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ALL THE SPECIAL FUNCTIONS
ARE FRACTIONAL DIFFERINTEGRALS
OF ELEMENTARY FUNCTIONS

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Abstract

In this survey we discuss an unified approach to the generalized hypergeometric functions based on a *generalized fractional calculus* developed in the monograph [11]. This generalization of the classical theory of the operators of integration and differentiation of fractional order deals with integral (differintegral) operators involving *Meijer's G- and Fox's H-functions* as kernel-functions. Their theory is fully developed and illustrated by various special cases and applications in different areas of the applicable analysis.

Usually, the special functions of mathematical physics are defined by means of power series representations. However, some alternative representations can be used as their definitions ([1],[5],[12],[15]). Let us mention the well-known *Poisson integrals* for the Bessel functions and the analytical continuation of the Gauss hypergeometric function via the *Euler integral formula*. The *Rodrigues differential formulas*, involving repeated or fractional differentiation are also used as definitions of the classical orthogonal polynomials and their generalizations.. As to the other special functions (most of them being ${}_pF_q$ - and ${}_p\Psi_q$ -functions), such representations are less popular and even unknown in the general case. There exist various integral and differential formulas, but unfortunately, quite peculiar for each corresponding special function and scattered in the literature without any common idea to relate them. Here, all the *generalized hypergeometric functions* are proved to be generalized fractional integrals or derivatives of three basic elementary functions. On this base, they are classified in three specific classes and several new integral and differential representations are found.

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1. Introduction

The generalized fractional calculus developed in [11] is based on the notion of *generalized operators of fractional integration* of Riemann-Liouville type

$$If(x) = x^\delta \int_0^1 \Phi(\sigma) \sigma^\gamma f(x\sigma) d\sigma \quad (1.1)$$

(see Kalla [6]), where $\Phi(\sigma)$ is an arbitrary elementary or special kernel-function. However, in order to develop a meaningful detailed theory with practical applications, we choose the kernel-functions as suitable special cases of the *Meijer's G-functions* and *Fox's H-functions*.

DEFINITION 1.1. (see [14],[18]) By a *Fox's H-function* we mean the generalized hypergeometric function defined by means of the contour integral

$$\begin{aligned} H_{p,q}^{m,n}(\sigma) &= H_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right. \right] \\ &= H_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) \sigma^s ds, \quad \sigma \neq 0, \end{aligned} \quad (1.2)$$

where the integrand in (1.2) has the form

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{k=1}^m \Gamma(b_k - B_k s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + B_k s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)}$$

and $\mathcal{L}$ is a suitable contour in $\mathbf{C}$; the orders $(m, n; p, q)$ are nonnegative integers such that $0 \leq m \leq q$, $0 \leq n \leq p$; the parameters $A_j, j = 1, \dots, p$ and $B_k, k = 1, \dots, q$ are positive and $a_j, j = 1, \dots, p$, $b_k, k = 1, \dots, q$ are arbitrary complex numbers such that

$$A_j(b_k + l) \neq B_k(a_j - l' - 1); \quad l, l' = 0, 1, 2, \dots; \quad j = 1, \dots, p, \quad k = 1, \dots, q.$$

In particular, when all $A_j = B_k = 1$, we obtain the so-called *Meijer's G-function* ([5,v.1]) :

$$H_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} (a_j, 1)_1^p \\ (b_k, 1)_1^q \end{matrix} \right. \right] = G_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right. \right], \quad (1.3)$$

namely,

$$G_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{k=1}^m \Gamma(b_k - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \sigma^s ds. \quad (1.4)$$

In Section 2 we define our *generalized fractional integrals and derivatives* using as kernel-functions peculiar cases of the above special functions with $m = p = q, n = 0$. This choice of the kernel-function $\Phi(\sigma)$ ensures a decomposition of these operators (called also *multiple Erdélyi-Kober operators*) into *products of commuting classical Erdélyi-Kober (E.-K.) operators*. Thus, complicated multiple integrals or differintegral expressions can be represented alternatively, by means of single integrals involving special functions. The beauty and succinctness of notations and properties of these functions allow the development of a *full chain of operational rules, mapping properties and convolutional structure of the generalized fractional integrals* as well as an *appropriate explicit definition of the corresponding generalized derivatives*. On the other hand, the frequent appearance of compositions of classical Riemann-Liouville and Erdélyi-Kober fractional operators in various problems of applied analysis gives *the key to the great number of applications and known special cases* of our *generalized fractional differintegrals*.

Section 3 deals with the *generalized hypergeometric functions* ${}_pF_q(x)$ being also special cases of the Meijer's G -functions (see [5],[14]):

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{p,q+1}^{1,p} \left[-x \left| \begin{matrix} 1 - a_1, \dots, 1 - a_p \\ 0, 1 - b_1, \dots, 1 - b_q \end{matrix} \right. \right]. \quad (1.5)$$

DEFINITION 1.2. By a *generalized hypergeometric function (g. h. f.)* ${}_pF_q(x)$ we mean the sum of the g.h.f. series:

$$\begin{aligned} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) &= {}_pF_q((a_i)_1^p; (b_j)_1^q; x) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \cdot \frac{x^k}{k!}, \quad \text{where } (a)_0 = 1, (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \end{aligned} \quad (1.6)$$

in the domain of its convergence: $\Omega = \{|x| < \infty\}$ for $p \leq q$ and $\Omega = \{|x| < 1\}$ for $p = q + 1$, or its analytical continuation in $\{|x| > 1, |\arg(1-x)| < \pi\}$ in the latter case. One may consider x also as a real variable $x \in [0, \infty)$.

We separate the ${}_pF_q$ -functions in three classes depending on whether $p < q, p = q$ or $p = q + 1$ and represent the functions of each class as *generalized fractional integrals or derivatives of three basic elementary functions*:

$$\cos_{q-p+1}(x) \text{ (if } p < q), \quad x^\alpha \exp x \text{ (if } p = q), \quad x^\alpha (1-x)^\beta \text{ (if } p = q + 1). \quad (1.7)$$

Abovementioned representations lead to several new integral and differential formulas for the ${}_pF_q$ -functions and allow their study in an unified way. Many interesting particular cases are mentioned.

In Section 4 we continue the same approach to the so-called *Wright's generalized hypergeometric functions* ([18],[19])

$$\begin{aligned} {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} ; x \right] &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1) \dots \Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1) \dots \Gamma(b_q + kB_q)} \frac{x^k}{k!} \\ &= H_{p,q+1}^{1,p} \left[-x \left| \begin{matrix} (1-a_1, A_1), \dots, (1-a_p, A_p) \\ (0, 1), (1-b_1, B_1), \dots, (1-b_q, B_q) \end{matrix} \right. \right]. \end{aligned} \quad (1.8)$$

Naturally,

$${}_p\Psi_q \left[\begin{matrix} (a_1, 1), \dots, (a_p, 1) \\ (b_1, 1), \dots, (b_q, 1) \end{matrix} ; x \right] = \frac{\prod_{i=1}^q \Gamma(b_i)}{\prod_{j=1}^p \Gamma(a_j)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x). \quad (1.9)$$

The results for the special functions (1.9) are essentially new and are published for the first time.

2. Generalized fractional calculus

We introduce the following generalizations of the Riemann-Liouville (R.-L.) fractional integrals of order $\delta > 0$:

$$R^\delta f(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-\tau)^{\delta-1} f(\tau) d\tau = \frac{x^\delta}{\Gamma(\delta)} \int_0^1 (1-\sigma)^{\delta-1} f(x\sigma) d\sigma, \quad (2.1)$$

having the form of operators (1.1).

DEFINITION 2.1. Let $m \geq 1$ be integer, $\beta > 0$, $\gamma_1, \dots, \gamma_m$ and $\delta_1 \geq 0, \dots, \delta_m \geq 0$ be arbitrary real numbers. By a *generalized (multiple, m -tuple) Erdélyi-Kober (E.-K.) operator of integration of multiorder $\delta = (\delta_1, \dots, \delta_m)$* we mean an integral operator

$$I_{\beta, m}^{(\gamma_k), (\delta_k)} f(x) = \int_0^1 G_{m, m}^{m, 0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\delta_k)_1^m \end{matrix} \right. \right] f(x\sigma^{\frac{1}{\beta}}) d\sigma. \quad (2.2)$$

Then, each operator of the form

$$Rf(x) = x^{\beta\delta_0} I_{\beta, m}^{(\gamma_k), (\delta_k)} f(x) \quad \text{with arbitrary } \delta_0 \geq 0, \quad (2.3)$$

is said to be a *generalized (m -tuple) operator of fractional integration of Riemann-Liouville type*, or briefly: a *generalized (R.-L.) fractional integral*.

