On Clarke–Ledyaev inequality

Milen Ivanov
Section of Mathematical Analysis, Department of Mathematics,
Sofia University, 5 James Bourchier blvd., 1164 Sofia, Bulgaria
email: milen@fmi.uni-sofia.bg

Nadia Zlateva
Bulgarian Academy of Sciences, Institute of mathematics and Informatics,
Acad. G. Bonchev str., bl. 8, 1113 Sofia, Bulgaria

Abstract

We prove that Clarke-Ledyaev multidirectional mean value inequality holds for lower semicontinuous function on smooth Banach space. As application we establish a formula for the Clarke-Rockafellar directional derivative of lower semicontinuous function.

1 Introduction

The aim of this paper is to extend Clarke-Ledyaev inequality (see [5]) to the setting of smooth Banach spaces. We relax the assumption of boundedness below of the function. As application we establish a formula for the Clarke-Rockafellar directional derivative of lower semicontinuous function on Fréchet smooth space generalizing that of Borwein and Priess [3].

Before stating the results we fix some notations. A bornology β on the Banach space X is family of bounded subsets of X together with the properties: $\{x\} \in \beta$ for arbitrary $x \in X$; if $A \in \beta$ and $B \subset A$ then $B \in \beta$. It is clear that the bornology G of all singletons is contained in any bornology and the bornology F of all bounded sets contains any other bornology.

[0]The work is partially supported by NSFR Grant No MM 701–97.

We denote by B_X the closed unit ball of the Banach space X, while B_X° is the open unit ball. For arbitrary $A \subset X$ we put $\|A\| = \sup_{a\in A} \|a\|$; $\overline{A}$ denotes the norm closure and for $\varepsilon > 0$ put $A_\varepsilon = A + \varepsilon B_X^\circ$, as $v_\varepsilon = \{v\}_\varepsilon$. We set $[A, B] = co\{A, B\}$ for $A, B \subset X$ (co stands for the convex hull).

Let us recall that a Banach space X is said to be β-smooth with respect to certain bornology β if there exists a Lipschitz differentiable bump (i.e. with non empty bounded support) function $b \in C^1_\beta(X) = \{f : X \to I\!R; f$ is Gâteaux differentiable and the derivative is a continuous mapping from X to the dual space X^*, equipped with the topology of uniform convergence on the members of the bornology $\beta\}$.

Our essential tool is the Smooth Variational Principle of Deville, Godefroy and Zizler.

Theorem 1.1 ([7]) *Let X be a β-smooth Banach space. Then for each proper lower semicontinuous and bounded below function $f : X \to I\!R \cup \{+\infty\}$ and every $\varepsilon > 0$, there exists a function $g \in C^1_\beta(X)$ such that $f + g$ attains its minimum on X and $\|g\|_\infty = \sup\{|g(x)|; x \in X\} < \varepsilon$ and $\|g'\|_\infty < \varepsilon$.*

If β is some bornology on X and $f : X \to I\!R \cup \{+\infty\}$ is proper and lower semicontinuous then the β–smooth subdifferential of f at x is

$$D^-_\beta f(x) = \{u'(x) : \ u \in C^1_\beta(X) \text{ and } f - u \text{ has a local minimum at } x\}$$

if $x \in \mathrm{dom} f$ and $D^-_\beta f(x) = \emptyset$ if $f(x) = \infty$. As usual, $\mathrm{dom}\, f = \{x \in X; f(x) \in I\!R\}$.

2 Main results

We now state the Clarke-Ledyaev inequality.

Theorem 2.1 *Let X be a β-smooth Banach space, $C \subset X$ be closed, convex and bounded set; the lower semicontinuous function $f : X \to I\!R \cup \{+\infty\}$ be bounded below on $[a, C] + \varepsilon B_X$ for some $\varepsilon > 0$ and $a \in \mathrm{dom} f$. Given real number r so that*

$$r < \lim_{\varepsilon\downarrow 0} \inf f(C_\varepsilon) - f(a)$$

and arbitrary $\delta > 0$, one can find $x \in [a, C] + \delta B_X$ and $p \in D^-_\beta f(x)$ such that $r < \inf p(C - a)$ and $f(x) < \inf f([a, C]) + |r| + \delta$.

