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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
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A PROOF OF 2ND ORDER POINTWISE CONVERGENCRE FOR A FINITE VOLUME SCHEME FOR A CLASS OF INTERFACE PROBLEMS WITH PIECEWISE CONSTANT COEFFICIENTS

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1869

BULGARIAN
ACADEMY
OF SCIENCES

PREPRINT

June 1998

No.2

A proof of 2nd order pointwise convergence for a finite difference scheme for a class of interface problems with piecewise constant coefficients

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Abstract

The paper presents a proof of 2nd order pointwise convergence for a new finite difference scheme for elliptic problems with discontinuous coefficients (so called interface problems). Cell-centred grid is exploited, i.e., the computational domain is divided into grid cells, and the values of the unknown function are related to the cell centers. It is assumed that the diffusivity coefficient is a constant within any grid cell, and that interfaces are aligned with the boundaries of the grid cells. The 2nd order pointwise convergence is proved under the assumption that the normal components of the fluxes of the solution are smooth enough at the midpoints of the finite volumes sides. Numerical experiments, confirming 2nd order pointwise convergence for the new scheme, are presented.

1 Introduction

There exist several approaches for discretizing interface problems. Let us briefly discuss finite difference and finite volume discretizations (existing finite element discretizations will be only mentioned).

The first schemes, suggested for solving interface problems, were based on some averaging of the discontinuous diffusivity coefficient. For a long time, arithmetic averaging was extensively used. However, it was shown (see, for example, [10, 14]) that harmonic averaging leads to more accurate numerical solution. Standard schemes, based on averaging for the diffusivity coefficient, usually exploit minimal stencil - 3 points in 1D case, 5 points in 2D case, 7 points in 3D case. Advantages of this approach are the minimal stencil, the good properties of the grid operator (it is symmetric, positive definite). A lower accuracy (which is even unsatisfactory in the case of arithmetic averaging) can be listed as a disadvantage for this approach. Note, that roughly speaking, standard FE discretization corresponds to arithmetic averaging of the diffusivity coefficient, while non-conforming FE approach corresponds to harmonic averaging.

Some schemes on extended stencils were proposed recently. Generally situated interfaces were considered in [6], while the great part of the papers in the field deal with interfaces, aligned with grid cells boundaries, or with grid nodes. Higher accuracy is main advantage of this approach. However, it may lead to scheme which does not satisfy maximum principle. Another scheme on extended stencil for 2D problems is suggested in ([5]), however there are some restrictions on the spatial grid steps there.

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Another approach is based on splitting the multidimensional interface problem into two subproblems. The first one concerns the solution of a dense problem in a space of a lower dimension aiming at computing the solution on the interface. The second one is to compute the solution in remaining grid points using already computed values on the interface. For details about this approach see, for example, [9, 7]). This approach is accurate, and the maximum principle is satisfied here. However, it is efficient when only one (or a very few) interfaces exist in the computational domain.

Recently an improvement to the harmonic averaging based scheme was presented in [3, 4]. The improved scheme is based on conjugate discretization for the normal components of the flux through the opposite finite volumes sides. The scheme demonstrates high accuracy in numerical experiments. In particular, second order pointwise converegence is observed in numerical experiments for problems where the normal components of the flux are smooth. However, only second order convergenece in W_2^1 is theoretically proven.

Here we present a new finite difference scheme for a class of interface problems. It is very close to the scheme from [3, 4], but now we are able to prove 2nd order pointwise convergence. We restrict our consideration to the problems with diffusivity coefficient being a constant within any subdomain. We also suppose, that interfaces are aligned with grid cells boundaries. However, we can consider as many interfaces as cells boundaries. Our scheme is modification of the harmonic averaging (HA) based scheme, and we will refer to it further as to MHA scheme. It has the same operator as HA, but the right hand side is modified. Thus, the operator of our scheme is symmetric, positive definite. The modification of the right hand side allows 2nd order pointwise convergence to be proved in the multidimensional case, and it also increases the accuracy of the numerical solution.

The reminder of the paper is organized as follows. The modified scheme (MHA) for 1D problems is presented in the next section, together with a proof of 2nd order pointwise convergence. The 2D case is discussed in the third section. The fourth section is devoted to results from numerical experiments. Finally, some conclusions are formulated.