Generalizing further the operators of fractional calculus, in [8], [11] we consider also operators involving Fox's H -functions instead of the G -functions in (2.2),(2.3). They are called in the same way, namely *generalized (multiple) E.-K. operators (fractional integrals)*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = \begin{cases} \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \end{matrix} \right. \right] f(x\sigma) d\sigma, & \text{if } \sum_{k=1}^m \delta_k > 0, \\ f(x), & \text{if } \delta_1 = \delta_2 = \dots = \delta_m = 0. \end{cases} \quad (2.4)$$

Thus, along with the *multiorder of integration* $(\delta_1, \dots, \delta_m)$ and the *multiweight* $(\gamma_1, \dots, \gamma_m)$, we introduce also a *multiparameter* $(\beta_1 > 0, \dots, \beta_m > 0)$ (different β_k 's) instead of the same $\beta > 0$ in the case with the kernel G -function. Note that due to relation, generalizing (1.3):

$$H_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} \left(a_1, \frac{1}{\beta} \right), \dots, \left(a_p, \frac{1}{\beta} \right) \\ \left(b_1, \frac{1}{\beta} \right), \dots, \left(b_q, \frac{1}{\beta} \right) \end{matrix} \right. \right] = \beta G_{p,q}^{m,n} \left[\sigma^\beta \left| \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right. \right], \quad \beta > 0, \quad (2.5)$$

operator (2.4) involving a H -function reduces to its simpler form (2.2), viz.

$$\text{for } \beta_1 = \beta_2 = \dots = \beta_m = \beta > 0: \quad I_{(\beta,\beta,\dots,\beta),m}^{(\gamma_k),(\delta_k)} = I_{\beta,m}^{(\gamma_k),(\delta_k)}. \quad (2.6)$$

Now let us define generalizations of the classical *Riemann-Liouville derivatives of fractional order* $\delta > 0$:

$$D^\delta f(x) = \begin{cases} \frac{d^\delta}{dx^\delta} f(x) = f^{(\delta)}(x), & \text{for integer } \delta, \\ \frac{d^\eta}{dx^\eta} R^{\delta-\eta} f(x), & \text{for noninteger } \delta \text{ with } \eta = [\delta] + 1, \end{cases} \quad (2.7)$$

corresponding to generalized fractional integrals (2.2), (2.4).

DEFINITION 2.2. With the same parameters as in Def. 2.1 and the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer,} \\ [\delta_k] + 1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \quad k = 1, \dots, m, \quad (2.8)$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} x \frac{d}{dx} + \gamma_r + j \right) \right]. \quad (2.9)$$

Then, the *multiple (m-tuple) Erdélyi-Kober fractional derivatives* of multiorder $\delta = (\delta_1 \geq 0, \dots, \delta_m \geq 0)$ are defined by means of the differintegral operators:

$$\begin{aligned} D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) &= D_\eta I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(x) \\ &= D_\eta \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} \gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \\ \gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \end{matrix} \right|_1^m \right] f(x\sigma) d\sigma. \end{aligned} \quad (2.10)$$

In the case (2.5) of equal β_k 's we obtain simpler representations involving the Meijer's G -function and corresponding to generalized fractional integrals (2.2):

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} = D_\eta I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} = \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma_r + j \right) \right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}. \quad (2.11)$$

More generally, the differintegral operators of the form

$$Df(x) = D_{\beta,m}^{(\gamma_k),(\delta_k)} x^{-\delta_0} f(x) = x^{-\delta_0} D_{\beta,m}^{(\gamma_k-\frac{\delta_0}{\beta}),(\delta_k)} f(x) \quad \text{with } \delta_0 \geq 0 \quad (2.12)$$

are called *generalized (multiple, m-tuple) fractional derivatives*.

Generalized derivatives (2.11),(2.12) are the counterparts of the generalized fractional integrals (2.2),(2.3).

The generalized fractional integrals and derivatives include as special cases a great number of operators of fractional (or integer but generalized) integration and differentiation, even in the *simpler cases related to the Meijer's G-function*. We consider separately the cases $m = 1, 2$ (when many generalized integration and differentiation operators introduced and used by various authors are included as special cases) and $m > 2$ (when our operators are less known).

i) $m = 1$. The kernel-function of (2.2) is the elementary function

$$G_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} \gamma + \delta \\ \gamma \end{matrix} \right| \right] = \begin{cases} (1 - \sigma)^{\delta-1} \sigma^\gamma / \Gamma(\delta), & 0 < \sigma < 1, \\ 0, & \sigma > 1. \end{cases} \quad (2.13)$$

Thus, for arbitrary $\beta > 0$, γ and $\delta > 0$ the generalized fractional integrals (2.2) coincide with the well-known *Erdélyi-Kober (E.-K.) operators (integrals)*

$$I_{\beta}^{\gamma,\delta} f(x) = \int_0^1 \frac{(1 - \sigma)^{\delta-1} \sigma^\gamma}{\Gamma(\delta)} f(x\sigma^{\frac{1}{\beta}}) d\sigma = I_{\beta,1}^{\gamma,\delta} f(x), \quad (2.14)$$

widely used in the applied analysis (see e.g. [17]) and incorporating the R.-L. fractional integrals (2.1) as well: $R^\delta f(x) = x^\delta I_1^{0,\delta} f(x)$.

For $m = 1$ the generalized fractional derivative (2.10), corresponding to the Erdélyi-Kober fractional integral (2.14), is called in [11] an *Erdélyi-Kober fractional derivative* and has the representation

$$\begin{aligned} D_{\beta}^{\gamma, \delta} f(x) &:= D_{\beta, 1}^{\gamma, \delta} f(x) = D_{\eta} I_{\beta}^{\gamma + \delta, \eta - \delta} f(x) \\ &= \left[\prod_{j=1}^{\eta} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma + j \right) \right] \int_0^1 \frac{(1 - \sigma)^{\eta - \delta - 1} \sigma^{\gamma + \delta}}{\Gamma(\eta - \delta)} f(x \sigma^{\frac{1}{\beta}}) d\sigma. \end{aligned} \quad (2.15)$$

Symbolically, it can be written as

$$D_{\beta}^{\gamma, \delta} f(x) := D_{\beta, 1}^{\gamma, \delta} f(x) = \left[x^{-\gamma} D^{\delta} x^{\gamma + \delta} f \left(x^{\frac{1}{\beta}} \right) \right]_{x \rightarrow x^{\frac{1}{\beta}}}, \quad (2.15')$$

where D^{δ} is the *R.-L. fractional derivative* (2.7), being also an E.-K. derivative:

$$D^{\delta} = \left(\frac{d}{dx} \right)^{\delta} = x^{-\delta} D_1^{-\delta, \delta} = D_1^{0, \delta} x^{-\delta}, \quad \delta > 0.$$

ii) $m = 2$. Then, the kernel-function of (2.2) is the *Gauss hypergeometric function* ${}_2F_1$, namely

$$\begin{aligned} G_{2,2}^{2,0} \left[\sigma \left| \begin{matrix} \gamma_1 + \delta_1, \gamma_2 + \delta_2 \\ \gamma_1, \gamma_2 \end{matrix} \right. \right] \\ = \begin{cases} \frac{\sigma^{\gamma_2} (1 - \sigma)^{\delta_1 + \delta_2 - 1}}{\Gamma(\delta_1 + \delta_2)} {}_2F_1(\gamma_2 + \delta_2 - \gamma_1, \delta_1; \delta_1 + \delta_2; 1 - \sigma) & \text{for } \sigma < 1, \\ 0 & \text{for } \sigma > 1, \end{cases} \end{aligned} \quad (2.16)$$

and the operators $I_{\beta, 2}^{(\gamma_k), (\delta_k)}$ are the so-called *hypergeometric fractional integrals*:

$$\begin{aligned} H f(x) &= I_{\beta, 2}^{(\gamma_1, \gamma_2), (\delta_1, \delta_2)} f(x) \\ &= \int_0^1 \frac{\sigma^{\gamma_2} (1 - \sigma)^{\delta_1 + \delta_2 - 1}}{\Gamma(\delta_1 + \delta_2)} {}_2F_1(\gamma_2 + \delta_2 - \gamma_1, \delta_1; \delta_1 + \delta_2; 1 - \sigma) f \left(x \sigma^{\frac{1}{\beta}} \right) d\sigma, \end{aligned} \quad (2.17)$$

introduced first by Love [13] and considered in different modifications by Tricomi, Kalla and Saxena, Saigo [16], McBride and many other authors.

iii) $m \geq 2$. In this case the generalized fractional integrals and derivatives have been used mainly in their alternative, multiple integral representations, without involving special kernel-functions. This has caused a lack of suitable tools to deal with easily.