This theorem is proved by Clarke and Ledyaev in the case when X is Hilbert space (see [5]). Aussel, Corvellec and Lasonde in [2] extended it for the abstract subdifferential defined by them in [1] which includes the β-smooth subdifferential but they imposed an additional requirement on the set C. The proof works for general C only if X possesses an equivalent norm such that the square of the distance to any convex set is β-smooth function (in particular if the dual norm is locally β-uniformly rotund). The case of Fréchet smooth Banach space is solved by Corvellec in [6].

Next we show that the assumption "f bounded below on $[a, C]_\varepsilon$" is redundant (although, by careful examination of the proof one can estimate $\|p\|$ by the lower bound of f or the modulus of continuity of f. Such estimates could sometimes be useful as shown in [9]).

Theorem 2.2 *Let X be a β-smooth Banach space, $C \subset X$ be closed, convex and bounded set. Given a lower semicontinuous function $f : X \to I\!R \cup \{+\infty\}$, $a \in \operatorname{dom} f$, real number r so that*

$$r < \lim_{\varepsilon \downarrow 0} \inf f(C_\varepsilon) - f(a)$$

and arbitrary $\delta > 0$, one can find $x \in [a, C] + \delta B_X$ and $p \in D_\beta^- f(x)$ such that $r < \inf p(C - a)$ and $f(x) < f(a) + |r| + \delta$.

In [3] Borwein and Preiss proved that for Lipschitz function f on a space with β-smooth norm holds $f^\circ(x, h) = \limsup_{y \to x} \langle D_\beta^- f(y), h \rangle$. This formula is no longer valid for general lower semicontinuous function (see the example in section 3).

Recall that if $x \in \operatorname{dom} f$, $v \in X$ then

$$f^\circ(x, v) = \sup_{\varepsilon > 0} \limsup_{t \downarrow 0,\ \rho_f(x,y) \to 0} \inf \frac{f(y + tv_\varepsilon) - f(y)}{t},$$

where $\rho_f(x, y) = \|x - y\| + |f(x) - f(y)|$, if x, $y \in \operatorname{dom} f$ and $\rho_f(x, y) = +\infty$, if not and by definition $v_\varepsilon = v + \varepsilon B_X^\circ$. The above formula is due to Rockafellar who use it as the right generalization of Clarke directional derivative to the setting of lower semicontinuous functions (see [4]). Define for $x \in \operatorname{dom} f$ and $v \in X$

$$f^\diamond(x, v) = \sup_{\varepsilon > 0} \inf_{\delta > 0} \sup\{\inf p(v_\varepsilon);\ p \in D_F^- f(y),\ \rho_f(x, y) < \delta\}$$

for all $x \in \operatorname{dom} f$ and $v \in X$.

Proposition 2.3 *Let X be a F-smooth space and the function $f : X \to I\!R \cup \{+\infty\}$ be lower semicontinuous. Then $f^\circ(x, v) = f^\diamond(x, v)$.*

3 Proofs of Theorems 2.1, 2.2

Proof of Theorem 2.1. We may assume that $a = 0$ and $f(0) = 0$ by considering instead of f the function $f_1(x) = f(x+a) - f(a)$. Also, by eventually taking smaller $\delta \in (0, \varepsilon)$ we may assume that $r < \inf f(C_\delta)$.
Since the space X is β-smooth there exists Leduc type function $\psi \in C^1_\beta(X)$ such that $\psi(0) = \psi'(0) = 0$ and $\psi(.) \geq \|.\|^2$, in particular one takes $\psi(.) = \|.\|^2$ if the norm is β-smooth (cf. Proposition II.5.1 from [7]).
Set $D = [0, C]$, $E = \overline{D_\delta}$ and let $M \in I\!R$ be such that $f(E) \dot{>} M$ (by the choice of δ we have that $E \subset [0, C]_\varepsilon$). Take $r_1 \in (r, \inf f(C_\delta))$ and $r_1 < r + 2^{-1}\delta$. Consider the following bounded below lower semicontinuous functions $f_n : X \times X \times I\!R \to I\!R \cup \{+\infty\}$:

