)$.

The fact that $P_n, n \in \mathbb{N}$, are polynomials of best uniform approximation plays an essential role in this statement. In general, this result is no longer valid for arbitrary polynomial sequences even behaiving in a "very good way" on the segment Δ. In fact, Bernstein showed (see [3]) that for each function $g \in C_R(\Delta)$, one can construct a sequence of polynomials P_n uniformly tending to g on Δ such that for every $\rho, \rho > 1$, the zeros of almost all P_n lie outside E_ρ. Later we shall extend Bernstein's result in direction interpolation polynomial sequences (Theorem 7).

Theorem 2 is the background for investigations connected with approximating sequences of rational functions. There are accomplished results for the case when the number of the finite poles is fixed. For instance, an analogue of Theorem 1 is valid (look [6]). For further analogous results we refer, for instance, to the papers [5],[6],[7],[8],[9] and [10].

In the present paper, we confine ourselves on rational functions with an unbounded number of the finite poles.

The first result in connection "an analogue of Montel's theorem to some rational sequences" was obtained by G. A. Baker and P. Graves-Morris in [11]; they considered the diagonal in the Pade - table of a given power series $f(z) = \sum_{n=0}^{\infty} f_n . z^n$. Under the assumption that all the rational Pade approximants $\pi_n(z)$ are holomorphic and different from zero in a disk $\mathcal{D}$ centered at the zero - point, they showed that the sequence $\{\pi_n(z)\}$ is normal (and, herewith, f admits continuation as an analytic function in $\mathcal{D}$).

Throughout the present paper, we will assume f not to be a rational function. Set

$$\varrho_n = \inf \|f - R\|_\Delta,$$

where the infimum is taken over the rational function of order (n, n). Let R_n be the function for which $\|f - R\|_\Delta = \varrho_n$. As in the case of polynomial approxiomation, the rational function $R_n, n \in N$, always exists and is unique determined by the alternation theorem of Chebyshev (see [2]). Further, it is a well known fact that

$$\varrho_n \to 0, \text{ as } n \to \infty$$

In 1988, Grothmann and Saff (see [12]) considered the rational functions R_n of best uniform approximation with an unbounded number of the finite poles and established an anologous result to Theorem 3; they proved, namely

Theorem 4: *Let $f \in C_R(\Delta)$. Assume that there is an ellipse $E_\rho, \rho > 1$, for which $R_n(z) \in \mathcal{H}(E_\rho)$ and $R_n(z) \neq 0$ for every $z \in E_\rho$ and each $n, n \in N$ starting with an integer.*

Then $\{R_n\} \in \mathcal{N}(E_\rho)$ and $f \in \mathcal{H}(E_\rho)$.

The result of Grothmann and Saff was extended for rational functions of best uniform approximation with a "small" number of zero-points in the domain under considertions. Let us denote, for each n, $n \in N$, and each compact set K in C, by $k(R_n, K)$ the number of the zeros of R_n on K. The following theorem is valid (see [13]).

Theorem 5: *Let $f \in C_R(\Delta)$. Assume that there is a domain U, $U \supset \Delta$ such that $R_n \in \mathcal{H}(U)$ for every n large enough. Assume further that $k(R_n, K) = o(n)$ as $n \to \infty$ for each compact subset K of U.*

Then $\{R_n\} \in \mathcal{N}(U)$ *and* $f \in \mathcal{H}(U)$.

Before formulating the next result, we denote for a number $n \in \mathbf{N}$ by $\tau_n(K)$ the number of the poles of R_n on the compact set K.

In the present paper, we prove the following theorem:

Theorem 6: *Let* $f \in C_R(\Delta)$. *Assume that there is a domain* U, $U \supset \Delta$, *such that* $k(R_n, K) = o(n)$ *and* $\tau_n(K) = o(n/\log n)$, *as* $n \to \infty$ *for each compact subset* K *of* U. *Assume , further, that there is a domain* $D \supset \Delta$ *such that* $R_n \in \mathcal{H}(V)$ *for every* n *starting with a number* n.