2 One - dimensional interface problems

2.1 Formulation of the problem

In order to illustrate our approach we consider the following 1-D problem: find $u(x)$ such that

$$\frac{\partial W}{\partial x} = f(x), \quad W = -k(x)\frac{\partial u}{\partial x}, \quad 0 < x < 1, \quad u(0) = u_0, \quad u(1) = u_1. \tag{1}$$

Here $k = k(x)$ is the known diffusion coefficient, $W(x)$ is the flux dependent variable, and $f(x)$ is the given source term. If the diffusion coefficient is discontinuous at a certain point ξ_ν (we call it interface point), then conditions for continuity of the function and of the flux through interface are added:

$$[u] = [W] = 0, \text{ for } x = \xi_\nu, \tag{2}$$

where $[u]$ denotes the difference of the right and left limits of u at the point of discontinuity. It is supposed that there are at most a finite number of interface points.

We introduce a cell-centered uniform grid $x_0 = 0.$, $x_1 = h/2$, $x_i = x_{i-1}+h$, $i = 2, 3, \cdots, N$, $x_{N+1} = 1$ with a step-size $h = 1/N$. Non-uniform grid can be treated in a similar way. The values of a function f defined on the grid are denoted by f_i. We define a finite volume $V_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ corresponding to the node x_i, where $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}h$, $x_{i-\frac{1}{2}} = x_i - \frac{1}{2}h$.

Further we assume that there are interfaces at the boundaries of any grid cell, i.e., at any point like $x_{i+\frac{1}{2}}$. We will consider a discretization for the above problem given by

$$L_h Y_i \equiv -\frac{1}{h}\left[k^H_{i+\frac{1}{2}}\frac{Y_{i+1}-Y_i}{h} - k^H_{i-\frac{1}{2}}\frac{Y_i - Y_{i-1}}{h}\right] = f_i + \frac{1}{4}\frac{k_i f_{i+1} - k_{i+1} f_i}{k_i + k_{i+1}} + \frac{1}{4}\frac{k_i f_{i-1} - k_{i-1} f_i}{k_i + k_{i-1}} = \Phi_i \quad (3)$$

where Y stands for the discrete solution, and coefficients $k^H_{i+\frac{1}{2}}$ are given by:

$$k^H_{i+\frac{1}{2}} = 2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right)^{-1}, \quad (4)$$

Remark. It is easy to see that the above scheme can be viewed as a finite volume one, if the following definition for discretization of the flux through interfaces is given:

$$W_{i+\frac{1}{2}} \approx k^H_{i+\frac{1}{2}}\frac{Y_{i+1}-Y_i}{h} + \frac{1}{4}\frac{k_i f_{i+1} - k_{i+1} f_i}{k_i + k_{i+1}}.$$

2.2 Pointwise 2nd order convergence

To prove the 2nd order pointwise convergence for the scheme (3) to the solution of the above interface problem (1), we use a technique which is often used in FEM. That is, we construct $v(x)$, a piecewise quadratic approximant to the solution u of the differential problem. Further, we show that $||u - v||_C = O(h^2)$ in grid points. Next, we show that $||Y - v||_C = O(h^2)$ and thus the proof is completed. Weiser and Wheeler [15] used similar technique to prove 2nd order pointwise convergence of finite volume discretization to Poisson equation on non-uniform grids.

2.2.1 Piecewise quadratic approximant

We construct $v(x)$ in three stages.

A. Constructing piecewise linear interpolant.

At this stage we construct v^L, a piecewise linear interpolant to the solution, satisfying interface conditions. More precisely, v^L is constructed under the following conditions:

$$v^L_i \equiv v^L(x_i) = u_i, \quad v^L_{i+1} \equiv v^L(x_{i+1}) = u_{i+1}$$

$$v^L_{i+\frac{1}{2}-0} = v^L_{i+\frac{1}{2}+0}, \quad k_i\frac{\partial v^L_{i+\frac{1}{2}-0}}{\partial x} = k_{i+1}\frac{\partial v^L_{i+\frac{1}{2}+0}}{\partial x}$$

Note, that such an interpolant is constructed in explicit form, for example, in [10], [14]. We do not use the explicit form, and because of this we do not list it here. We will need further the obvious fact that

$$v^L_{i+\frac{1}{2}-0} = u_{i+\frac{1}{2}-0} + O(h^2) \quad (5)$$

We presented above the construction of the piecewise linear interpolant on (x_i, x_{i+1}). The piecewise linear interpolant in all domain $(0,1)$ is constructed in a straightforward way.