$$f_n(x, y, t) = \begin{cases} f(x) + n\psi(ty - x) - r_1 t &, \quad x \in E,\ y \in C,\ t \in [0,1] \\ +\infty &, \quad \text{otherwise} \end{cases}$$

It is trivial that the product of β-smooth spaces is β-smooth (the product of bump functions is again bump). Apply the Smooth Variational Principle of Deville, Godefroy and Zizler to obtain $g_n \in C^1_\beta(X \times X \times I\!R)$, $x_n \in E$, $y_n \in C$ and $t_n \in [0, 1]$, so that $\max\{\|g_n\|_\infty, \|g'_n\|_\infty\} < 2^{-n}$ and the function $h_n := f_n + g_n$ attains its minimum at (x_n, y_n, t_n).

Claim. *For all large enough $n \in I\!N$ we have that $\|x_n - t_n y_n\| < \delta$ and $t_n < 1$.*

Proof of the claim. It is clear that

$$f_n(x_n, y_n, t_n) \leq \inf f_n(E, C, [0,1]) + 2\|g_n\|_\infty < f_n(0,0,0) + 2^{1-n} = 2^{1-n},$$

i.e. $2^{1-n} > f(x_n) + n\psi(t_n y_n - x_n) - r_1 t_n > M - |r_1| + n\|t_n y_n - x_n\|^2$, so $\|t_n y_n - x_n\| \underset{n\to\infty}{\longrightarrow} 0$.

If we assume that the set $N := \{n \in I\!N;\ t_n = 1\}$ is infinite, then for all $n \in N$ it will hold that $2^{1-n} > f_n(x_n, y_n, 1) = f(x_n) + n\psi(y_n - x_n) - r_1 \geq f(x_n) - r_1$. But, as mentioned above $\|y_n - x_n\| < \delta$ for all large enough $n \in N$, so $x_n \in C_\delta$ and $f(x_n) > \inf f(C_\delta)$. From the last computation we have that $\inf f(C_\delta) < r_1 + 2^{1-n}$ for all $n \in N$ and consequently $\inf f(C_\delta) \leq r_1$, which is a contradiction. The claim is proved.

Let us continue with the proof of the theorem. Since $t_n y_n \in D$, from the claim we have that $x_n \in D_\delta$ and that the function $h_n(\cdot, y_n, t_n) = f(\cdot) + n\psi(t_n y_n - \cdot) + g_n(\cdot, y_n, t_n)$ attains its local minimum at x_n. By definition for all large enough n we have $p_n := n\psi'(t_n y_n - x_n) - (g_n)'_x(x_n, y_n, t_n) \in D^-_\beta f(x_n)$.

We shall show that $\inf p_n(C) > r$ for all large $n \in I\!N$. Fix $y \in C$. Since C is convex $y_n + s(y - y_n) \in C$ for all $s \in [0,1]$. Therefore

$$h_n(x_n, y_n + s(y - y_n), t_n) \geq h_n(x_n, y_n, t_n),$$

i.e.

$$n\psi(t_n(y_n + s(y - y_n)) - x_n) - n\psi(t_n y_n - x_n) \geq$$

$$g_n(x_n, y_n, t_n) - g_n(x_n, y_n + s(y - y_n), t_n) \geq -2^{-n}s\|y - y_n\|,$$

where the last inequality holds because the function g_n is 2^{-n}–Lipschitz.