Then the family $\{R_n\}$ *is normal everywhere in* U *except for a set of zero* m_1 *- measure.*

If there is an integer $m, m \in \mathbf{N}$ *such that* $\tau_n(K) \leq m$ *for every compact set* K *in* U *then* f *admits a continuation as a meromorpic function in* U, *having not more that* m *poles there.*

(The poles are counted with regard to their multiplicities.)

Let now $\{T\} := \{T_n(z)\}_{n=1}^{\infty} = \{\prod_{k=1}^{n}(z - x_{k,n}\}$ be the sequence of the Chebyshev polynomials on the segment Δ. Denote by $X = \{y_N\}_{N=1}^{\infty}$ the correspondimg sequence of Newton type, and T_n^*, $n = 1, 2, ...$ - the Chebyshev polynomials of Newton type. We recall that $y_N := x_{n,n}$ for $N = 1 + ... + n$ and $y_{N+l} := x_{n+1,l}$ for $N = 1 + ... + n$ and $1 \leq l \leq n$; $T_n^*(z) := \prod_{k=1}^{n}(z - y_k)$.

Let $f \in \mathcal{H}(\Delta)$ and n, m be a pair of fixed nonnegative integers. There exist polynomals p_n and q_m of degree $\leq n$ and $\leq m$, respectively such that the function

$$\Psi(z) := (f(z).g_m(z) - p_n(z))/T_{n+m+1}^*(z)$$

is analytic on Δ. We now set $r_{n,m} := p_n/q_m$. The function $r_{n,m}$ is called an interpolting rational function of order (n, m) to f of Newton type, corresponding to T, or T-interpolating rational function of order (n, m). Obviously, for each pair of integers $(n, m), n, m \in \mathbf{N}$ the rational function $r_{n,m}$ always exists and is uniquely determined.

This construction is a particular case of a general construction introduced by E. B. Saff (see [5]).

We prove the following theorems.

Theorem 7: *Let the function* f *be holomorphic on the segment* Δ. *Denote by* P^* *the sequence* $\{P^*{}_n\}, n \in \mathbf{N}$ *of the* T*-interpolating polynomials. Assume that there is an open set* U, $U \bigcap \Delta \not\equiv 0$ *such that for each compact subset* K *of* U *we have* $k(P^*{}_n, K) = o(n)$ *as* $n \to \infty$.

Then $\{P^*{}_n\} \in \mathcal{N}(U)$ *and* $f \in \mathcal{H}(U)$

In what follows we shall write $r_n := r_{n,n}$. Let

$$r_n(z) := p_n(z)/q_n(z)$$

where both polynomials do not have common divisors. Denote for $n \in \mathbf{N}$ by $\mathcal{P}_n$ the set of the finite poles of the rational function r_n. We say that $\mathcal{P}_n \to \infty$, as $n \to \infty$ if the infinity is the only concentration point of the set $\{\mathcal{P}_n\}, n = 1, 2, ..$.

Theorem 8: *Assume $f \in \mathcal{H}(\Delta)$ and $\mathcal{P}_n \to \infty$, as $n \to \infty$. Assume that there exists an open set $U, U \bigcap \Delta \not\equiv 0$ such that $k(r_n, K) = o(n)$ for each compact subset K of U. Then the family $\{r_n\} \in \mathcal{N}(U)$ and $f \in \mathcal{H}(U)$.*

Preliminary Results

Lemma 1: (Lemma of Bernstein-Walsh, see [4]). *Let B be a domain in C and $g \in H(B) \bigcap C(\overline{B})$. Suppose $\alpha_i \in B, i = 1, \ldots, n+1$ and denote by S_n the polynomial of degree not more than n which interpolates the function g at the points $\alpha_i, i = 1, \ldots, n+1$.*

Then, for each $z, z \in B$, it is valid that

$$(g - S_n)(z) = \frac{1}{2\pi i} \int_{\partial B} \prod_{i=1}^{n+1} \frac{z - \alpha_i}{t - \alpha_i} \cdot \frac{g(t)}{t - z}\, dt.$$

Lemma 2: *For each integer $n, n \in \mathbf{N}$, let $T_n(z) = \prod_{k=1}^{n}(z - x_{k,n})$ be the Chebyshev polynomial of degree n for the segment Δ.*

Then

a) the points $\{x_{n,k}\}$ are uniformly distributed on Δ with respect to its equillibrium meassure.