In a series of papers (e.g. [3]) Dimovski introduced and studied the *hyper-Bessel differential operators* of arbitrary integer order $m > 1$ of the form

$$B = x^{\alpha_0} \frac{d}{dx} x^{\alpha_1} \dots \frac{d}{dx} x^{\alpha_m} = x^{-\beta} \prod_{k=1}^m \left(x \frac{d}{dx} + \beta \gamma_k \right), \quad \beta > 0. \quad (2.18)$$

These operators, generalizing the 2nd order Bessel differential operator are widely used in the differential equations of mathematical physics (see Ch.3, [11]). Dimovski developed a detailed theory, including operational calculi, integral transforms etc. Operators (2.18) and their linear right inverse operators L , the so-called *hyper-Bessel integral operators* have been shown in [11] to be generalized (m -tuple, arbitrary $m > 1$) fractional derivatives and integrals of integer multi-order ($\delta_1 = 1, \dots, \delta_m = 1$), namely:

$$L = \frac{x^\beta}{\beta^m} I_{\beta,m}^{(\gamma_k),(1)} f(x), \quad B = \beta^m D_{\beta,m}^{(\gamma_k),(1)} x^{-\beta} = \frac{\beta^m}{x^\beta} D_{\beta,m}^{(\gamma_k^{-1}),(1)}. \quad (2.19)$$

A variety of useful *transmutation operators*, related to operators (2.18),(2.19) like the *Poisson-Sonine-Dimovski transforms* are examples of m -tuple generalized fractional integrals too, for all the details see Ch. 3, [11].

The *multiple Dzrbashjan-Gelfond-Leontiev operators* (Section 5.4, [11]), are typical examples of the more complicated generalized fractional (differ)integrals, (2.4),(2.10) involving Fox's H -functions.

In [11] the generalized operators of fractional integration and differentiation have been considered in different *functional spaces*, like weighted spaces of continuous, Lebesgue integrable or analytic functions. Here we need the definition of the latter spaces only.

DEFINITION 2.3. Let μ be arbitrary real, the variable x be real or complex, running resp. over the interval $[0, \infty)$ or in the domain $\Omega \subset \mathbf{C}$, starlike with respect to the origin $x = 0$ and let $\mathcal{H}(\Omega)$ stand for the space of analytic functions in Ω . Denote

$$\mathcal{H}_\mu(\Omega) = \left\{ f(x) = x^\mu \tilde{f}(x); \tilde{f}(x) \in \mathcal{H}(\Omega) \right\}, \quad \mathcal{H}_0(\Omega) := \mathcal{H}(\Omega). \quad (2.20)$$

To study the generalized fractional integrals, we use essentially the *theory of the G - and H -functions*, appearing as kernel-functions of (2.2),(2.4). To this end we refer to the classical book [5, Ch.5], also to [14],[18],[19] or [11, Appendix]. Note also, that the $G_{m,m}^{m,0}$ - and $H_{m,m}^{m,0}$ -functions have three regular singular points $\sigma = 0, 1$ and ∞ , they vanish for $|\sigma| > 1$ and are analytic functions in the unit disk $|\sigma| < 1$. Their asymptotic behaviour near $\sigma = 0, 1$ is already well-known (see e.g. [14] or Appendix, [11]) and ensures the correctness of definitions (2.2),(2.4) in the above spaces, under suitable conditions on parameters.

Operators (2.4) can be rewritten in the form

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = \frac{1}{x} \int_0^x H_{m,m}^{m,0} \left[\frac{t}{x} \middle| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \end{matrix} \right] f(t) dt, \quad (2.21)$$

and thus, it can be put in the form of a *convolutional type integral transform*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^\infty k \frac{x}{t} f(t) \frac{dt}{t} = (k \circ f)(x), \quad (2.20)$$

where $\circ$ denotes the Mellin convolution.

Most of the basic results for the operators of the generalized fractional calculus have been stated in [11] separately for the cases of G - and H -functions and for different kind of functional spaces. Here we expose them in the general case (2.4) only. The corresponding reductions for the simpler case (2.2) with G -functions are easily seen (for proofs and details see also [9]).

THEOREM 2.4. *Let the conditions*

$$\gamma_k > -\frac{\mu}{\beta_k} - 1, \quad \delta_k > 0, \quad k = 1, \dots, m \quad (2.21)$$

be satisfied. Then, the multiple E.-K. operator (2.4) maps the class $\mathcal{H}_\mu(\Omega)$ into itself, preserving the power functions up to constant multipliers:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{x^p\} = c_p x^p, \quad p > \alpha \quad \text{where} \quad c_p = \prod_{k=1}^m \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)} \quad (2.22)$$

The image of a power series

$$f(x) = x^\mu \sum_{n=0}^{\infty} a_n x^n = x^\mu (a_0 + a_1 x + \dots) \in \mathcal{H}_\mu(\Delta_R), \quad \Delta_R = \{|x| < R\},$$

where $R = \left\{ \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \right\}^{-1}$, is given by the series

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = x^\mu \sum_{n=0}^{\infty} \left\{ a_n \prod_{k=1}^m \frac{\Gamma\left(\gamma_k + \frac{n+\mu}{\beta_k} + 1\right)}{\Gamma\left(\gamma_k + \delta_k + \frac{n+\mu}{\beta_k} + 1\right)} \right\} x^n, \quad (2.23)$$

having the same radius of convergence $R > 0$ and the same signs of the coefficients.

From the properties of the G - and H -functions some immediate corollaries of definitions (2.2),(2.4) follow.

THEOREM 2.5. *Suppose conditions (2.21) are satisfied. Then, in $\mathcal{H}_\mu(\Omega)$ the following basic operational rules of multiple E.-K. fractional integrals (2.4) hold:*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{\lambda f(cx) + \eta g(cx)\} = \lambda \left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f \right\}(cx) + \eta \left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} g \right\}(cx) \quad (2.24)$$

(bilinearity);

$$I_{(\beta_1, \dots, \beta_m), m}^{(\gamma_1, \dots, \gamma_s, \gamma_{s+1}, \dots, \gamma_m), (0, \dots, 0, \delta_{s+1}, \dots, \delta_m)} f(x) = I_{(\beta_{s+1}, \dots, \beta_m), m-s}^{(\gamma_{s+1}, \dots, \gamma_m), (\delta_{s+1}, \dots, \delta_m)} f(x) \quad (2.25)$$

(if $\delta_1 = \delta_2 = \dots = \delta_s = 0$, then the multiplicity reduces to $(m-s)$);

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} x^\lambda f(x) = x^\lambda I_{(\beta_k),m}^{(\gamma_k + \frac{\lambda}{\beta_k}),(\delta_k)} f(x), \quad \lambda \in \mathbf{R} \quad (2.26)$$

(generalized commutability with power functions);

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} I_{(\varepsilon_j),n}^{(\tau_j),(\alpha_j)} f(x) = I_{(\varepsilon_j),n}^{(\tau_j),(\alpha_j)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) \quad (2.27)$$

(commutability of operators of form (2.4));

$$\text{the left-hand side of (2.27)} = I_{((\beta_k)_1^m, (\varepsilon_j)_1^n), m+n}^{((\gamma_k)_1^m, (\tau_j)_1^n), ((\delta_k)_1^m, (\alpha_j)_1^n)} f(x) \quad (2.28)$$

(compositions of m -tuple and n -tuple integrals (2.4) are $(m+n)$ -tuple integrals);

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\sigma_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = I_{(\beta_k),m}^{(\gamma_k),(\sigma_k+\delta_k)} f(x), \quad \text{if } \delta_k > 0, \sigma_k > 0, k = 1, \dots, m \quad (2.29)$$

(law of indices, product rule or semigroup property);

$$\left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \right\}^{-1} f(x) = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} f(x) \quad (2.30)$$

(formal inversion formula).

Above inversion formula follows from index law (2.29) for $\sigma_k = -\delta_k < 0, k = 1, \dots, m$ and definition (2.4) for zero multiorder of integration, since:

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = I_{(\beta_k),m}^{(\gamma_k),(0,\dots,0)} f(x) = f(x).$$

But symbols (2.4) have not yet been defined for negative multiorders of integration $-\delta_k < 0, k = 1, \dots, m$. The problem is to propose an appropriate meaning for them and hence to avoid the divergent integrals appearing in (2.30). The situation is the same as in the classical case when the R.-L. and E.-K. operators of fractional order $\delta > 0$ are inverted by appealing to an additional differentiation of suitable integer order $\eta = [\delta] + 1$. Now, we make use of the following differential formula for the kernel H -function ([11], Lemma 5.1.7 or Lemma B.3, Appendix, for the G -function): Let $\eta_k \geq 0, k = 1, \dots, m$ be arbitrary integers, then

$$H_{m,m}^{m,0} \left[\frac{t}{x} \left| \begin{matrix} (a_k, \frac{1}{\beta_k})_1^m \\ (b_k, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] = D_\eta H_{m,m}^{m,0} \left[\frac{t}{x} \left| \begin{matrix} (a_k + \eta_k, \frac{1}{\beta_k})_1^m \\ (b_k, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right], \quad (2.31)$$

with differential operator D_η being polynomial of $x \frac{d}{dx}$ of degree $\eta = \eta_1 + \dots + \eta_m$:

$$D_\eta = \prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} x \frac{d}{dx} + a_r - 1 + j \right).$$

This formula helps to increase the parameters $a_k, k = 1, \dots, m$ of the H -function in the upper row by arbitrary integers $\eta_k \geq 0, k = 1, \dots, m$, by using a suitable operator D_η . Choosing appropriately the necessary parameters, as in Def. 2.2, we

can prove now that $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ (2.10) is in fact, a *generalized fractional derivative* with a linear right inverse operator $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, namely:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = f(x), \quad f \in \mathcal{H}_\mu(\Omega). \quad (2.32)$$

Now we state the *basic result* for the generalized fractional integrals (2.2),(2.4) suggesting their alternative name “multiple (m -tuple)” fractional integrals.