Let $K := \operatorname{diam} C$. Since ψ is at least Gâteaux differentiable at $t_n y_n - x_n$, we divide by $s > 0$ and take a limit as $s \to 0$ to obtain that

$$-2^{-n}K \leq \langle n\psi'(t_n y_n - x_n), y - y_n\rangle$$

for arbitrary $y \in C$. But $\|p_n - n\psi'(t_n y_n - x_n)\| \leq \|g_n'\|_\infty < 2^{-n}$, so

$$\inf p_n(C - y_n) \geq -2^{1-n}K. \tag{3.1}$$

For fixed large n and small enough $s > 0$ we have according to the claim that $t_n + s < 1$, so

$$h_n(x_n, y_n, t_n + s) \geq h_n(x_n, y_n, t_n),$$

i.e. $n\psi((t_n + s)y_n - x_n) - n\psi(t_n y_n - x_n) \geq sr_1 - 2^{-n}s$. Since ψ is Gâteaux differentiable at $t_n y_n - x_n$, when dividing by $s > 0$ and taking $s \to 0$ we obtain $r_1 - 2^{-n} \leq \langle n\psi'(t_n y_n - x_n), y_n\rangle$. As $\|p_n - n\psi'(t_n y_n - x_n)\| < 2^{-n}$ and $y_n \in C$, we have

$$p_n(y_n) > r_1 - (\|C\| + 1)2^{-n}. \tag{3.2}$$

From (3.1) and (3.2) we conclude that $\inf p_n(C) > r_1 + O(2^{-n})$, so for $n \in I\!N$ large enough $\inf p_n(C) > r$.

To estimate $f(x_n)$ we note that for arbitrary $x \in D$, such that $x = ty$, where $y \in C$ and $t \in [0,1]$, we can write $f_n(x_n, y_n, t_n) \leq \inf f_n(E, C, [0,1]) + 2\|g_n\|_\infty \leq f_n(x, y, t) + 2^{1-n} = f(x) - tr_1 + 2^{1-n}$, since $\psi(0) = 0$. Then $f(x_n) - t_n r_1 \leq f_n(x_n, y_n, t_n) \leq f(x) - tr_1 + 2^{1-n}$ and $f(x_n) \leq f(x) + |r_1| + 2^{1-n}$, since $|t - t_1| \leq 1$. Take the infimum over $x \in D$ and recall that $r_1 \in (r, r + 2^{-1}\delta)$ to see that for n large enough $f(x_n) \leq \inf f(D) + |r| + \delta$. ■

Proof of Theorem 2.2: We set again $a = 0$, $f(0) = 0$. It is clear that, if $0 \in C$, then we can apply Theorem 2.1. Hence we need to consider only the case when $0 \notin C$. Of course, without loss of generality, we may assume that $0 \notin C_\delta$ (recall that C is closed) and $r < \inf f(C_\delta)$. Set $D = [0, C]$.

Choose $r_1 \in (r, \inf f(C_\delta))$, $r_1 < r + 2^{-2}\delta$ and fix $\varepsilon \in (0, 2^{-1})$ so small that $\varepsilon < 2^{-2}\delta$; $\frac{\varepsilon}{1-2\varepsilon}\|C_\varepsilon\| < \delta$; $(1-\varepsilon)^{-1}\overline{C_\varepsilon} \subset D_\delta$ and $r_1 < (1-\varepsilon)^{-1} \inf f(C_\delta)$ in the case when $\inf f(C_\delta) < 0$.

For any $y \in X$ define the set

$$A(y) = \{(t, x) \in [0, \varepsilon] \times C_\varepsilon;\ f(y + tx) \leq f(y) + r_1 t\}.$$

Remark that for every $y \in X$ the set $A(y)$ is non-empty, since $(0, x) \in A(y)$ for any $x \in C_\varepsilon$. Define the function $s(y) = \sup\{\|tx\|; (t, x) \in A(y)\}$. From the definitions it is clear that for every $y \in X$ it is fulfilled that $0 \leq s(y) \leq \varepsilon\|C_\varepsilon\|$.

