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Weighted limit of the Caratheodory metric in a h-extendible boundary point

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Weighted limit of the Caratheodory metric in a h-extendible boundary point of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$

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1 Introduction

I. Graham [8] obtained weighted boundary limits of the Caratheodory and the Kobayashi metrics for strongly pseudoconvex domains. Later on, D. Ma [12] refined Graham's results.

The sharp bounds in terms of small/large constants for these metrics of smooth bounded pseudoconvex domains of finite type in $\mathbb{C}^2$, smooth bounded convex domains of finite type in $\mathbb{C}^n$ and decoupled domains in $\mathbb{C}^n$ were obtained by D. Catlin [4], J.-H. Chen [5] and G. Herbort [9], respectively. Particular cases of of these results are contained in the papers [1] of E. Bedford and J. E. Fornaess and [13] of M. R. Range. In this paper we study the nontangencial limit of the Caratheodory metric $C_D(z, X)$ in a h-extendible boundary point z_0 of a smooth bounded pseudoconvex domain D in $\mathbb{C}^n$. A boundary point is said to be h-extendible [14] (or semiregular [7]), if Catlin's multitype [3] coincides with D'Angelo's type [6]. This class of points includes points of finite type in $\mathbb{C}^2$, convex points of finite type in $\mathbb{C}^n$ and decoupled points in $\mathbb{C}^n$.

Our main result is in the spirit of the Yu results [15] for the generalized Kobayashi-Royden metrics and the Boas-Straube-Yu ones [2] for the Bergman kernel, metric and curvature. We associate to D and z_0 a model domain E. We prove that, when $z \to z_0$ in a nontangencial cone in D with vertex at z_0, the ratio $C_D(z, X)/C_E(\hat{z}_0, \hat{X}) \to 1$, where $\hat{z}_0$ is a fixed interior point of E and the vector $\hat{X}$ depends on X and $\operatorname{dist}(z, \partial D)$.

It is shown in Lemma 4 that the model E is c-hyperbolic. As consequences we obtain sharp lower and upper nontangencial bounds for the Caratheodory metric of D near z_0 and of small perturbations of E, which recover the aforementioned results in [4, 5, 9], when $z \to z_0$ in a cone.

We also compute the Caratheodory metric of a class of models. As a consequence we get a precise weighted limit of this metric in a strongly pseudoconvex boundary point of a C^∞-smooth weakly pseudoconvex domain. D. Ma [12] obtained a similar result for strongly pseudoconvex domains of class C^2 (using the Henkin-Ovrelid L^∞ estimates for $\bar\partial$).

To prove our main result, we associate to each closed to z_0 point $z \in D$ domains $E_{\pm z}$ and a neigbourhood U_z of z_0 with $E_{+z} \cap U_z \subset D \cap U_z \subset E_{-z}$ (more precisely , we consider biholomorphic images of D instead of D itself). Applying a scaling method, we obtain in Lemma 2 the stability results $\lim\limits_{z \to z_0} C_{E_{\pm z} \cap U_z}(z, X)/C_E(\hat{z}_0, \hat{X}) = 1$. Using, in addition, $\bar\partial$-technique, Kohn's global regularity and a certain bumping property equivalent to h-extendibility, we get in Lemma 3 the following localization result: $\liminf\limits_{z \to z_0} C_D(z, X)/C_{E_{-z} \cap U_z}(z, X) \geq 1$. (Thus we overcome the difficulty that L^∞ estimates for $\bar\partial$ do not exist in general.) These facts and the inequality $C_D(z, X) \leq C_{E_{+z} \cap U_z}(z, X)$ prove our main result.

The method, described above, can be also applied to the generalized Kobayashi-Royden metrics, the Bergman and another invariants.

2 Definitions and statement of the main result

Let D be a domain in C^n, $z \in D$ and $X \in \mathbb{C}^n$. Denote by $H(D)$ the space of holomorphic functions on D. The Caratheodory metric is defined by

$$C_D(z,X) = \sup\{|Xf| : f \in H(D), \sup_D |f| \leq 1 \text{ and } f(z) = 0\}.$$

Note that the condition $f(z) = 0$ is superfluous.

Let z_0 be a smooth boundary point of D. The Catlin multitype [3] of z_0 is a certain biholomorphically invariant n-tuple of rational numbers $(m_1, m_2, \ldots, m_n)$ with $1 = m_1$ and $2 \leq m_2 \leq m_3 \leq \ldots \leq m_n \leq \infty$. It follows from the main theorem of [3] that $m_{n+1-k} \leq \Delta_k$ for each $1 \leq k \leq n$, where Δ_k denotes the D'Angelo k-type [6] of z_0 (roughly speaking the maximal order of contact of k-dimensional varieties with the boundary of D at z_0).

The point z_0 is said to be h-extendible [14] (or semiregular [7]) if $m_{n+1-k} = \Delta_k < \infty$ for each $1 \leq k \leq n$.