B. Constructing an auxiliary quadratic interpolant.

Denote $R(x) = u(x) - v^L(x)$. Further we construct $v^Q(x), x \in (0,1)$ under the conditions that any its piece $v^{Q,i}(x), x \in V_i$, satisfies the following three relations:

$$v^{Q,i}_{i-\frac{1}{2}+0} \equiv v^{Q,i}(x_{i-\frac{1}{2}+0}) = R_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} - v^L_{i-\frac{1}{2}}$$

$$v^{Q,i}_{i+\frac{1}{2}-0} \equiv v^{Q,i}(x_{i+\frac{1}{2}-0}) = R_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - v^L_{i+\frac{1}{2}}$$

$$-\frac{\partial}{\partial x}\left(k_i \frac{\partial v^{Q,i}_i}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right) = f_i$$

C. Constructing piecewise quadratic approximant.

Now we consider piecewise quadratic approximant, $v(x)$,

$$v(x) = v^L(x) + v^Q(x)$$

By constructing, this piecewise quadratic approximant satisfies the following relations:

$$v_{i+\frac{1}{2}} = v^L_{i+\frac{1}{2}} + v^Q_{i+\frac{1}{2}} = v^L_{i+\frac{1}{2}} + u_{i+\frac{1}{2}} - v^L_{i+\frac{1}{2}} = u_{i+\frac{1}{2}}, \qquad v_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} \tag{6}$$

$$-\frac{\partial}{\partial x}\left(k_i \frac{\partial v_i}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right) = f_i \tag{7}$$

It follows from here that the approximant is continuous on the interfaces, i.e. $[v(x)] = 0$ for $x = x_{i+\frac{1}{2}}, x = x_{i-\frac{1}{2}}$. Also, it is obvious, that $-\frac{\partial}{\partial x}\left(k_i \frac{\partial v(x)}{\partial x}\right) = f_i$ for any $x \in V_i$, and that the third derivative is zero inside any grid cell V_i.

Lemma 1 *Let the solution $u(x)$ be sufficiently smooth within any cell V_i. Then the flux of the approximant v approximates the flux of the solution u on the interfaces with second order. More precisely, the following relations hold*

$$k_i \frac{\partial v_{i+\frac{1}{2}-0}}{\partial x} = k_i \frac{\partial u_{i+\frac{1}{2}-0}}{\partial x} - \frac{h^2}{12} k_i \frac{\partial^3 u_{i+\frac{1}{2}-0}}{\partial x^3} + O(h^3), \quad k_i \frac{\partial v_{i-\frac{1}{2}+0}}{\partial x} = k_i \frac{\partial u_{i-\frac{1}{2}+0}}{\partial x} - \frac{h^2}{12} k_i \frac{\partial^3 u_{i-\frac{1}{2}+0}}{\partial x^3} + O(h^3)$$

Proof. Consider the piece of the approximant $v^i(x)$ in a general form: $v^i(x) = a^i x^2 + q^i x + r^i$. The proof of the above relation is based on direct calculation of coefficients a^i, q^i, r^i. Substituting $v^i(x)$ in (6,7), we obtain

$$v^i(x_{i+\frac{1}{2}}) \equiv a^i(x_i + \frac{h}{2})^2 + q^i(x_i + \frac{h}{2}) + r^i = u_{i+\frac{1}{2}}$$

$$v^i(x_{i-\frac{1}{2}}) \equiv a^i(x_i - \frac{h}{2})^2 + q^i(x_i - \frac{h}{2}) + r^i = u_{i-\frac{1}{2}}$$

$$-2k_i a^i = -\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right) = f_i$$

The last equation implies $a^i = \frac{1}{2k_i}\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right)$. Substituting this in the first and the second equation, and subtracting the second equation from the first one, we obtain

$$hq_i = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} - 2hx_i \frac{1}{2k_i}\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right)$$