Consider sequences $\{y_n\}_{n=0}^{\infty} \subset X$, $\{x_n\}_{n=1}^{\infty} \subset C_\varepsilon$ and $\{t_n\}_{n=1}^{\infty} \subset [0, \varepsilon]$ such that $y_0 = 0$ and for $n \geq 1$ $(t_n, x_n) \in A(y_{n-1})$ and $t_n\|x_n\| \geq 2^{-1}s(y_{n-1})$; $y_n = \sum_{i=1}^{n} t_i x_i$. Such sequences exist by induction.

We claim that $\sum_{i=1}^{\infty} t_i \leq 1 - \varepsilon$. To this end assume that there exists $n \in I\!N$ such that $1 - \varepsilon \leq \sum_{i=1}^{n} t_i$ and take the smallest $n \in I\!N$ with this property. Then $1 - \varepsilon \leq \sum_{i=1}^{n} t_i < 1$.

Put $z = (\sum_{i=1}^{n} t_i)^{-1} \sum_{i=1}^{n} t_i x_i$. It is clear that $z \in C_\varepsilon$, since the latter is convex. $\|z - y_n\| = \left|1 - (\sum_{i=1}^{n} t_i)^{-1}\right| \left\|\sum_{i=1}^{n} t_i x_i\right\| \leq \frac{\varepsilon}{1-\varepsilon}\|y_n\|$ wherefore $\|y_n\| - \|z\| \leq \frac{\varepsilon}{1-\varepsilon}\|y_n\|$ and $\|y_n\| \leq \frac{1-\varepsilon}{1-2\varepsilon}\|z\| \leq \frac{1-\varepsilon}{1-2\varepsilon}\|C_\varepsilon\|$. Then $\|z - y_n\| \leq \frac{\varepsilon}{1-2\varepsilon}\|C_\varepsilon\| < \delta$ that is $y_n \in C_\delta$. By our construction we have

$$f(y_n) \leq f(y_{n-1}) + t_n r_1 \leq \ldots \leq f(0) + r_1 \sum_{i=1}^{n} t_i < r_1 < \inf f(C_\delta) \leq f(y_n),$$

which yields a contradiction.

Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded ($\|x_n\| \leq \|C_\varepsilon\|$) the sum $\sum_{i=1}^{\infty} t_i x_i$ absolutely converges. Put $y = \sum_{i=1}^{\infty} t_i x_i = \lim_{n\to\infty} y_n$. From the convexity of C_ε it is clear that the points $z_n = (\sum_{i=1}^{n} t_i)^{-1} \sum_{i=1}^{n} t_i x_i = (\sum_{i=1}^{n} t_i)^{-1} y_n \in C_\varepsilon$. Evidently

$z_n \underset{n\to\infty}{\longrightarrow} (\sum_{i=1}^{\infty} t_i)^{-1} y$, so $y \in \sum_{i=1}^{\infty} t_i \overline{C_\varepsilon} \subset D_\delta$.