Let the multitype of z_0 be finity, i.e. $m_n < \infty$ and let $r = r(z)$ be a local definig function for D near z_0. By the definition of Catlin's multitype there are local coordinates $w = \Phi(z)$ near z_0 such that $\Phi(z_0) = 0$ and

$$r(w) = r_\circ \Phi^{-1}(w) = Rew_1 + P('w) + O(\sigma^{1+\alpha}(w))$$

for some positive constant α. Here $'w = (w_2, \ldots, w_n)$, $\sigma(w) = |w_1| + |w_2|^{m_2} + \ldots + |w_n|^{m_n}$ and $P = P_{r,\Phi}$ is a $(1/m_2, \ldots, 1/m_n)$- homogeneous polynomial with no pluriharmonic therms, i.e. $P \circ \pi_t('w) = tP('w)$ for each $t > 0$ and $'w \in \mathbb{C}^n$, where $\pi_t(w) = (tw_1, t^{1/m_2}w_2, \ldots, t^{1/m_n}w_n)$.

If D is pseudoconvex near z_0, then P must be plurusubharmonic (hence each $m_k, 2 \leq k \leq n$, is even). Then the definition of h-extendible point is equivalent to each of the following two conditions [14, 7]:

I: There is a $(1/m_2, \ldots, 1/m_n)$-homogeneous, strictly plurisubharmonic, C^∞-smooth function $\tilde{P}$ on $\mathbb{C}^{n-1} \setminus \{0\}$ such that $\tilde{P} < P$ on $\mathbb{C}^{n-1} \setminus \{0\}$.

II: The boundary point 0 of the domain $E = E_{r,\Phi} = \{w : Rew_1 + P('w) < 0\}$ is of finite type, i.e. $\Delta_1 < \infty$ (hence each boundary point of E is of finite type).

The function $P - \tilde{P}$ is called a bumping and the domain E - a model of D at z_0.

We shall prove the following result.

Theorem: *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. and let $E = E_{r,\Phi}$ be a model of D at a h-extendible boundary point z_0. If Λ is a cone with vertex at z_0 and axis the interior normal to ∂D at z_0, then*

$$\lim_{z \to z_0, z \in \Lambda} \frac{C_D(z,X)}{C_E(\hat{z}_0, \hat{X})} = 1$$

uniformly in all vector fields X, where $\hat{z}_0 = (-1,'0)$ and $\hat{X} = (\pi_{1/r(z)})_* \Phi_* X$.

Note that, since ∂D is smooth, there is a neigbourhood V of z_0 that $\Lambda \cap V \subset D$. So, the above limit makes sense.

The nontangencial approaching can not be removed in general. It is essentional even in the simplest case, when D is a ball [8].

It is not a priori obvious that $C_E(\hat{z}_0, \hat{X}) \neq 0$. In Lemma 4 we prove an estimate for C_E, which implies that E is c-hyperbolic.

3 Proof of the main result by several lemmas

Proof: We shall use [15] that there exist a holomorphic polynomial $Q('w)$ and a real-valued polynomial $S('w)$ such that $Q('w) = O(|'w|^2), S('w) = 0$ and, if $v = \Psi(w) = (w_1 + Q('w),' w)$, $r'(w) = r(w)(1 + S(w))$, then

$$r'(v) = r' \circ \Psi^{-1}(u) = Rev_1 + P('v) + O(|v_1|^2 + \sigma^{1+2\gamma}('v))$$

for some positive constant γ.

Let us perform the change of variables $u = \Psi_\pm(v) = (v_1 \pm A{v_1}^2,' v)$. Set

$$r'_\pm(v) = r(v)(1 \pm 2A(Rev_1 - P('v)), r_\pm(u) = r'_\pm \circ \Psi_\pm^{-1}(u), \Phi_\pm = \Psi_\pm \circ \Psi \circ \Phi, U_{\epsilon,\delta} = \{u : |u_1| < \delta, \sigma('u) < \epsilon\}.$$

We may choose a positive constants a and A such that $\Phi_\pm(D \cap \Phi_\pm^{-1}(U_{a,a})) = \{u \in U_{a,a} : r_\pm(u) < 0\}$, where

$$r_\pm(u) = Reu_1 + P('u) + R_\pm(u),$$

$R_-(u) \le (\sigma^{1+\gamma}('u) - A|u_1|^2)/2$ and $R_+(u) \ge (A|u_1|^2 - \sigma^{1+\gamma}('u)/2$ in $U_{a,a}$.