For the polynomials $T_n^*, n = 1, 2, \ldots$ of Newton type for T we have

b)

$$\lim_{n\to\infty} \|T_n^*\|_\Delta^{1/n} = 1/2$$

and

c)

$$\lim_{n\to\infty} |T_n^*(z)|^{1/n} = (1/2).|\phi(z)|$$

locally uniformly inside $C - \Delta$.

The proof of this lemma follows from results of J. L. Walsh and M. Tsuji (see [4] and [15], respectively).

Let now the functions $F_n(z), n = 1, 2, \ldots$ be analytic in a domain B except perhaps for branch points, and let $|F_n(z)|$ be single-valued there. We say that the harmonic function $v(z)$ is harmonic majorant for $F_n(z)$ in B, if for every continuum $M, M \subset B$ there holds

$$\limsup_{n\to\infty} \|F_n(z)\|_M \leq \exp(\max_M v(z))$$

If a strict equality holds, then $v(z)$ is called an exact harmonic majorant. The next lemma is valid:

Lemma 3: (see [14]). *Let B be a domain in $\mathbf{C}$ and let the functions $F := \{F_n(z)\}, n = 1, 2, \ldots$ be as above. Assume that the harmonic function $v(z)$ is a harmonic majorant for the sequence $\{F_n(z)^{1/n}\}$.*

If there is a continuum M such that a strict equality holds, e.g.

$$\limsup_{n\to\infty} \|F_n^{1/n}\|_M = \exp(\max_M v(z))$$

then $v(z)$ is an exact harmonic majorant for the family $F_n^{1/n}$ in the domain B.

Further, for each compact subset K of B, we have

$$k(F_n, K) = o(n) \text{ as } n \to \infty$$

Lemma 4 (see [16]): *Let M be a regular set in $\mathbf{C}$ and B be a domain, $B \supset M$. Set S_n be a rational function of order (n, n), $S_n \in \mathcal{H}(B)$.*

Then for every compact subset $K, K \subset B$, there is a constant $\lambda, \lambda > 1$ such that

$$\|S_n\|_K \leq \|S_n\|_M.\lambda^n$$

Proofs

Proof of Theorem 6.

Theorem 6 will be proved under the additional condition that $D \subset U$. Otherwise, the statement results from Theorem 5.

Let

$$R_n = P_n/Q_n$$

where both polynomials P_n and Q_n do not have common divisors and Q_n is monic.

We say the integer $n, n \in \mathbf{N}$ is normal if the number of the poles of the rational function $R_n(z)$ in the extended complex plane $\overline{\mathbf{C}}$ is equal exactly to n. Denote by $\Xi, \Xi \in \mathbf{N}$, the sequence of the normal integers; $\Xi = \Xi(f)$. It is easy to verify that Ξ is infinite iff the function f is not a rational function. For such a function, for every $n, n \in \mathbf{N}$ the equality $R_n(z) \equiv R_{n_k}(z)$, where $n_k := \max\{n', n' \leq n, n' \in \Xi\}$ is true. Therefore, in what follows, we shall assume for sake of simplicity that $\deg P_n = \deg Q_n = n$ for any n. In view to the above considerations, the generality of the considerations will be preserved.

We now notice that in the conditions of the theorem, the domain U is symmetric with respect to the real axis.

As it is known

$$R_n(z) \to f(z), \text{ as } n \to \infty \tag{1}$$

uniformly on Δ.

Further, for each $n, n \in \mathbf{N}$, there follows from the alternation theorem of Chebyshev the validity of the representation

$$(R_{n+1} - R_n)(z).Q_n(z).Q_{n+1}(z) = A_n.w_{2n+1}(z) \tag{2}$$

where A_n is a suitable constant and $w_{2n+1}(z)$ is a monic polynomial of degree $2n+1$; all the zeros of $w_{2n+1}(z)$ are simple and belong to the interval Δ.