Now, assume that we can find $t \in (0, \varepsilon]$ and $x \in C_\varepsilon$ such that $f(y + tx) < f(y) + r_1 t$. Put $\overline{x_n} = x + t^{-1}(y - y_n)$ and note that $\overline{x_n} \underset{n\to\infty}{\longrightarrow} x \in C_\varepsilon$. Since C_ε is open set we have $\overline{x_n} \in C_\varepsilon$ for sufficiently large n. Observe that $f(y_n + t\overline{x_n}) = f(y + tx) < f(y) + r_1 t$ and the lower semicontinuity of f at y gives $f(y_n + t\overline{x_n}) < f(y_n) + r_1 t$ for sufficiently large n. Thus $(t, \overline{x_n}) \in A(y_n)$ and $\underline{\lim}_{n\to\infty} s(y_n) \geq \underline{\lim}_{n\to\infty} t\|\overline{x_n}\| = t\|x\| > 0$, since $x \in C_\varepsilon \subset C_\delta \not\ni 0$ and $t > 0$. But from the definition of (t_n, x_n) we have that $s(y_{n-1}) \leq 2t_n\|x_n\| \underset{n\to\infty}{\longrightarrow} 0$, which yields a contradiction. Therefore, for arbitrary $t \in (0, \varepsilon]$ and $x \in C_\varepsilon$ it follows that $f(y + tx) \geq f(y) + r_1 t$. By the lower semicontinuity of f at y it is bounded below on the set $y + 2\delta_1 B_X \subset D_\delta$ for some $\delta_1 \in (0, 2^{-1}\delta)$. Choose $t \in (0, \varepsilon]$ so small that $tC \subset \delta_1 B_X$, that is possible for the boundedness of C. We will apply Theorem 2.1 to the point y and the set $y + tC$ with δ_1. To this end we see that $f(y) \leq \lim_{n\to\infty} f(y_n) \leq |r_1| \sum_{i=1}^{\infty} t_i \leq (1 - \varepsilon)|r_1|$. Also, f is bounded below on $[y, y + tC] + \delta_1 B_X$ by the choice of δ_1 and $\liminf_{\gamma\downarrow 0} f(y + tC + \gamma B_X) - f(y) \geq \inf f(y + tC_\varepsilon) - f(y) \geq r_1 t > rt$ by our previous consideration. According to Theorem 2.1 there are $x \in [y, y + tC] + \delta_1 B_X \subset D_\delta$ and $p \in D_\beta^- f(x)$ such that $\inf p(tC) > rt$, i.e. $\inf p(C) > r$ and $f(x) < f(y) + |r|t + \delta_1 \leq (1 - \varepsilon)|r_1| + |r|\varepsilon + 2^{-1}\delta \leq |r| + \delta$ because of the choice of r_1, δ_1 and ε.

The proof is completed. ∎

Remark: Obviously, if the set C is compact or the function f is uniformly continuous then $\lim_{\varepsilon\downarrow 0} \inf f(C_\varepsilon) = \inf f(C)$ and one can simply take the latter in Theorems 2.1, 2.2. It is natural to ask whether such replacement can be done in general and the answer is no. Consider $X = l^2 = \left\{\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}; \sum_{n=0}^{\infty} x_n^2 < \infty\right\}$ and the subspace Y of the vectors $\{x_n\}_{n=0}^{\infty}$ with $x_0 = 0$. In [8] it is constructed a continuous function $f : X \to \mathbb{R}$ such that $\forall y \in B_X \cap Y \Rightarrow f(y) = 0$, but $\forall x \in X, p \in D_F^- f(x) \Rightarrow \inf p(B_X \cap Y) < -2^{-1}$.

4 Formula for Clarke-Rockafellar directional derivative

Proof of Proposition 2.3. Without loss of generality we take $x = 0 \in \mathrm{dom} f$. Fix $v \in X$.

Let $r < f^\circ(0, v)$. There is $\varepsilon > 0$ such that $r < \inf_{\delta>0} \sup\{\inf p(v_\varepsilon),\ p \in D_F^- f(y),\ \rho_f(0, y) < \delta\}$. Hence we can choose $x_n \xrightarrow[n\to\infty]{}_f 0$ (this means by definition that $\rho_f(x_n, 0) \xrightarrow[n\to\infty]{} 0$), $p_n \in D_F^- f(x_n)$ with $\inf p_n(v_\varepsilon) > r$ for all $n \in I\!N$.

By definition $f(x_n + h) \geq f(x_n) + p_n(h) + \alpha_n(h).\|h\|$, where $\alpha_n(h) \xrightarrow[\|h\|\to 0]{} 0$. Thus

$$\inf \frac{f(x_n + tv_\varepsilon) - f(x_n)}{t} \geq \inf p_n(v_\varepsilon) + \inf \alpha_n(tv_\varepsilon)\|v_\varepsilon\|.$$

Choose $t_n > 0$ so small that $t_n < 2^{-n}$ and $\inf \alpha_n(t_n\|v_\varepsilon\| B_X) > -\dfrac{2^{-n}}{\|v_\varepsilon\|}$ (this is possible, since $\alpha_n(h) \xrightarrow[\|h\|\to 0]{} 0$).