Set

$$E_{\pm\epsilon} = \{u : Reu_1 + P('u) \pm \epsilon^\gamma \sigma('u) < 0\} \text{ and}$$
$$F_{-\epsilon} = \{u : Reu_1 + P('u) + A|u_1|^2/3 - \epsilon^\gamma \sigma('u) < 0\}.$$

Note that for each $0 < \epsilon \le a$ and $0 < \delta \le a$ we have

$$\Phi_+(D \cap \Phi_+^{-1}(U_{\epsilon,\delta})) \subset F_{-\epsilon} \subset E_{-\epsilon} \text{ and } \Phi_-(D \cap \Phi_-^{-1}(U_{\epsilon,\delta})) \supset E_{+\epsilon} \cap U_{\epsilon,\delta}.$$

We need of the following lemmas.

Lemma 1: *If f and g are positive functions with $\lim_{u \to 0} f(u)/g(u) = 1$, then*

$$\lim_{u \to 0} \frac{C_E(\tilde{z}_0, \pi_{g_*} \Psi_{\pm *} \Psi_* Y)}{C_E(\tilde{z}_0, \pi_{f_*} Y)} = 1$$

uniformly in all $Y \in \mathbb{C}^n$.

Lemma 2: *If $u \to 0$ such that $Reu_1 < 0$ and $\sigma('u)/Reu_1 \to 0$, and $\epsilon = \epsilon(u)$ is such that $\sigma('u) < \epsilon$ and $\epsilon^{1+\gamma}/Reu_1 \to 0$, then all sufficiently small $u \in E_{-\epsilon} \cap U_{\epsilon,\infty}$ and*

$$\liminf_{u \to 0} \frac{C_{E_{-\epsilon} \cap U_{\epsilon,\infty}}(u, Y)}{C_E(\tilde{z}_0, (\pi_{-1/Reu_1})_* Y)} \ge 1$$

uniformly in all vector fields Y.

If, in addition, $Reu_1/\epsilon \to 0$ and $\delta = \delta(u)$ is such that $|Imu_1| < \delta$ and $Reu_1/(\delta - |Imu_1|) \to 0$, then all sufficiently small $u \in E_{+\epsilon} \cap U_{\epsilon,\delta}$ and

$$\limsup_{u \to 0} \frac{C_{E_{+\epsilon} \cap U_{\epsilon,\delta}}(u, Y)}{C_E(\tilde{z}_0, (\pi_{-1/Reu_1})_* Y)} \le 1$$

uniformly in all vector fields Y.

Lemma 3: *If $u \to 0$ and $\epsilon = \epsilon(u)$ is such that $\sigma_{\ln \epsilon/\epsilon}('u) \to 0, Reu_1 \ln \epsilon/\epsilon \to 0$ and $u \in F_{-\epsilon} \cap W_\epsilon$, then $\Phi_+^{-1}(u) \in D$ for all sufficiently small u, and*

$$\liminf_{u \to 0} \frac{C_D(\Phi_+^{-1}(u), \Phi_{+*}^{-1} Y)}{C_{F_{-\epsilon} \cap W_\epsilon}(u, Y)} \ge 1$$

uniformly in all vector field Y, where $W_\epsilon = U_{\sqrt{\epsilon},\epsilon}$.

Lemma 4: *There exists a positive constant c such that for any sufficiently small ϵ it holds*

$$C_{E_{-\epsilon}}(u,Y) \geq c||(\pi_{1/\sigma(Reu_1,'u)})_* Y||$$

for all $u \in E_{-\epsilon}$ and $Y \in \mathbb{C}^n$.

Consequently, if $b\sigma('u) \max_{\sigma('\cdot)=1} P('\cdot) \leq -Reu_1$ $(b > 1)$, there exist a positive constant c' and C such that

$$c' \leq C_{E_{-\epsilon}}(u,Y)||(\pi_{-1/Reu_1})_* Y|| \leq C$$

for all $u \in E_{-\epsilon}$ and $Y \in \mathbb{C}^n$. (Since $P \not\equiv 0$ is plurisubharmonic, the maximum is positive, hence, $u \in E$ and $Reu_1 < 0$.)

Now we ready to prove the theorem. Note that there is a neigbourhood V of z_0 and a cone $\Gamma = \{u : Reu_1 + b||u|| < 0\}$ $(0 < b < 1)$ such that $\Lambda \cap V \subset D$ and $\Phi_\pm(\Lambda \cap V) \subset \Gamma$. Let $z \in \Lambda \cap V$. Set $\delta = a, u^\pm = \Phi_\pm(z)$ and $\epsilon = (-Reu_1^\pm)^{1-\gamma/2}$. We may assume $\gamma < 2$. Then, since $u^\pm \to 0$ and $u^\pm \in \Gamma$, the conditions of the Lemmas 2 and 3 satisfied. Using that $\lim_{u^\pm \to 0} r \circ \Phi_\pm^{-1}(u^\pm)/r_\pm(u^\pm) = 1$ and $\lim_{u^\pm \to 0, u^\pm \in \Gamma} r_\pm(u^\pm)/Reu_1^\pm = 1$, we obtain