THEOREM 2.6. (Composition/Decomposition theorem) *Under the conditions (3.6), the classical E.-K. fractional integrals of form (2.14): $I_{\beta_k}^{\gamma_k, \delta_k}$, $k = 1, \dots, m$, commute in $\mathcal{H}_\mu(\Omega)$ and their product*

$$\begin{aligned} I_{\beta_m}^{\gamma_m, \delta_m} \left\{ I_{\beta_{m-1}}^{\gamma_{m-1}, \delta_{m-1}} \dots (I_{\beta_1}^{\gamma_1, \delta_1} f(x)) \right\} &= \left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k, \delta_k} \right] f(x) \\ &= \int_0^1 \underbrace{\dots}_m \int_0^1 \left[\prod_{k=1}^m \frac{(1 - \sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f(x \sigma_1^{\frac{1}{\beta_1}} \dots \sigma_m^{\frac{1}{\beta_m}}) d\sigma_1 \dots d\sigma_m \end{aligned} \quad (2.33)$$

can be represented as an m -tuple E.-K. operator (2.4), i.e. by means of a single integral involving the H -function:

$$\begin{aligned} \left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k, \delta_k} \right] f(x) &= I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) \\ &= \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \end{matrix} \right. \right] f(x\sigma) d\sigma, \quad f \in \mathcal{H}_\mu. \end{aligned} \quad (2.34)$$

Conversely, under the same conditions, each multiple E.-K. operator of form (2.4) can be represented as a product (2.33).

Let us note that the same proposition holds for the generalized fractional derivatives (2.10),(2.11) as well: they are products of E.-K. derivatives (2.15), namely:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{\beta_1}^{\gamma_1, \delta_1} D_{\beta_2}^{\gamma_2, \delta_2} \dots D_{\beta_m}^{\gamma_m, \delta_m}. \quad (2.35)$$

Combining (2.30),(2.32), (2.34) and (2.35), we can make the *next step in clarifying the structure of a great number of operators*: generalized or classical, fractional or integer order integrations, differentiations or differintegrations. Namely, in [11] we introduce an *unified theory based on the common notion “generalized fractional differintegrals”*. By now, operators $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ with all $\delta_k \geq 0, k = 1, \dots, m$ have been considered as (fractional) integrals while those with all $\delta_k < 0, k = 1, \dots, m$ have been undertaken as formal denotations for the generalized fractional derivatives (cf.(2.30) and (2.32)): $I_{(\beta_k),m}^{(\gamma'_k + \delta'_k), (-\delta'_k)} =$

$D_{(\beta_k),m}^{(\gamma'_k),(\delta'_k)}$, i.e. $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)}$. Now, having the decomposition theorem in mind, we may consider both symbols $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ as *generalized fractional differintegrals*. If not all of the components of multiorder of “differintegration” $\delta = (\delta_1, \dots, \delta_m)$ are of the same sign, we simply interpret them as “mixed” products of E.-K. fractional integrals and derivatives. For example, if $\delta_1 < 0, \dots, \delta_s < 0, \delta_{s+1} = \dots = \delta_{s+j} = 0, \delta_{s+j+1} > 0, \dots, \delta_m > 0$, then

$$\begin{aligned} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} &:= D_{(\beta_1, \dots, \beta_s),s}^{(\gamma_1+\delta_1, \dots, \gamma_s+\delta_s),(-\delta_1, \dots, -\delta_s)} I_{(\beta_{s+j+1}, \dots, \beta_m),m-s-j}^{(\gamma_{s+j+1}, \dots, \gamma_m),(\delta_{s+j+1}, \dots, \delta_m)} \\ &= \prod_{i=1}^s D_{\beta_i}^{\gamma_i+\delta_i, -\delta_i} \prod_{k=s+j+1}^m I_{\beta_k}^{\gamma_k, \delta_k}, \text{ a } (m-j)\text{-tuple fractional differintegral.} \end{aligned} \quad (2.36)$$

Theorem 2.6 gives the key to the numerous applications of the generalized fractional calculus operators. Some of them, especially to the theory of the special functions, are briefly mentioned in the next two sections.

3. Representations of the generalized hypergeometric functions ${}_pF_q$.

In [10],[11] we propose an unified approach to the generalized hypergeometric functions (1.6) and derive new or newly written integral, differential and differintegral representations of these special functions by means of the generalized fractional calculus. A suitable classification of the ${}_pF_q$ -functions is also introduced. *The idea is based on the following simple facts:* i) most of the special functions of mathematical physics are nothing but modifications of the g.h.f-s ${}_pF_q$; ii) each ${}_pF_q$ -function can be represented as an E.-K. fractional differintegral of a ${}_{p-1}F_{q-1}$ -function (see (3.1)); iii) a finite number (q) of steps ii) leads to one of the basic g.h.f-s ${}_0F_{q-p}$ (for $q-p = 1$: Bessel function), ${}_1F_1$ (confluent h.f.), ${}_2F_1$ (Gauss h.f.); iv) the above 3 basic g.h.f-s can be considered as fractional differintegrals of the 3 elementary functions (1.7), depending on whether $p < q, p = q$ or $p = q + 1$; v) the compositions of E.-K. operators arising in iii) give generalized (q -tuple) fractional integrals or derivatives, according to Th.2.6. Thus, we obtain the following general

PROPOSITION 3.1. *All the generalized hypergeometric functions ${}_pF_q$ can be considered as generalized (q -tuple) fractional differintegrals (2.2),(2.3),(2.10), (2.12) of one of the elementary functions (1.7), depending on whether $p < q$, $p = q$, $p = q + 1$.*

To establish the above proposition we need the following basic lemma.

LEMMA 3.2. ([1],[12]) *Let $|x| < \infty$ ($|x| < 1$ for $p = q + 1$), then*

$$\begin{aligned}
& [\Gamma(a_p)/\Gamma(b_q)] {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \\
&= \begin{cases} I_{1,1}^{a_p-1, b_q-a_p} \{ {}_{p-1}F_{q-1}(a_1, \dots, a_{p-1}; b_1, \dots, b_{q-1}; x) \} & \text{if } b_q > a_p, \\ D_{1,1}^{b_q-1, a_p-b_q} \{ {}_{p-1}F_{q-1}(a_1, \dots, a_{p-1}; b_1, \dots, b_{q-1}; x) \} & \text{if } b_q < a_p. \end{cases} \quad (3.1)
\end{aligned}$$

The three cases $p < q, p = q, p = q + 1$ are to be considered separately, first of them being more complicated and involving some auxiliary definitions and results.

1st case: $p < q$

DEFINITION 3.3. In [3] Delerue introduced a generalization of the Bessel function $J_\nu(x)$ for a multiindex $\nu = (\nu_1, \dots, \nu_m)$, $m \geq 1$:

$$J_{\nu_1, \dots, \nu_m}^{(m)}(x) = \frac{(x/m+1)^{\nu_1+\dots+\nu_m}}{\Gamma(\nu_1+1)\dots\Gamma(\nu_m+1)} {}_0F_m((\nu_k+1)_1^m; -(x/m+1)^{m+1}), \quad (3.2)$$

referred to as a *hyper-Bessel function of order m* . As a special case, the so-called (*generalized*) *cosine function of order $(m+1)$* follows:

$$\cos_{m+1}(x) = {}_0F_m\left(\left(\frac{k}{m+1}\right)_1^m; -(x/m+1)^{m+1}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k(m+1)}}{(k(m+1))!}, \quad (3.3)$$

generalizing the elementary cosine function $\cos x = \cos_2(x)$, $m = 1$. For more details on (3.2), (3.3), see [11, Appendix]; [5, v.3], [14].