Then $\inf \dfrac{f(x_n + t_n v_\varepsilon) - f(x_n)}{t_n} > r - 2^{-n}$ and

$$\limsup_{t\downarrow 0,\ \rho_f(0,y)\to 0} \inf \frac{f(y + tv_\varepsilon) - f(y)}{t_n} \geq \overline{\lim_{n\to\infty}} \inf \frac{f(x_n + t_n v_\varepsilon) - f(x_n)}{t_n} \geq r,$$

hence $f^\circ(0, v) \geq f^\circ(0, v)$.

From the other side, let $r < f^\circ(0, v)$. Fix $\varepsilon > 0$ so small that

$$\limsup_{t\downarrow 0,\ \rho_f(y,0)\to 0} \inf \frac{f(y + tv_\varepsilon) - f(y)}{t} > r.$$

Find sequences $x_n \xrightarrow[n\to\infty]{}_f 0$, $t_n \downarrow 0$ such that

$$\inf \frac{f(x_n + t_n v_\varepsilon) - f(x_n)}{t_n} > r, \quad \forall n \in I\!N.$$

Take $\varepsilon_1 < \varepsilon$ and put $C^n = x_n + t_n \overline{v_{\varepsilon_1}}$. Then $\lim_{\delta\downarrow 0} \inf f((C^n)_\delta) > t_n r + f(x_n)$. We apply Theorem 2.2 to x_n and C^n to get $y_n \in [x_n, C^n]_{2^{-n}}$, $p_n \in D_F^- f(y_n)$ so that

$$p_n(C^n - x_n) > r.t_n \iff p_n(\overline{v_{\varepsilon_1}}) > r$$

and $f(y_n) < f(x_n) + |r|t_n + 2^{-n}$. It is clear that $y_n \underset{n\to\infty}{\longrightarrow} 0$ and $\overline{\lim_{n\to\infty}} f(y_n) \leq \overline{\lim_{n\to\infty}} f(x_n) = f(0)$, since $x_n \underset{n\to\infty}{\longrightarrow}_f 0$. So $\lim_{n\to\infty} f(y_n) = 0$ by lower semicontinuity of f. We have found $y_n \underset{n\to\infty}{\longrightarrow}_f 0$, $p_n \in D_F^- f(y_n)$ such that $p_n(v_{\varepsilon_1}) > r$. It is easy to see that for every $\delta > 0$

$$\sup\{\inf p(v_{\varepsilon_1}) \ : \ p \in D_F^- f(y),\ \rho_f(y,0) < \delta\} \geq \overline{\lim_{n\to\infty}} \inf p_n(v_{\varepsilon_1}) \geq r.$$

So $f^\circ(0,v) \geq r$.

Finally $f^\circ(0,v) = f^\circ(0,v)$. ■

Example: Consider the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} x^{\frac{1}{2}} - |y|^{\frac{1}{3}} & , \ \text{if} |y| < x^{\frac{3}{2}},\ x \geq 0 \\ 0 & , \ \text{otherwise.} \end{cases}$$

It is easy to compute that $D_F^- f(x,y) = (0,0)$ if $x \leq 0$, or $x > 0$ and $|y| > x^{\frac{3}{2}}$; $D_F^- f(x,y) = (2^{-1}x^{-\frac{1}{2}}, -3^{-1}y^{-\frac{2}{3}}\operatorname{sgn} y)$ if $y \neq 0$, $|y| < x^{\frac{3}{2}}$, $x > 0$; $D_F^- f(x,y) = \{\alpha(2^{-1}x^{-\frac{1}{2}}, -3^{-1}y^{-\frac{2}{3}}\operatorname{sgn} y) \ : \ \alpha \in [0,1]\}$ if $0 \neq |y| = x^{\frac{3}{2}}$; $D_F^- f(x,y) = \emptyset$ if $y = 0$, $x > 0$.