In the conditions of the theorem, it is obvious that $f \not\equiv 0$. (Otherwise $R_n \equiv 0$ for every $n \in \mathbf{N}$). Let Δ' be a subinterval of Δ such that $f(x) \neq 0$ there. By (1), there are two positive constants C_1 and C_2 such that for every $z, z \in \Delta'$ and all n sufficiently large $(n > n_1)$ the inequalities

$$C_1 \leq |R_n(z)| \leq C_2, n > n_1 \tag{3}$$

are valid.

Everywhere afterwards we shall denote by $C_i, i = 1, 2, ...$ positive constants which do not depend on i.

We fix the simply connected domains V and W, such that $\Delta \subset V \subset W \subset U$. We set now $P_n(z) = \gamma_n(z).p'_n(z)$ and $Q_n(z) = \eta_n(z).q'_n(z)$, where $p'_n(z).q'_n(z) \neq 0$ for $z \in W$. Both polynomials $\gamma_n(z)$ and $\eta_n(z)$ are normalized to be monic; set $\deg \gamma_n = m_n$, $\deg \eta_n = s_n$. Let $X_n, n \in N$, be the regular branch of the function $(\gamma_n.R_n^{-1}.\eta_n^{-1})^{1/n}(z)$ in V for which $X_n(\infty) > 0$. The functions X_n are holomorphic on V and, the family $\{X_n\}$ is with respect to lemma 4 and to (3), uniformly bounded on the set $\overline{V}$. Further, taking account of the conditions of the theorem, we see that

$$\|X_n(z)\|_{\Delta}^{\prime 1/n} \to 1 \text{ as } n \to \infty$$

From here, we obtain

$$|X_n(z)|^{1/n} \to 1 \text{ as } n \to \infty$$

uniformly on Δ'. Indeed, suppose to the contrary that there is a subinterval $\Delta"$ of Δ' such that for $z \in \Delta"$, $\limsup_{n\to\infty} |X_n(z)|^{1/n} < 1$. Let $|\Delta"|$ be its lenght; $|\Delta"| > 0$. It follows from our assumption and from the conditions of the theorem that

$$\liminf_{n\to\infty} \|\eta_n\|_{\Delta"}^{1/n} > 1 \tag{4}$$

In fact, let θ be an arbitrary positive number; for every $n \in \mathbf{N}$ large enough we have

$$C_3^{m_n} \geq \|\gamma_n\|_{\Delta"} \geq (\exp \theta |\Delta"|/4)^{m_n}$$

so that, with respect to the conditions of the theorem, (4) holds. On the other hand, for the polynomils $\eta_n(z)$ we have, for n sufficiently large

$$C_3^{s_n} \geq \|\eta_n\|_{\Delta"} \geq (\exp \theta |\Delta"|/4)^{s_n},$$

which is an obvious contradiction to (4).

Taking now account of the symmetry of the functions $R_n, n \in \mathbf{N}$ accordingly the real axes and applying Vitali's theorem, we obtain that

$$X_n(z)^{1/n} \to 1 \text{ as } n \to \infty$$

uniformly on $\overline{V}$. Then on each compact subset K of V the inequality

$$\limsup \|(R_{n+1} - R_n).\eta_n\eta_{n+1}\|_K^{1/n} \leq 1$$

holds. Denote

$$\Theta_n(z) := (R_{n+1} - R_n)(z).\eta_n(z)\eta_{n+1}(z);$$

the functions $|\Theta_n|^{1/n}$ are subharmonic on $\overline{V}$. Let us assume that $\limsup_{n\to\infty} \|\Theta_n\|_\Delta^{1/n} = 1$, then, by Lemma 3, the family $\{\Theta_n^{1/n}\}$ has the zero-function as an exact harmonic majorant in V (and on ∂V). The further application of Lemma 3 yields that $k_n(\Theta_n, K) = o(n)$ as $n \to \infty$ on any compsact subset K in V. But, for the compact set Δ, we have, in accordance to (2), that $k_n(\Theta_n, \Delta) = 2n + 1$. Therefore, the assumption that $\limsup_{n\to\infty} \|\Theta_n\|_\Delta^{1/n} = 1$ is not correct; thus

$$\limsup_{n\to\infty} \|\Theta_n\|_K^{1/n} < 1 \tag{5}$$

for any compact set in $\overline{V}$.