$$\liminf_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_E(\hat{z}_0,\hat{X})} = \liminf_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_E(\hat{z}_0,(\pi_{-1/Reu_1^+})_*\Phi_{+*}X)} \text{ by Lemma 1}$$

$$\geq \liminf_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_{E_{-\epsilon}\cap U_{\epsilon,\delta}}(u_+,\Phi_{+*}X)} \text{ by Lemma 2}$$

$$\geq 1 \text{ by } F_{-\epsilon} \subset E_{-\epsilon} \text{ and Lemma 3.}$$

Similarly,

$$\limsup_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_E(\hat{z}_0,\hat{X})} = \limsup_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_E(\hat{z}_0,(\pi_{-1/Reu_1^-})_*\Phi_{-*}X)} \text{ by Lemma 1}$$

$$\leq \limsup_{z\to z_0, z\in\Lambda} \frac{C_D(z,X)}{C_{E_{+\epsilon}\cap U_{\epsilon,\delta}}(u_-,\Phi_{-*}X)} \text{ by Lemma 2}$$

$$\leq 1 \text{ since } \Phi_-^{-1}(E_{+\epsilon}\cap U_{\epsilon,\delta}) \subset D.$$

The theorem is proved.

4 Proof of the lemmas

Proof of Lemma 1: Let $Y = \sum_{j=1}^n y_j \frac{\partial}{\partial u_j}(0)$. Then $Y_1(u) = \pi_{f_*}Y = \sum_{j=1}^n f^{1/m_j}(u) y_j \frac{\partial}{\partial u_j}(0)$ and $Y_2(u) = \pi_{g_*}\Psi_{\pm *}\Psi_* Y = g(u)(1+o(1))y_1 \frac{\partial}{\partial u_1}(0) + \sum_{j=2}^n g^{1/m_j}(u) y_j \frac{\partial}{\partial u_j}(0)$. Since the Caratheodory pseudometric is homogenic, subaditive, continuous and hyperbolic on E (Lemma 4), the lemma follows from the equality $\lim_{u\to 0}(Y_1(u) - Y_2(u))/||Y_1(u)|| = 0$.

Proof of Lemma 2: The inclusions are obviously.

Fix for a moment the point u. Since π_t $(t>0)$ is an automorphism of $E_{\pm\epsilon}$, we have $\pi_{-1/Reu_1}(E_{\pm\epsilon}\cap U_{\epsilon,\delta}) = E_{\pm\epsilon}\cap\tilde{U}_{\epsilon,\delta}$, where $\tilde{U}_{\epsilon,\delta}=\pi_{-1/Reu_1}(U_{\epsilon,\delta})$. Set

$$\tilde{E}_{\pm\epsilon}=\{v: Rev_1+P('v)\mp\epsilon^{1+\gamma}/Reu_1<0\}.$$

Using that $E_{-\epsilon}\cap\tilde{U}_{\epsilon,\infty}\subset\tilde{E}_{-\epsilon}$ and $\tilde{E}_{+\epsilon}\cap\tilde{U}_{\epsilon,\delta}\subset E_{+\epsilon}$, we obtain

$$C_{E_{-\epsilon}\cap\, U_{\epsilon,\infty}}(u,Y)\geq C_{\tilde{E}_{-\epsilon}}(\tilde{u},\tilde{Y}) \text{ and } C_{E_{+\epsilon}\cap\, U_{\epsilon,\delta}}(u,Y)\leq C_{\tilde{E}_{+\epsilon}\cap\tilde{U}_{\epsilon,\delta}}(\tilde{u},\tilde{Y}),$$

where $\tilde{u}=\pi_{-1/Reu_1}(u)=(-1-iImu_1/Reu_1,\pi_{-1/Reu_1}('u))$ and $\tilde{Y}=(\pi_{-1/Reu_1})_*Y$.

Since $\Omega_{\pm\epsilon}(v)=(v_1+(iImu_1\mp\epsilon^{1+\gamma})/Reu_1,'v)$ is a biholomorphic mapping from $\tilde{E}_{-\epsilon}$ to E, it follows that

$$C_{\tilde{E}_{-\epsilon}}(\tilde{u},\tilde{Y})=C_E(\hat{z}_{-\epsilon},\tilde{Y}) \text{ and } C_{\tilde{E}_{+\epsilon}\cap\tilde{U}_{\epsilon,\delta}}(\tilde{u},\tilde{Y})=C_{E\cap V_{\epsilon,\delta}}(\hat{z}_{+\epsilon},\tilde{Y}),$$

where $V_{\epsilon,\delta}=\Omega_\epsilon(\tilde{U}_{\epsilon,\delta})=\{v: |v_1+iImu_1/Reu_1|<-\delta/Reu_1, \sigma('v)<-\epsilon/Reu_1\}$ and $\hat{z}_{\pm\epsilon}=\Omega_{\pm\epsilon}(\tilde{u})=(-1\pm\epsilon^{1+\gamma}/Reu_1,\tilde{u})$.