In [4] Dimovski and Kiryakova proved a *generalization of the Poisson integral representation of the Bessel function*

$$J_\nu(x) = \frac{2}{\sqrt{\pi}} \frac{(\frac{x}{2})^\nu}{\Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos xt \, dt, \quad \nu > -\frac{1}{2}, \quad (3.4)$$

based on the Poisson-Dimovski transformation (see [11], Ch.3). In terms of the generalized fractional calculus, G - and ${}_pF_q$ - functions, the generalized Poisson integral can be written in the modified form ([11], Th. 4.1.1, Cor. 4.1.4):

$$\begin{aligned}
{}_0F_m((b_k)_1^m; -x) &= c I_{1,m}^{(k/(m+1)-1), (b_k-k/(m+1))} \{ \cos_{m+1}((m+1)x^{1/(m+1)}) \} \\
&= c \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \left(\frac{b_k}{m+1} \right)_1^m \right. \right] \sigma^{-1} \cos_{m+1}((m+1)(x\sigma)^{1/(m+1)}) d\sigma, \quad (3.5)
\end{aligned}$$

where $c = \sqrt{(m+1)/(2\pi)^m} \prod_{j=1}^m \Gamma(b_j)$ and the condition $b_k \geq k/(m+1)$, $k = 1, \dots, m$ is supposed. Otherwise, if some of the b_k 's does not satisfy it, the corresponding component in the generalized fractional differintegral should be considered as an E.-K. derivative (2.15) (see [11], Th.4.1.6). As an illustration, if

$b_k := \nu_k + 1 = k/(m+1) - \eta_k$ with integers $\eta_k > 0, k = 1, \dots, m$, then (3.5) turns into a purely differential expression for the “spherical” hyper-Bessel functions (3.2) ([11],(4.1.42)), reducible in the case $m = 1$ to the spherical Bessel functions:

$$J_{-\eta-\frac{1}{2}}(x) = \frac{(2x)^{\eta+\frac{1}{2}}}{\sqrt{\pi}} \frac{d\eta}{(dx^2)^\eta} \left\{ \frac{\cos x}{x} \right\}, \quad \eta = 0, 1, 2, \dots \quad (3.6)$$

Observe now that by p steps (3.1), a ${}_pF_q$ -function, $p < q$ can be reduced to a hyper-Bessel function, i.e. to a ${}_0F_{q-p}$ -function:

$${}_pF_q((a_k)_1^p; (b_l)_1^q; x) = \left[\prod_{j=1}^p \frac{\Gamma(b_{q-p+j})}{\Gamma(a_j)} \right] I_{1,p}^{(a_k-1), (b_{q-p+k}-a_k)} \left\{ {}_0F_{q-p}((b_l)_1^{q-p}; x) \right\}. \quad (3.7)$$

This intermediate differintegral relation combined with (3.5) gives the following particular form of Proposition 3.1 in the case $p < q$.

THEOREM 3.4. *Each ${}_pF_q$ -function, $p < q$ is a generalized q -tuple fractional (differ)integral of $\cos_{q-p+1}(x)$, namely:*

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x) = AI_{1,q}^{(\gamma_k), (\delta_k)} \left\{ \cos_{q-p+1} \left((q-p+1)x^{\frac{1}{q-p+1}} \right) \right\}, \quad (3.8)$$

with $A = \sqrt{q-p+1}/(2\pi)^{q-p} \left[\prod_{j=1}^q \Gamma(b_j) / \prod_{i=1}^p \Gamma(a_i) \right]$ and parameters γ_k, δ_k :

$$\gamma_k = \begin{cases} \frac{k}{q-p+1} - 1, \\ a_{k-q+p} - 1, \end{cases}; \delta_k = \begin{cases} b_k - \frac{k}{q-p+1}, & \text{for } k = 1, \dots, q-p \\ b_k - a_{k-q+p}, & \text{for } k = q-p+1, \dots, q. \end{cases} \quad (3.9)$$

If the conditions

$$b_k > \frac{k}{q-p+1}, \quad k = 1, \dots, q-p; \quad b_{q-p+k} > a_k > 0, \quad k = 1, \dots, p \quad (3.10)$$

are satisfied, then relation (3.8) gives a Poisson type integral representation:

$$\begin{aligned} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x) &= A \int_0^1 G_{q,q}^{q,0} \left[\sigma \left| \frac{(b_k)_{k=1}^q}{\frac{k}{q-p+1} \frac{q-p}{k=1}}, (a_{k-q+p})_{k=q-p+1}^q \right. \right] \\ &\times \sigma^{-1} \cos_{q-p+1} \left[(q-p+1)(x\sigma)^{\frac{1}{q-p+1}} \right] d\sigma = A \int_0^1 \int_0^1 \prod_{k=1}^{q-p} \left[\frac{(1-t_k)^{b_k - \frac{k}{q-p+1} - 1}}{\Gamma(b_k - \frac{k}{q-p+1})} \right. \\ &\times \left. t_k^{\frac{k}{q-p+1} - 1} \right] \prod_{k=q-p+1}^q \left[\frac{(1-t_k)^{b_k - a_{k-q+p} - 1}}{\Gamma(b_k - a_{k-q+p})} t_k^{a_{k-q+p} - 1} \right] \\ &\times \cos_{q-p+1} \left[(q-p+1)(xt_1 \dots t_q)^{\frac{1}{q-p+1}} \right] dt_1 \dots t_q. \end{aligned} \quad (3.11)$$

If (3.10) are not true at least for one of the indices k , then $I_{1,q}^{(\gamma_k), (\delta_k)}$ in (3.8) is considered as a generalized fractional derivative.

Relation (3.8) including integral representations (3.5),(3.11) and their differintegral analogues, expresses the ${}_pF_q$ -functions, $p < q$ as fractional differintegrals (integrals or derivatives) of the generalized cosine function. This fact generalizes the well-known representations like (3.4),(3.6) of the Bessel functions via the cosine and thus, suggests the name *Bessel type g.h.f-s* for the functions ${}_pF_q$ with $p < q$. The Bessel function itself is the simplest special function of this *class*.

2nd case: $p = q$

Applying relation (3.1) to a function ${}_pF_p$ consequently $(p - 1)$ times, we reach to a ${}_1F_1$ -function which on its side is representable as an Erdélyi-Kober operator of the elementary function $x^a \exp x$:

$$\frac{\Gamma(a)}{\Gamma(b)} {}_1F_1(a; b; x) = x^{1-a} I_1^{0, b-a} \{x^{a-1} e^x\} = \begin{cases} I_1^{a-1, b-a} \{e^x\} & \text{if } b > a \\ D_1^{b-1, a-b} \{e^x\} & \text{if } b < a. \end{cases} \quad (3.12)$$

Thus, combining (3.1) and (3.12), by p steps we obtain the form of Proposition 3.1 in this case.

THEOREM 3.5. *If $p = q$, each g.h.f. ${}_pF_p(x)$ is an p -tuple fractional integral or derivative of the elementary function $\{x^{a_1-1} e^x\}$, namely:*

$${}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; x) = \Gamma' x^{1-a_1} I_{1,p}^{(\gamma_k), (\delta_k)} \{x^{a_1-1} e^x\}, \quad (3.13)$$

where $\gamma_k = a_k - a_1$, $\delta_k = b_k - a_k$, $k = 1, \dots, p$ and $\Gamma' = \prod_{j=1}^p [\Gamma(b_j)/\Gamma(a_j)]$. If

$$b_k > a_k > 0, \quad k = 1, \dots, p, \quad (3.14)$$

this relation yields the following integral representation:

$$\begin{aligned} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; x) &= \Gamma' \int_0^1 G_{p,p}^{p,0} \left[\sigma \left| \begin{matrix} (b_k)_1^p \\ (a_k)_1^p \end{matrix} \right. \right] \sigma^{-1} \exp(x\sigma) d\sigma \\ &= \Gamma' \int_0^1 \dots \int_0^1 \prod_{k=1}^p \left[\frac{(1-t_k)^{b_k-a_k-1} t_k^{a_k-1}}{\Gamma(b_k - a_k)} \right] \exp(xt_1 \dots t_p) dt_1 \dots dt_p. \end{aligned} \quad (3.15)$$

Above theorem justifies separating of all the g.h.f-s ${}_pF_p$ in a *class of g.h.f-s of confluent type*, involving the confluent hypergeometric function ${}_1F_1(a; b; x) = \Phi(a; b; x)$ as a simplest case.

For parameters not satisfying (3.14), relation (3.13) gives differintegral expressions. For example, we can introduce “spherical” g.h.f-s of confluent type (by analogy with (3.6)) representable by pure differential operators of $\exp x$.