Therefore, $\limsup_{(x,y)\to(0,0)} \langle D_F^- f(x,y),(1,0)\rangle \geq \limsup_{n\to\infty} \langle D_F^- f\left(\frac{1}{n},\frac{1}{n^2}\right),(1,0)\rangle \geq \lim_{n\to\infty} 2^{-1}n^{\frac{1}{2}} = \infty$ and nevertheless $f^\circ((0,0),(1,0)) = 0$.

To see the last equality note that obviously $f^\circ((0,0),(1,0)) \geq 0$ and assume that there is $r > 0$ such that $f^\circ((0,0),(1,0)) > r$. According to Proposition 2.3 $f^\circ((0,0),(1,0)) > r$. By definition there exist $\varepsilon > 0$ and sequences $(x_n, y_n) \underset{n\to\infty}{\longrightarrow} (0,0)$, $(a_n, b_n) \in D_F^- f(x_n, y_n)$ such that $\inf\langle (a_n, b_n), (1,0)_{2\varepsilon}\rangle > r$. Since $(1, \pm\varepsilon) \in (1,0)_{2\varepsilon}$, we have in particular that

$$a_n - \varepsilon|b_n| > r. \tag{4.1}$$

It is easy to see that for all $n \in \mathbb{N}$ we have that $x_n > 0$ and $|y_n| \leq x_n^{\frac{3}{2}}$, since otherwise we would have $(a_n, b_n) = (0,0)$. From the shape of the subdifferential we derive that for all $n \in \mathbb{N}$ there exists $\alpha_n \in [0,1]$ such that $(a_n, b_n) = \alpha_n(2^{-1}x_n^{-\frac{1}{2}}, -3^{-1}y_n^{-\frac{2}{3}}\operatorname{sgn} y_n)$. From (4.1) it follows that $0 < r < \alpha_n(2^{-1}x_n^{-\frac{1}{2}} - \varepsilon 3^{-1}|y_n|^{-\frac{2}{3}})$. In particular $\alpha_n > 0$ and $2^{-1}x_n^{-\frac{1}{2}} > \varepsilon 3^{-1}|y_n|^{-\frac{2}{3}}$. Then $\frac{2\varepsilon}{3}x_n^{\frac{1}{2}} < |y_n|^{\frac{2}{3}} \leq x_n$, i.e. $x_n^{\frac{1}{2}} > \frac{2\varepsilon}{3} > 0$, but $x_n \underset{n\to\infty}{\longrightarrow} 0$, which yields a contradiction.

References

[1] D. Aussel, J.-N. Corvellec and M. Lassonde. Mean value theorem and subdifferential criteria for lower semicontinuous functions, TAMS, 347 (1995), 4147-4161.

[2] D. Aussel, J.-N. Corvellec and M. Lassonde. Nonsmooth constrained optimization and multidirectional mean value inequalities, preprint, Université des Antilles et de la Guyane, 1996.

[3] J. Borwein and D. Preiss. A smooth variational principle with applications to subdifferentiability and differentiability of convex functions, Trans. Am. Math. Soc., 303 (1987), 517-527.

[4] F. Clarke. Optimization and Nonsmooth Analysis, Wiley, New York, 1983.

[5] F. Clarke and Yu. Ledyaev. Mean value inequalities in Hilbert space, Trans. Am. Math. Soc., 344 No. 1 (1994), 307-324.

[6] J.-N. Corvellec. constrained variational principle and mean value inequality in Banach spaces with a smooth bump function, preprint.

[7] R. Deville and G. Godefroy and V. Zizler. Smoothness and Renormings in Banach Spaces, Longman, 1993.

[8] R. Deville and M. Ivanov. Smooth Variational Principles with constraints, Arch. Math., 69 (1997), 418-426.

[9] M. Ivanov and N. Zlateva. Abstract subdifferential calculus and semi-convex functions. Serdica Math. J., 23 (1997), 35-58.