Since $\sigma('\tilde{u})=-\sigma('u)/Reu_1\to 0$ and $\epsilon^{1+\gamma}/Reu_1\to 0$, we have $\hat{z}_{\pm\epsilon}\to\hat{z}_0$. Then

$$C_E(\hat{z}_{-\epsilon},\tilde{Y})\to C_E(\hat{z}_0,\tilde{Y})$$

by the continuity of the Carathedory metric. This prove the first inequality of the lemma.

To obtain the second inequality, note that $V_{\epsilon,\delta}\subset\{v: |v_1|<(|Imu_1|-\delta)/Reu_1, \sigma('v)<-\epsilon/Reu_1\}$. Then it is not difficult to prove by normal familly arguments that

$$C_{E\cap V_{\epsilon,\delta}}(\hat{z}_{+\epsilon},\tilde{Y})\to C_E(\hat{z}_0,\tilde{Y})$$

under the assumptions of the lemma. This completes its proof.

Proof of Lemma 3: The inclusion is obviously.

We shall use the existence of a holomorhic function q on $\mathbb{C}^{n-1}$ with $q('0)=1$ and $|q('u)|\leq C_1\exp(P('u)-\gamma_1\sigma('u))$ for some positive constants C_1 and γ_1 [14, 15]. (Its construction is given in the proof of Lemma 4.) It is shown in Lemma 5.3 [7] (see also Theorem 3.4 [14]) that there exist a real number s, $0<s<1$, such that the function $p=(1-s)\sum_{k=0}^{\infty}s^k(\exp(\cdot_1)q)_\circ\pi_{2^k}$ is peak for $E_{-\epsilon}$ at the point 0 $(\epsilon^\gamma<\gamma_1)$.

Let $\epsilon\leq\epsilon_0:=\min(1/2,(\gamma_1/2)^{1/\gamma},a^2)$. Let f be an arbitrary holomorphic function on $F_{-\epsilon}\cap W_\epsilon$ with $\sup_{F_{-\epsilon}\cap W_\epsilon}|f|\leq 1$. Let χ be a C^∞-smooth function on $\mathbb{C}^n$ such that $\chi\equiv 0$ on $W_{1/2}$ and $\chi\equiv 1$ on $W_{1/3}$. Set $\chi_\epsilon(u)=\chi(u_1/\sqrt{\epsilon},\pi_{1/\epsilon}('u))$, $p_\epsilon(u)=p_\circ\pi_{-m\ln\epsilon/\epsilon}(u)$ (the positive number m we shall choose later) and $g_\epsilon=fp_\epsilon\bar\partial\chi_\epsilon$.

Set $G_\epsilon=\Phi_+(D\cap\Phi_+^{-1}(W_{\epsilon/2})\setminus W_{\epsilon/3}$. Since $G_\epsilon\subset\subset F_{-\epsilon}\cap W_\epsilon$, we may extend trivially $\tilde{g}_\epsilon=g_\epsilon\circ\Phi_+$ as a C^∞- smooth $\bar\partial$-closed $(0,1)$-form on the whole $\bar{D}$.

By Kohn's global regularity [11] and Sobolev's Lemma, there is a C^∞-smooth function h_ϵ on D with $\bar\partial h_\epsilon=\tilde{g}_\epsilon$ and $||h_\epsilon||_{C^1(D)}\leq C_2||\tilde{g}_\epsilon||_{C^{n+1}(D)}$ for some positive constant C_2, which depends only on D. Here $||h||_{C^k(D)}=\max\sup_D\mathcal{D}h$, where maximum is taken on all differentiation $\mathcal{D}$ in z and $\bar{z}$ of order at most k and $||\tilde{g}||_{C^k(D)}$ denotes the greatest of the norms of the coefficients of $\tilde{g}$.

It is not difficult to see that

$$||\tilde{g}_\epsilon||_{C^{n+1}(D)}\leq(n+1)!||g_\epsilon||_{C^{n+1}(G_\epsilon)}\max_{1\leq k\leq n+1}||\Psi||^k_{C^{n+1}(\Phi_+^{-1}(W_{\epsilon/2}))}.$$

Using the Leibniz formula, we obtain

$$||g_\epsilon||_{C^{n+1}(G_\epsilon)}\leq 2^{n+1}||p_\epsilon||_{C^{n+1}(G_\epsilon)}||\bar\partial\chi_\epsilon||_{C^{n+1}(W_{\epsilon/2})}\leq 2^{n+1}||p_\epsilon||_{C^4(G_\epsilon)}||\epsilon^{-2}||\bar\partial\chi_\epsilon||_{C^{n+1}(\mathbb{C}^n)}.$$

The Cauchy inequalities show that

$$||p_\epsilon||_{C^{n+1}(G_\epsilon)} \leq (n+1)!\mathrm{dist}^{-n-1}(G_\epsilon, \partial(F_{-\epsilon} \setminus W_{\epsilon/4})) \sup_{F_{-\epsilon}\setminus W_{\epsilon/4}} |p_\epsilon|.$$

It follows from the definitons of G_ϵ, $F_{-\epsilon}$ and $W_{\epsilon/4}$ that there is a positive constant C_3 such that $\mathrm{dist}(G_\epsilon, \partial(F_{-\epsilon} \setminus W_{\epsilon/4})) \geq C_3\epsilon^{1+\gamma}$.