We now are going to prove that the family $R_n(z)$ is normal everywhere in V except for a set of zero m_1−measure; herewith the function f admits a holomorphic continuation there and, if $\deg \eta_n(z) \le m$, then - as a meromorphic function having not more that m poles in V (multiplicities included).

For our goal, we write

$$\eta_n(z) = \prod_{k=1}^{l_n} (z - \nu_{n,k});$$

in the conditions of the theorem,

$$l_n = o(n/\log n) \text{ as } n \to \infty. \tag{6}$$

For a fixed positive number ε, let

$$U_{n,k} = \{z, |z - \nu_{n,k}| < \varepsilon/2l_n.n^2\}$$

and

$$\Omega_\varepsilon = \bigcup_{n=n_1}^{\infty} \bigcup_{k=1}^{l_n} U_{n,k}.$$

Further, for $z, z \notin \Omega_\varepsilon$, we have

$$|\eta_n(z)| \ge (\varepsilon/2.l_n.n^2)^{l_n}. \tag{7}$$

We introduce $m_1(.)$ as follows: for each set e in C, we set $m_1(e) = \inf \sum |U_{n,k}|$, where the infimum is taken over all covering disks $U_{n,k}$; $|U_{n,k}|$ is the corresponding radius.

By our definition, we have

$$m_1(\Omega_\varepsilon) < \sum \varepsilon/2n^2 < \varepsilon.$$

We notice that Ω_ε is an open set and $\Omega_{\varepsilon_2} \subset \Omega_{\varepsilon_1}$, if $\varepsilon_2 < \varepsilon_1$.

For each compact set K, let $K(\varepsilon) = K - \Omega_\varepsilon(z)$.

Let $\rho' := \sup\{\rho, \mathcal{E}_\rho \subset V\}$. Select a positive number ε such that $\varepsilon < (1/2).(\rho' - 1/\rho')$ and $V(\varepsilon) \supset \Delta$. After (5) there is a positive number $q, q < 1$ such that for every $z \in V(\varepsilon)$ and n sifficiently large ($n > n_2$)

$$|(R_{n+1} - R_n).\eta_n.\eta_{n+1}|(z) \leq q^n, \tag{8}$$

(We have applied the maximum principle for holomorphic functions.)

Coming back to our considerations, we obtain from (6) and from (7)

$$|(R_{n+1} - R_n)(z)| \leq C_4.q^n.(2.l_n.n^2/\varepsilon)^{l_n}.(2.l_{n+1}.(n+1)^2/\varepsilon)^{l_{n+1}}, z \in V(\varepsilon), n > n_2.$$

Hence, by the conditions of the theorem, the series $\sum(R_{n+1} - R_n)(z)$, and herewith, the sequence $\{R_n(z)\}$, converges uniformly on $\overline{V}(\varepsilon)$.

We now recall that $R_n(z) \to f(z)$ on Δ (see (1)). Then, by the theorem for the uniqueness of the holomorphic functions, f is holomorphic on $\overline{V}(\varepsilon)$.

Suppose now that there is a sequence $\Lambda, \Lambda \subset \mathbf{N}$ for which the sequence $\eta_n(z)$ converges inside V. From the generalized Montel's theorem, it follows that this is certainly the case when $\deg \eta_n \leq m$. Denote the limit function by $\eta(z)$; we notice that $\eta_n(z) \in \mathcal{H}(V)$. Set $\Phi_n(z) = (R_n.\eta_n)(z)$. By the theorem for the uniqueness of the holomorphic functions, the functions $\Phi_n(z)$ converge uniformly on $\overline{V}$ itself. Let $\Phi(z)$ be the limit function; obviously, $\Phi(z) \in \mathcal{H}(V)$. From the representation $f(z) = \Phi(z)/\eta(z)$, it follows that f is meromorphic in V and the number of the poles in V does not exceed m. Also, it is obviously to see that all the poles of f in V lie among the zeros of $\eta(z)$.

To complete the proof of the theorem, we have to exhaust W by V and U by W and to apply the previous results.

Theorem 6 is proved.

The proofs of the coming theorems repeats the main idea laid out above. We will confine ourselves only on sketching the most important steps.

Proof of Theorem 7.