COROLLARY 3.6. Let all the differences $a_k - b_k = \eta_k, k = 1, \dots, p$ be nonnegative integers. Then, the (differ)integral operator in (3.13) turns into a differential operator D_η of integer order $\eta = \eta_1 + \dots + \eta_k \geq 0$ and of form (2.9), namely:

$${}_pF_p(b_1 + \eta_1, \dots, b_p + \eta_p; b_1, \dots, b_p; x) \quad (3.16)$$

$$= \left[\prod_{j=1}^p \frac{\Gamma(b_j)}{\Gamma(b_j + \eta_j)} \right] \left[\prod_{k=1}^p \prod_{j=1}^{\eta_k} \left(x \frac{d}{dx} + b_k + j - 1 \right) \right] \{ \exp x \} = Q_p(x) \{ \exp x \}.$$

Differential representation (3.16) gives an example how differential formulas for the “spherical” g.h.f-s introduced in [11] can be used for their explicit calculation, especially in the case $p = q$ in the form $Q_p(x) \{ \exp x \}$, where $Q_p(x)$ is a p -degree polynomial. A special case of (3.16) with $b_k = \eta_k = 1, k = 1, \dots, p$ and $Q_p(x) = \left(x \frac{d}{dx} \right)^p$ can be seen in [14], p.593.

3rd case: $p = q + 1$

The generalized hypergeometric functions ${}_pF_q, p = q + 1$ are said to be *g.h.f-s of Gauss type* (see [11], Ch.4) and are considered for $|x| < 1$. In this case the starting specific result is the representation of the *Gauss hypergeometric function*:

$$\frac{\Gamma(a_1)}{\Gamma(b_1)} {}_2F_1(a_1, a_2; b_1; x) = I_1^{a_2-1, b_1-a_2} \{ (1-x)^{-a_1} \} = x^{1-a_2} I_1^{0, b_1-a_2} \{ x^{a_2-1} (1-x)^{-a_1} \} \quad (3.17)$$

for $b_1 > a_2 > 0$, or with an Erdélyi-Kober fractional derivative if $a_2 > b_1 > 0$.

Since by $(q-1)$ steps (3.1) a ${}_pF_q$ -function reduces to a ${}_2F_1$ -function and the composition of fractional differintegrals in (3.1) and (3.17) gives a q -tuple integral or derivative, we obtain the third form of Proposition 3.1.

THEOREM 3.7. In the unit disk $|x| < 1$ the *g.h.f-s of Gauss type* ${}_pF_q, p = q + 1$ are q -tuple generalized fractional differintegrals of elementary functions of the form $x^\alpha (1-x)^\beta$, namely:

$${}_{q+1}F_q(a_1, \dots, a_{q+1}; b_1, \dots, b_q; x) = \Gamma'' x^{1-a_2} I_{1,q}^{(a_{k+1}-1)_1^q, (b_k-a_{k+1})_1^q} \{ x^{a_2-1} (1-x)^{-a_1} \}. \quad (3.18)$$

with $\Gamma'' = \prod_{j=1}^q [\Gamma(b_j)/\Gamma(a_{j+1})]$. This means that for

$$b_k > a_k > 0, \quad k = 1, \dots, m \quad (3.19)$$

the following Poisson type integral representations hold:

$${}_{q+1}F_q(\pm x) = \Gamma'' \int_0^1 G_{q,q}^{q,0} \left[\sigma \left| \begin{matrix} (b_k) \\ (a_{k+1}) \end{matrix} \right. \right] \sigma^{-1} (1 \mp x\sigma)^{-a_1} d\sigma$$

$$= \left[\prod_{j=1}^q \frac{\Gamma(b_j)}{\Gamma(a_{j+1}) \Gamma(b_j - a_{j+1})} \right] \int_0^1 \dots \int_0^1 \prod_{j=1}^q \left[(1 - t_k)^{b_k - a_{k+1} - 1} t_k^{a_{k+1} - 1} \right] \times (1 \mp x t_1 \dots t_q)^{-a_1} dt_1 \dots dt_q. \quad (3.20)$$

The repeated integral form of (3.20) can be found also in [14], p. 438.

COROLLARY 3.8. For $q = 1$ representation (3.20) coincides with (3.17), written as the known Euler formula for the Gauss function ([5,v.1]):

$${}_2F_1(a_1, a_2; b_1; x) = \frac{\Gamma(b_1)}{\Gamma(a_2) \Gamma(b_1 - a_2)} \int_0^1 \frac{(1 - t)^{b_1 - a_2 - 1} t^{a_2 - 1}}{(1 - xt)^{a_1}} dt, \quad (3.21)$$

valid in $|x| < 1$.

This formula proposes a way for an analytical continuation of ${}_2F_1(x)$ outside the unit disk to the domain $|\arg(1 - x)| < \pi$, where the right-hand side of (3.21) represents an analytical function of x . For the same reasons, formulas (3.20) can serve as analytical extensions of the g.h.f-s ${}_{q+1}F_q(x)$, $q \geq 1$ outside $|x| < 1$.

The case with parameters not satisfying condition (3.19) yields generalized fractional derivatives in (3.18) and provides also useful corollaries. By analogy with the previous two cases, we introduce the notion *spherical g.h.f-s of Gauss type* when all the differences $a_k - b_k = \eta_k$, $k = 1, \dots, q$ are nonnegative integers and ${}_{q+1}F_q(x)$ is representable by a purely differential operator of a function $(1 - x)^\beta$ (a special case can be seen in [14], p. 572).

Another interesting case concerns the so-called *hypergeometric polynomials*

$${}_{p+1}F_q(-n, a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{k=0}^n \frac{(-n)_k (a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} \quad (3.22)$$

when $p = q$. By taking $a_{q+1} = -n$, $n \geq 0$ -integer and $a_k > b_k > 0$, $k = 1, \dots, q$, the fractional derivative form of (3.18) turns into the *Rodrigues type formula*

$$\begin{aligned} & \left[\prod_{j=1}^q \frac{\Gamma(a_j)}{\Gamma(b_j)} \right] {}_{p+1}F_q(-n, a_1, \dots, a_q; b_1, \dots, b_q; x) = D_{1,q}^{(b_k-1), (a_k-b_k)} \{(1-x)^n\} \\ & = x^{1-a_q} D_{1,q}^{(b_k-a_q), (a_k-b_k)} \{x^{a_1-1} (1-x)^n\} = x^{1-b_q} D^{a_q-b_q} x^{a_p-b_{q-1}} D^{a_{p-1}-b_{q-1}} \\ & \times \dots x^{a_3-b_2} D^{a_2-b_2} x^{a_2-b_1} D^{a_1-b_1} \{x^{a_1-1} (1-x)^n\}. \end{aligned} \quad (3.23)$$

Special cases of (3.23) yield some *classical Rodrigues formulas*. For example: $p = q = 1$ with $a_1 = n + 1$, $b_1 = 1$ and $x \rightarrow \frac{1-x}{2}$ yields the *Rodrigues formula for the Legendre polynomials*:

$$\begin{aligned}
P_n(x) &= (-1)^n {}_2F_1 \left(-n, n+1; 1; \frac{1-x}{2} \right) \\
&= \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \left[\frac{1-x}{2} \frac{1+x^n}{2} \right] = \frac{1}{2^n n!} \frac{d^n}{dx^n} \{ (x^2-1)^n \};
\end{aligned} \tag{3.24}$$

and $p = q = 2$ with $a_1 = n+1, b_1 = 1, a_2 = \zeta, b_2 = p$ ($\zeta > p > 0$) gives the *Rodrigues formula for the Rice polynomials*, viz.

$$R_n(x) = {}_3F_2(-n, n+1, \zeta; 1, p; x) = \frac{\Gamma(p)}{n! \Gamma(\zeta)} \left[\frac{d^n}{dx^n} x^{1-p} \left(\frac{d}{dx} \right)^{\zeta-p} \right] \{ x^n (1-x)^n \}. \tag{3.25}$$

All the results mentioned in this section show that there are, essentially, *three kinds of g.h.f-s* ${}_pF_q$, similar in properties and reducible to the three elementary functions (1.7) and to the three “initial”(simplest) g.h.f-s ${}_0F_1$, ${}_1F_1$, ${}_2F_1$. Thus, results for them can be obtained by the tools of the generalized fractional calculus.

4. New results on the Wright's generalized hypergeometric functions ${}_p\Psi_q$

In the previous section we have reviewed results on the generalized hypergeometric functions using fractional differintegrals with G -functions only. Now we show that the same theory works for the *Wright's g.h.f-s* ${}_p\Psi_q(x)$, $p < q, p = q, p = q+1$ with operators and representations involving the Fox's H -functions. In this way, we give a solution to the *Open problem E.4*, stated in [11].

Although the ${}_pF_q$ -functions (1.6) encompass almost all the known special functions of mathematical physics, there still exist important examples of generalized hypergeometric functions that could not be included in their scheme and for arbitrary parameters, are not Meijer's G -functions at all. Happily enough, these cases are representable by means of the Fox's H -functions ([14],[18],[19]). Let us mention the *Mittag-Leffler type functions* ([5,v.3],[18],[11,Appendix]):

$$\begin{aligned}
E_\rho(x; \mu) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu + \frac{k}{\rho})} \\
&= x^{\rho(\mu+1)} H_{1,2}^{1,1} \left[-x \left| \begin{matrix} (0, 1) \\ (0, 1), (1 - \mu, \frac{1}{\rho}) \end{matrix} \right. \right], \quad \rho > 0, \quad \mu \in \mathbf{C},
\end{aligned}$$

which have found useful applications as solutions of fractional order differential and integral equations (see e.g. Tuan and Al-Saqabi [20]) and in many other problems of analysis. They reduce to G -functions only for rational values of the parameter ρ .