On other hand,

$$\sup_{F_{-\epsilon}\setminus W_{\epsilon/4}} |p_\epsilon| \leq C_1 \sup_{F_{-\epsilon}\setminus W_{\epsilon/4}} \exp((\gamma_1\sigma('u) - P('u) - Reu_1)m\ln\epsilon/\epsilon)$$

$$\leq C_1 \sup_{\mathbb{C}^n\setminus W_{\epsilon/4}} \exp(((\gamma_1 - \epsilon^\gamma)\sigma('u) + A|u_1|^2/3)m\ln\epsilon/2\epsilon) \leq C_1\epsilon^{mC_4},$$

where $4C_4 = \min(\gamma_1/2, A/3)$.

Now it is clear that we may choose sufficiently large m such that $|h_\epsilon|_{C^1(D)} \leq \epsilon$ for each $\epsilon \leq \epsilon_0$.

Set $f_\epsilon = (fp_\epsilon\chi_\epsilon) \circ \Phi_+ - h_\epsilon$. We have that f_ϵ is a holomorphic function on D and $\sup_D |f_\epsilon| \leq 1+\epsilon$. Moreover, if $u \in W_{\epsilon/3}$ and $f(u) = 0$ then

$$|\Phi_+^{-1}{}_* Y f_\epsilon| = |p_\epsilon(u)Y_* f - \Phi_+^{-1}{}_* Yh_\epsilon| \leq |p_\epsilon(u)| \cdot |Y_* f| - ||\mathcal{D}\Phi_+^{-1}(u)||n^{3/2}||Y||\epsilon.$$

Here $||\mathcal{D}\Phi_+^{-1}(u)||$ denotes the greatest of the partial derivatives of Φ_+^{-1} in the point u. Since f is arbitrary, we conclude that

$$|p_\epsilon(u)|C_{F_{-\epsilon}\cap W_\epsilon}(u, Y) \leq (1+\epsilon)C_D(\Phi_+^{-1}(u), \Phi_+^{-1}{}_* Y) + ||\mathcal{D}\Phi_+^{-1}(u)||n^{3/2}||Y||\epsilon.$$

It follows from the hypothesis of the lemma that $|p_\epsilon(u)| \to 1$. Using Lemma 4, we obtain that $\lim_{u\to 0} ||Y||/C_{F_{-\epsilon}}(u, Y) = 0$. Now we get the needfull inequality letting $u \to 0$.

Proof of Lemma 4: Since $\pi_t(u)$ and $T_r(u) = (u_1 + ir,' u)$ (r is an real number) are automorhisms of $E_{-\epsilon}$, it suffices to prove that $C_{E_{-\epsilon}}(u, Y) \geq c$ on the set $\{(u, Y) : \sigma(u) = 1, ||Y|| = 1\}$ and $C_E(u, Y) \leq C$ on the set $\{(u, Y) : u_1 = -1, b\sigma('u) \max_{\sigma('\cdot)=1} P('.) \leq 1, ||Y|| = 1\}$.

Since the second set is a compact subset of $E \times \mathbb{C}^n$, the second inequality follows from the continuity of the Caratheodory metric.

Now we shall prove the first inequality. Set $g('u) = \exp(\sum_{k=2}^n u_k\bar{y}_k) + 2\bar{y}_1 - 1$. Let χ be a C^∞-smooth function on $\mathbb{C}^{n-1}$ with $\chi('u) = 0$ if $||'u|| \geq 3/4$ and $\chi('u) = 1$ if $||'u|| \leq 1/2$. Since the function $\varphi = \tilde{P} + (2n+2)\ln|'u|$ is plurisubharmonic, there is a C^∞-smooth function f with $\bar{\partial}f = \bar{\partial}(\chi g)$ and $\int_{\mathbb{C}^{n-1}} |f|^2 e^{-2\varphi}(1 + |'u|^2)^{-2} \leq 2\int_{\mathbb{C}^{n-1}} |\bar{\partial}(\chi g)|^2 e^{-2\varphi}$ [9]. It can be shown in the same way as in the proof of Theorem 3.4 [14] (see also Theorem 3.11 [15]), that $|f|^2 \leq C_1\exp(P - \gamma_1\sigma')$ for some positive constants C_1 and γ_1. Moreover, it follows from the convergence of the integral that $f('0) = \partial f('0) = 0$. Set $h(u) = \exp(Reu_1/2)(f - \chi g)('u)$. Then h is a holomorhic function on $\mathbb{C}^n$, $|Yh(0)| = ||Y||^2$ and, if $\epsilon^\gamma < \gamma_1$, $\sup_{E_{-\epsilon}} |h| \leq C_2$ for some constant C_2. By continuity there is a positive constant a such that $|Yh(u)| \geq ||Y||^2/2$ if $\sigma(u) \leq a$. Set $h_1 = h \circ \pi_{1/a}/C_1$. Then h_1 is a holomorhic function on $\mathbb{C}^n$, $\sup_{E_{-\epsilon}} |h_1| \leq 1$ and $|Yh_1(u)| \geq a||Y||^2/2C_1$ if $\sigma(u) \leq 1$. This ends the proof of Lemma 4.