We first notice that in the conditions of the theorem, for every $n, n \in \mathbf{N}$, the representation

$$P^*_{n+1}(z) - P^*_n(z) = A_n.T^*_{n+1}(z) \tag{9}$$

holds, with A_n a suitable constant.

We recall that f is holomorphic on the segment Δ. Let ρ be such a number that the function f is holomorphic on $\mathcal{E}_\rho$. For each integer $n, n \in \mathbf{N}$ and $z \in \mathcal{E}_\rho$ we have, accordingly to Lemma 1

$$f(z) - P^*_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} f(t)T^*_{n+1}(z)/T^*_{n+1}(t)(t-z)\,dt$$

From Lemma 2, b) and c), we get

$$\limsup_{n\to\infty} \|f - P^*_n\|_\Delta^{1/n} \leq 1/\rho \tag{10}$$

so that

$$P_n^*(z) \to f \text{ as } n \to \infty$$

uniformly on Δ.

Hence, the family $P_n^*(z)^{1/n}$, being normal in the domain U, converges locally uniformly inside U, to 1.

Using now lemma 2, a) and repeating the considerations in the proof of Theorem 6, we come to the conclusion of Theorem 7.

Proof of Theorem 8.

In the conditions of the theorem, we have

$$r_{n+1}(z) - r_n(z) = A_n.T_{2n+1}^*(z)/q_n(z).q_{n+1}(z)$$

with A_n a suitable constant.

Provided $f, r_n, \in \mathcal{H}(E_\rho), n > n'$ we get from Lemma 1,

$$f(z) - r_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \{f(t)T_{2n+1}^*(z)q_n(t)/q_n(z)T_{2n+1}^*(t)(t-z)\}\,dt \tag{11}$$

In the conditions of the theorem, it is clearly true that for any $\tau, \tau > 1$

$$\liminf_{n\to\infty}\{\|q_n\|_{\Gamma_\tau}/\min_{t\in\Delta}|q_n(t)|\}^{1/n} = 1$$

From here and from (11), we obtain

$$\limsup_{n\to\infty}\|f - r_n\|_\Delta^{1/n} \le 1/\rho(U) \tag{12}$$

where $\rho(U) := \sup(\rho, E_\rho \subset U)$.

Hence, the sequence r_n converges to f on Δ.

Using now the previous considerations, we see that Theorem 8 is true.

We observe that the rational sequences we have dealed with converge to f on the segment Δ with the speed of a geometric progression, and the speed is given by $\sup(\rho, E_\rho \subset U)$. In fact, for the rational T - interpolating sequences, this follows from Bernstein - Walsh's lemma (compare (10) and (12)). As for the rational function $\{R_n\}$, this statement follows from Theorem 6, Theorem 1 and from the definition of the rational functions of best uniform approximation.

This fact is of importance for the case when there is no information about the behaviour of the zeros of the approximating rational functions so that Montel;s theorem is not applicable. In the case of a "geometric" convergence on Δ, the sequences necessarily converge inside some larger domain containing Δ. The validity of this statement follows from

Theorem 9:(see [17]) *Let $f \in C_R(\Delta)$. Suppose that there is a sequence of rational functions $\{r^*_{n,m_n}(z)\}$ with $\lim_{n\to\infty} n/m_n \to 1$ and $\limsup_{n\to\infty} \|r^*_n(z) - f\|_\Delta^{1/n} \leq 1/\rho_1$ for a number $\rho_1, \rho_1 > 1$. Assume further that $r^*_n \in \mathcal{H}(E_{\rho_1})$ for every $n \in \mathbf{N}$.*

Then, the sequence $\{r^*_n\}$ converges, as $n \to \infty$, locally uniformly inside E_{ρ_2}, where ρ_2 is given by $\rho_2 = (\rho_1^2 + 1)/\rho_1^{3/2} + 1)$. (and, herewith, the functions f admits a holomorphic continuation in the ellipse E_{ρ_2}.

This result refers to rational sequences with an unbounded number of the finite poles; as for polynomial sequences or sequences with a fixed number of the poles, the convergence everywhere inside the ellipse E_{ρ_1} follows from Walsh's results (see [3]).

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