Other examples are the ${}_p\Psi_q$ -*Wright's generalized hypergeometric functions* (1.8),(1.9), generalizing the ${}_pF_q$ -functions. Interesting particular cases of them are the so-called *Wright's generalized Bessel functions* ([21])

$$J_\nu^\mu(x) := {}_0\Psi_1[(\nu+1, \mu); -x] = H_{0,2}^{1,0}[x|(0,1), (-\nu, \mu)]$$

$$= \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(\nu+k\mu+1)k!}. \quad (4.1)$$

For $\mu = 1$, (4.1) turn into the Bessel functions. The above functions are also misnamed as *Bessel-Maitland functions*, by the second name Maitland of E. M. Wright [21].

To develop an approach to the ${}_p\Psi_q$ -functions, similar to that for the ${}_pF_q$ -functions of Section 3, we are to deal with compositions of fractional integrals and derivatives, similar to the E.-K. operators (2.14),(2.15) but involving 4 instead of 3 arbitrary parameters. The generalized fractional integration operators, suitable for our aims, turn to involve as kernels the Wright's generalized Bessel functions (4.1) instead of the elementary kernels (2.13) of (2.14).

Generalizing the operators and results of Kiryakova [11] and Kalla, Kiryakova [8], in [7] Kalla and Galue have already introduced such operators and their compositions have been shown representable by single integrals, closely looking like (2.4). However, their kernel $H_{m,m}^{m,0}$ -functions involve *two different groups of positive numbers* $\{\beta_k\}_1^m$ and $\{\lambda_k\}_1^m$ in the bottom and top rows of parameters. Kalla and Galue have established also properties of these new operators, completely analogous to that of operators (2.4),(2.10), mentioned in Section 2. Therefore, we give here only the basic definitions and omit the details.

DEFINITION 4.1. Let $\beta > 0, \lambda > 0, \delta \geq 0, \gamma$ be real parameters. We define the *Wright-Erdélyi-Kober (W-E-K) operators of fractional integration* by

$$W_{\beta,\lambda}^{\gamma,\delta} f(x) := I_{\beta,\lambda,1}^{\gamma,\delta} f(x) = \lambda \int_0^1 \sigma^{\lambda(\gamma+1)-1} J_{\gamma+\delta-\lambda(\gamma+1)/\beta}^{-\lambda/\beta}(\sigma^\lambda) f(x\sigma) d\sigma, \quad (4.2)$$

where J_ν^μ stands for the Wright's Bessel function (4.1).

If $\lambda = \beta$, (4.2) reduce to the Erdélyi-Kober operators (2.14). For $\gamma > -\frac{\mu}{\lambda} - 1, \delta \geq 0, \beta \geq \lambda > 0$, the W-E-K operators are considered in the classes $\mathcal{H}_\mu(\Omega)$, (2.20) with $\Omega = \{|x| < R, |\arg \frac{1}{x}| < \frac{\pi}{2}(\frac{1}{\lambda} - \frac{1}{\beta})\}$.

DEFINITION 4.2. With integer $m \geq 1$ and real parameters $\beta_k > 0, \lambda_k > 0, \delta_k \geq 0, \gamma_k, k = 1, \dots, m$, we define the *multiple Wright-Erdélyi-Kober (W-E-K) fractional integrals* by

$$If(x) = I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)} f(x)$$

$$= \begin{cases} \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\lambda_k}, \frac{1}{\lambda_k})_1^m \end{matrix} \right. \right] f(x\sigma) d\sigma, & \text{if } \sum_{k=1}^m \delta_k > 0, \\ f(x), & \text{if } \delta_k = 0, \lambda_k = \beta_k, k = 1, \dots, m. \end{cases} \quad (4.3)$$

It is suggested that

$$\gamma_k > -\frac{\mu}{\lambda_k} - 1, \delta_k \geq 0, \beta_k \geq \lambda_k > 0, \quad (4.4)$$

to ensure that $A = \sum_{k=1}^m \frac{1}{\lambda_k} - \sum_{k=1}^m \frac{1}{\beta_k} \geq 0$ and also to preserve the functional space

$$\mathcal{H}_\mu(\Omega) \text{ with } \Omega = \{|x| < R, |\arg \frac{1}{x}| < \frac{\pi}{2}A\}.$$

The following two relations, following from [18], (2.2.4) and (2.6.1):

$$\begin{aligned} H_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} \gamma + \delta + 1 - \frac{1}{\beta}, \frac{1}{\beta} \\ \gamma + 1 - \frac{1}{\beta}, \frac{1}{\beta} \end{matrix} \right. \right] &= \beta G_{1,1}^{1,0} \left[\sigma^\beta \left| \begin{matrix} \gamma + \delta + 1 - \frac{1}{\beta} \\ \gamma + 1 - \frac{1}{\beta} \end{matrix} \right. \right] \\ &= \beta \sigma^{\beta-1} \frac{(1 - \sigma^\beta)^{\delta-1}}{\Gamma(\delta)} \sigma^{\beta\gamma} \end{aligned} \quad (4.5)$$

$$H_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} \gamma + \delta + 1 - \frac{1}{\beta}, \frac{1}{\beta} \\ \gamma + 1 - \frac{1}{\beta}, \frac{1}{\beta} \end{matrix} \right. \right] = \lambda \sigma^{\lambda(\gamma+1-\frac{1}{\lambda})} J_{\gamma+\delta-\frac{\lambda}{\beta}(\gamma+1)}^{-\lambda/\beta}(\sigma^\lambda), \quad (4.6)$$

show that if $\beta_k = \lambda_k$, then the new "multiple W-E-K" fractional integrals coincide with the m -tuple E-K operators (2.4).

Under conditions (4.4), operational rules, similar to (2.22) - (2.30), are satisfied. Symbols $I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}$ with some of the $\delta_k, k = 1, \dots, m$ negative are interpreted as *multiple W-E-K fractional derivatives* $D_{(\beta_k),(\lambda_k),m}^{(\gamma_k+\delta_k),(-\delta_k)}$, namely:

$$\begin{aligned} I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)} f(x) &= \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} x \frac{d}{dx} + \gamma_r + \delta_r + j - \frac{1}{\beta_r} \right) \right] I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k+\eta_k)} f(x). \end{aligned} \quad (4.7)$$

By analogy with [11], Kalla and Galue [7] have proved by mathematical induction the decomposition relationship

$$I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)} f(x) = \left[\prod_{k=1}^m W_{\beta_k,\lambda_k}^{\gamma_k,\delta_k} \right] f(x) = \left[\prod_{k=1}^m I_{\beta_k,\lambda_k,1}^{\gamma_k,\delta_k} \right] f(x), \quad (4.8)$$

justifying the name "multiple (m -tuple)" operators for (4.3).

The basic result established below can be stated by the following

PROPOSITION 4.3. *All the Wright's generalized hypergeometric functions ${}_p\Psi_q(x)$ can be represented as multiple (q -tuple) Wright-Erdélyi-Kober fractional integrals or derivatives of one of the three basic functions*

$$\cos_{q-p+1}(x), \exp x, {}_1\Psi_0(x) = H_{1,1}^{1,1}(-x) \quad (4.9)$$

depending on whether $p < q, p = q, p = q + 1$.

Naturally, functions (4.9) generalize the three basic functions (1.7), corresponding to the case of ${}_pF_q$ -functions and generalized fractional differintegrals involving the simpler G -functions.

The following lemma is in the base of the results in this section.

LEMMA 4.4. Each ${}_{p+1}\Psi_{q+1}$ -function is representable as a Wright-Erdélyi-Kober fractional integral or derivative of a ${}_p\Psi_q$ -function, namely

$$\begin{aligned} & {}_{p+1}\Psi_{q+1} \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p), (\alpha_{p+1}, A_{p+1}) \\ (\beta_1, B_1), \dots, (\beta_q, B_q), (\beta_{q+1}, B_{q+1}) \end{matrix} \middle| x \right] \\ &= I_{\frac{1}{B_{q+1}}, \frac{1}{A_{p+1}}, 1}^{\alpha_{p+1}-1, \beta_{q+1}-\alpha_{p+1}} {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| x \right] \end{aligned} \quad (4.10)$$

for $|x| < \infty$ ($|x| < 1$ if $p = q + 1$).