5 Corollaries and explicit formulas

We obtain immediately from c-hyperbolicity of E and the continuity of the Caratheodory metric the following sharp bounds for $C_D(z,X)$ (in terms of small/large constans).

Corollarry 1: *Under the assumption of the theorem, there exist positive constants c and C such that*

$$c \leq \liminf_{z\to z_0, z\in\Lambda} C_D(z,X)/||\hat{X}|| \leq \limsup_{z\to z_0, z\in\Lambda} C_D(z,X)/||\hat{X}|| \leq C$$

uniformly in all vector fields X.

If the vector field X is nontangencial to ∂D at z_0, the following refinement of the theorem holds.

Corollary 2: *Under the assumptions of the theorem, we have*

$$(i) \qquad \lim_{z\to z_0, z\in\Lambda} C_D(z,X)r(z)/C_E(\hat{z}_0, Y_1) = 2||\bar{\partial}r(z_0)||$$

uniformly in all vector fields X, for which $||X|| \leq c||X_n||$ $(0 < c \leq 1)$; *here* X_n *denotes the normal component of X at* z_0 *and* $Y_1 = (||X_n||, 0, \ldots, 0)$;

$$(ii) \qquad \lim_{z\to z_0, z\in\Lambda} C_D(z,X)r(z)/||X|| = 2||\bar{\partial}r(z_0)||C_E(\hat{z}_0, Y_1(z_0))/||X(z_0)||$$

for each vector field X, for which $X_n/||X||$ *is continuos in* z_0.

Proof: It is not difficult to see that $(\Psi_* X)_1 = 2||\bar{\partial}r(z_0)||X_n + o(1)X$. Then $Y = \hat{X}r(z)/2||\bar{\partial}r(z_0)|| = Y_1 + o(1)X$. Since the Caratheodory pseudometric is homogenic, subaditive, continuous and hyperbolic on E, (i) and (ii) follow from the equalities $\lim_{z\to z_0}(Y(z) - Y_1(z))/||Y_1(z)|| = 0$ and $\lim_{z\to z_0} Y(z)/||X(z)|| = Y_1(z_0)/||X(z_0)||$, respectively.

We can also compute $C_E(\hat{z}_0, X)$ explicitly for certain class of models.

Proposition: *Let a domain E in* $\mathbb{C}^n$ *has the form* $E = \{w : Rew_1 + P('w) < 0\}$, *where P is a* $(\alpha_2, \ldots \alpha_n)$*-homogenous, convex, circular function (i.e.* $P(\zeta' z) = P('z)$ *for all* $'z \in \mathbb{C}^{n-1}$ *and* ζ *with* $|\zeta| = 1$*). Assume that the boundary point 0 of E has finite multitype* $(m_1, \ldots, m_n)$. *Then*

(i) P is a polynomial and $\alpha_k = m_k = \Delta_{n+1-k}$ *for each* $1 \leq k \leq n$, *where* $\alpha_1 = 1$; *in particular,* $0 \in \partial E$ *is a h-extendible point.*

Let $X \in \mathbb{C}^n$. *Then*

(ii) $C_E(\hat{z}_0, X_1) = ||X_1||/2$;

(iii) if $X \not\equiv 0$, *the equation* $|\lambda X_1/2|^2 + P(\lambda' X) = 1$ *has a unique positive solution* $\lambda(X)$; *if* $X = 0$, *we set* $\lambda(X) = \infty$;

(iv) $C_E(\hat{z}_0,' X) = \lambda^{-1}('X)$

(v) if, in addition, P is circular in each variables,
$C_E(\hat{z}_0, X) = \lambda^{-1}(X)$.

Proof. The equalities $m_k = \Delta_{n+1-k}$ follow from Propostion 5. [16]. In particular, the point $0 \in \partial E$ has finite type. Hence $P(0, \ldots, 0, w_k, 0, \ldots, 0) \not\equiv 0$ for each $2 \leq k \leq n$. Then it is easy to see that $\alpha_k = m_k$ and, hence, P is a polynomial.

Now, since the Caratheodory metric of a convex domain coincides with its Kobayashi-Royden metric by Lempert's theorem, we obtain the remain of the proposition, using similar arguments as in the proofs of Corollary 5.4. [14] and the corollary in [2].

Remarks.

1. If we replace the assumptions for convexity and circularity of P with the weaker ones - pseudoconvexity and positivity, then (i) and (ii) also hold (since $P/2$ is a bumping function and $\pi(F) \subset E \subset F$, where $F = \{z : Rez_1 < 0\}$ and $\pi(z) = (z_1,'0)$).

2. If E is a model of a domain D at a point $z_0 \in \partial D$ of finite multitype, then
(a) the multitypes of $z_0 \in \partial D$ and $0 \in \partial E$ coincide;
(b) when D is pseudoconvex near z_0, $z_0 \in \partial D$ is h-extendible iff $0 \in \partial E$ is so.
(c) when D is convex near z_0, E is convex (see the proof of Proposition 2. [16]).

As a consequence of Proposition (v) and the theorem we obtain the following

Corollary 3: *Let z_0 be a strongly pseudoconvex point of a smooth bounded pseudoconvex domain D in $\mathbb{C}^n$. If $r = r(z)$ is a defining function of D near z_0, Λ is a cone with vertex at z_0 and axis the interior normal to ∂D at z_0, then*

$$\lim_{z \to z_0; z \in \Lambda} C_D^{-2}(z, X)(-L_r(z_0, X_t)/r(z) + ||X_n||^2 ||\bar{\partial} r(z_0)||^2/r^2(z)) = 1$$

uniformly in all vector fields X. Here $L_r(z_0, X)$ is the Levi form of r at z_0 and X is splited into its normal and tangencial components X_n and X_t at z_0.

Proof: Note that there exists a unitary transformation A such that

$$r \circ \Phi_1^{-1}(v) = Rev_1 + \tilde{L}_r(0,'v) + \sum_{i,j=2}^{n} b_{ij} v_i v_j + o(|v_1| + |'v|^2),$$

where $\Phi_1(z) = 2||\bar{\partial} r(z_0)||A(z - z_0)$ and $\tilde{L}_r(0, v) = L_r(z_0, \Phi_1^{-1} v)$. Let $\Phi_2(v) = (v_1 + \sum_{i,j=2}^{n} b_{ij} v_i v_j,' v)$ and $\Phi = \Phi_2 \circ \Phi_1$. Then

$$r \circ \Phi^{-1}(w) = Rew_1 + \tilde{L}_r(0,' w) + o(|w_1| + |'w|^2),$$

which shows that the complex ellipsoid

$$E = \{w : Rew_1 + \tilde{L}_r(0,' w) < 0\}$$

is a model of D at z_0. Applying the theorem, we obtain

$$\lim_{z \to z_0; z \in \Lambda} C_D^2(z, X)(-\tilde{L}_r(0, (\Phi * X)_t)/r(z) + ||(\Phi * X)_n||^2 ||\bar{\partial} r(z_0)||^2/r^2(z)) = 1,$$

since $C_E^2(\tilde{z}_0, Y) = \tilde{L}_r(0, Y_t) + ||Y_n||^2/4$ by Proposition (v). Now the corollary follows from the equalities $\tilde{L}_r(0, (\Phi_* X)_t) = L_r(0, X_t)$, $(\Phi_* X)_n = 2||\bar{\partial} r(z_0)||(X_n + o(|z|)X_t)$ and $\lim_{z \to z_0; z \in \Lambda} o(|z|)/r(z) = 0$.

Now we shall consider the case, when z approaches a strongly pseudoconvex point z_0 in arbitrary way. Note that the Levi form is continuous. In particular, it is strictly positive near z_0. Then, checking the proofs of Corollary 3 and the theorem, it is easily to see that, if the aperture of the cone Λ is fixed, the approaching is uniformly in z_0. Observe that, if $z \in D$ is close to z_0, then z lies in the cone $\Lambda_{\pi(z)}$ with vertex the projection $\pi(z)$ of z at ∂D. Thus we obtain the following result, which is more precise than Colorarry 2.

Corollary 3': *Let D, z_0 and r be as in Colorarry 3. Then*

$$\lim_{z \to z_0} C_D^{-2}(z, X)(-L_r(\pi(z), X_t(\pi(z))/r(z) + ||X_n(\pi(z))||^2 ||\bar{\partial} r(\pi(z))||^2/r^2(z)) = 1$$

uniformly in all vector fields X. Here $L_r(\pi(z), X)$ is the Levi form of r at $\pi(z)$ and X is splited into its normal and tangencial components $X_n(\pi(z))$ and $X_t(\pi(z))$ at $\pi(z)$.

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