The proof of above relation, as well the proofs of the particular statements below follow easily by representing all the functions as H -functions and application of the key formula for an integral of a product of two H -functions ([14],[18],[11,App.]):

$$\begin{aligned} & \int_0^\infty x^{\alpha-1} H_{u,v}^{s,t} \left[\sigma x \middle| \begin{matrix} (c_i, C_i)_1^u \\ (d_l, D_l)_1^v \end{matrix} \right] H_{p,q}^{m,n} \left[\omega x^r \middle| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] dx \\ &= \sigma^{-\alpha} H_{p+v, q+u}^{m+t, n+s} \left[\frac{\omega}{\sigma^r} \middle| \begin{matrix} (a_j, A_j)_1^n, (1-d_l-\alpha D_l, r D_l)_1^v, (a_j, A_j)_{n+1}^p \\ (b_k, B_k)_1^m, (1-c_i-\alpha C_i, r C_i)_1^u, (b_k, B_k)_{m+1}^q \end{matrix} \right]. \end{aligned} \quad (4.11)$$

The cases $p < q, p = q, p = q + 1$ are considered separately, like in Section 3.

1st case: $p < q$

DEFINITION 4.5. By analogy with the hyper-Bessel functions (3.2), we call the g.h.f-s

$$\begin{aligned} & {}_0\Psi_m \left[\begin{matrix} (\beta_1, B_1), \dots, (\beta_m, B_m) \end{matrix} \middle| -x \right] \\ &= H_{0,m+1}^{1,0} \left[x \middle| \begin{matrix} (0, 1), (1-\beta_1, B_1), \dots, (1-\beta_m, B_m) \end{matrix} \right] \\ &= \sum_{k=0}^\infty \frac{x^k}{\Gamma(\beta_1 + k B_1) \dots (\beta_m + k B_m) \cdot k!} \end{aligned} \quad (4.12)$$

Wright's hyper-Bessel functions. The denotation, corresponding to (3.4),(3.2) and (4.1) should be $J_{\beta_1-1, \dots, \beta_m-1}^{B_1, \dots, B_m}(-x)$.

The result, analogous to that provided by the Poisson-Dimovski transformation takes now the following form: each Wright's hyper-Bessel function ${}_0\Psi_{q-p}$ can be represented by means of a Poisson type integral of the $\cos_{p-q+1}$ -function, namely:

$$\begin{aligned} & {}_0\Psi_m \left[\begin{matrix} (\beta_1, B_1), \dots, (\beta_{q-p}) \end{matrix} \middle| -x \right] \\ &= I_{\left(\frac{1}{B_k}, (1), q-p\right)}^{\left(\frac{k}{q-p+1}-1, (\beta_k - \frac{k}{q-p+1})\right)} \left\{ \cos_{q-p+1} \left((q-p+1)x^{\frac{1}{q-p+1}} \right) \right\} \end{aligned} \quad (4.13)$$

Thus, combining the p -times application of (4.10), composition / decomposition property (4.8) and generalized Poisson integral representation (4.13), we obtain the following

THEOREM 4.6. Each ${}_p\Psi_q$ -function, $p < q$ is a generalized q -tuple W - E - K fractional (differ)integral of $\cos_{q-p+1}(x)$, namely:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| -x \right] = I_{(\frac{1}{B_k}), (\lambda_k), q}^{(\gamma_k), (\delta_k)} \left\{ \cos_{q-p+1} \left((q-p+1)x^{\frac{1}{q-p+1}} \right) \right\}, \quad (4.14)$$

with parameters

$$\begin{aligned} \gamma_k &= \begin{cases} \frac{k}{q-p+1} - 1, \\ \alpha_{k-q+p} - 1, \end{cases} ; \delta_k = \begin{cases} \beta_k - \frac{k}{q-p+1}, \\ \beta_k - \alpha_{k-q+p}, \end{cases} ; \\ \lambda_k &= \begin{cases} 1, & k = 1, \dots, q-p \\ \frac{1}{A_{k-q+p}}, & k = q-p+1, \dots, q. \end{cases} \end{aligned} \quad (4.15)$$

If the conditions

$$\begin{aligned} \beta_k &> \frac{k}{q-p+1}, \quad k = 1, \dots, q-p; \quad \beta_k > \alpha_{k-q+p} > 0, \quad k = q-p+1, \dots, q, \\ B_k &\geq 1, \quad k = 1, \dots, q-p; \quad B_k \geq A_{k-q+p}, \quad k = q-p+1, \dots, q \end{aligned} \quad (4.16)$$

are satisfied, then relation (4.14) gives a Poisson type integral representation; otherwise the operator in the R.H.S. should be interpreted as a multiple W - E - K derivative (4.7) and (4.14) turns into a new Rodrigues type formula.

Theorem 4.6 suggests the name *Bessel type Wright's g.h.f.s* for the functions ${}_p\Psi_q$ with $p < q$. The Bessel function and the Wright's Bessel function are the simplest special functions of this class.

2nd case: $p = q$

Applying relation (4.10) to a function ${}_p\Psi_p$ consequently $(p-1)$ times, one get to a ${}_1\Psi_1$ -function which on its side is representable as a Wright-Erdélyi-Kober operator of $\exp x$:

$${}_1\Psi_1 \left[\begin{matrix} (\alpha_1, A_1) \\ (\beta_1, B_1) \end{matrix} \middle| x \right] = I_{\frac{1}{B_1}, \frac{1}{A_1}, 1} \{ \exp x \} \quad (4.17)$$

Combining (4.10) and (4.17), we obtain the form of Proposition 4.3 in this case.

THEOREM 4.7. If $p = q$, each g.h.f. ${}_p\Psi_p(x)$ is an p -tuple W - E - K fractional integral or derivative of the exponent function, namely:

$${}_p\Psi_p \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_p, B_p) \end{matrix} \middle| x \right] = I_{(\frac{1}{B_k}), (\frac{1}{A_k}), p} \{ \exp x \}, \quad (4.18)$$

provided

$$\beta_k > \alpha_k > 0 \quad \text{and} \quad B_k \geq A_k > 0, \quad k = 1, \dots, p. \quad (4.19)$$

If for some indices k the ordinance between α_k and β_k is not as in (4.19), representation (4.18) turns into differintegral one.

Above result justifies separating of all the g.h.f-s ${}_p\Psi_p$ in a class of Wright's g.h.f-s of confluent type, involving the confluent hypergeometric function ${}_1F_1(a; b; x) = \Phi(a; b; x)$ as a simplest case.

3rd case: $p = q + 1$

We call the ${}_{q+1}\Psi_q$ -functions Wright's g.h.f-s of Gauss type and consider them for $|x| < 1$. The q -times application of (4.10) gives the following proposition.

THEOREM 4.8. In the unit disk $|x| < 1$ each Wright's g.h.f. of Gauss type ${}_p\Psi_q$, $p = q + 1$ is a q -tuple Wright-Erdélyi-Kober fractional differintegral of the $H_{1,1}^{1,1}$ -function, generalizing the binomial series from Theorem 3.7, namely:

$$\begin{aligned} {}_{q+1}\Psi_q \left[\begin{matrix} (\alpha_0, A_0), (\alpha_1, A_1), \dots, (\alpha_q, A_q) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| x \right] &= I_{(\frac{1}{B_k}), (\frac{1}{A_k}), q} \left\{ {}_1\Psi_0 \left[\begin{matrix} (\alpha_0, A_0) \\ - \end{matrix} \middle| x \right] \right\} \\ &= I_{(\frac{1}{B_k}), (\frac{1}{A_k}), q} \left\{ H_{1,1}^{1,1} \left[-x \middle| \begin{matrix} (1 - \alpha_0, A_0) \\ (0, 1) \end{matrix} \right] \right\}, \end{aligned} \quad (4.20)$$

under the same conditions (4.19).

When (4.20) is an integral, it generalizes the Euler integral formula (3.21) for the Gauss function. In particular, for $A_0 = 1$ the basic function in the case $p = q + 1$ reduces to the binomial series:

$${}_1\Psi_0 \left[\begin{matrix} (\alpha_0, 1) \\ - \end{matrix} \middle| x \right] = H_{1,1}^{1,1} \left[-x \middle| \begin{matrix} (1 - \alpha_0, 1) \\ (0, 1) \end{matrix} \right] = G_{1,1}^{1,1} \left[-x \middle| \begin{matrix} 1 - \alpha_0 \\ 0 \end{matrix} \right] = (1 - x)^{-\alpha_0}. \quad (4.21)$$

The technical details, concerning the results of Section 4, together with some interesting particular cases and applications will be given in a separate paper.

The surveyed approach allows, by using the tools of the generalized fractional calculus, to transfer the known results for the basic functions (1.7), (4.9) to the complicated generalized hypergeometric functions ${}_pF_q(x)$, ${}_p\Psi_q(x)$, by separating the special functions in three classes of functions with similar properties.

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