Thus, the flux of the approximant is given by

$$k_i \frac{\partial v^i(x)}{\partial x} = k_i 2a^i x + k_i q^i = (x - x_i)\frac{\partial}{\partial x}\left(k_i \frac{\partial u_i}{\partial x}\right) + k_i \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} \tag{8}$$

Consider now the value of this flux at $x_{i+\frac{1}{2}-0}$. Expanding $u_{i-\frac{1}{2}}$ in a series around $x_{i+\frac{1}{2}}$, and after that expanding $\frac{\partial^2 u_{i+\frac{1}{2}}}{\partial x^2}$ in a series around x_i, we obtain:

$$k_i\frac{\partial v^i_{i+\frac{1}{2}-0}}{\partial x} = (x_i + \frac{h}{2} - x_i)\frac{\partial}{\partial x}\left(k_i\frac{\partial u_i}{\partial x}\right) + k_i\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} =$$

$$= \frac{h}{2}\frac{\partial}{\partial x}\left(k_i\frac{\partial u_i}{\partial x}\right) + k_i\frac{1}{h}\left(u_{i+\frac{1}{2}} - u_{i+\frac{1}{2}} + h\frac{\partial u_{i+\frac{1}{2}}}{\partial x} - \frac{h^2}{2}\frac{\partial^2 u_{i+\frac{1}{2}}}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u_{i+\frac{1}{2}}}{\partial x^3} + O(h^4)\right) =$$

$$= \frac{h}{2}\frac{\partial}{\partial x}\left(k_i\frac{\partial u_i}{\partial x}\right) + k_i\frac{\partial u_{i+\frac{1}{2}}}{\partial x} - \frac{h}{2}k_i\frac{\partial^2 u_i}{\partial x^2} - \frac{h^2}{4}k_i\frac{\partial^3 u_i}{\partial x^3} + \frac{h^2}{6}k_i\frac{\partial^3 u_{i+\frac{1}{2}}}{\partial x^3} + O(h^3) =$$

$$= k_i\frac{\partial u_{i+\frac{1}{2}}}{\partial x} - \frac{h^2}{4}k_i\frac{\partial^3 u_{i+\frac{1}{2}}}{\partial x^3} + \frac{h^2}{6}k_i\frac{\partial^3 u_{i+\frac{1}{2}}}{\partial x^3} + O(h^3) =$$

$$= k_i\frac{\partial u_{i+\frac{1}{2}}}{\partial x} - \frac{h^2}{12}k_i\frac{\partial^3 u_{i+\frac{1}{2}}}{\partial x^3} + O(h^3)$$

Thus the first estimate from Lemma 1 is proved. The second one is proved in the same way, and the proof of the **Lemma 1** is completed.

2.2.2 An estimate for the error of the approximant

In this subsection we prove the following Lemma:

Lemma 2 *The approximant $v(x)$ approximates the solution $u(x)$ with second order in any grid point x_i.*

Proof To prove this, let us consider

$$|u_i - v_i| = |u_i - v_i^L - v_i^Q| = |u_i - u_i - v_i^Q| = |-v_i^Q|$$

That is, we have to prove that $v_i^Q = O(h^2)$. Let write the piece of the auxiliary approximant $v^Q(x)$ belonging to V_i in a general form:

$$v^{Q,i}(x) = a^i x^2 + b^i x + c^i, \qquad x \in (x_{i+\frac{1}{2}}, x_{i+\frac{1}{2}})$$

Now consider

$$v^{Q,i}(x_{i+\frac{1}{2}}) \equiv a^i(x_i + \frac{h}{2})^2 + b^i(x_i + \frac{h}{2}) + c^i = R(x_{i+\frac{1}{2}}) = O(h^2) \tag{9}$$

$$v^{Q,i}(x_{i-\frac{1}{2}}) \equiv a^i(x_i - \frac{h}{2})^2 + b^i(x_i - \frac{h}{2}) + c^i = R(x_{i-\frac{1}{2}}) = O(h^2) \tag{10}$$

$$-2k_i a^i = f_i \tag{11}$$

From third equation above we have

$$a^i = -\frac{f_i}{2k_i},$$

and the summation of the first and the second equations gives us

$$2a^i x_i^2 + 2a^i\frac{h^2}{4} + 2b^i x_i + 2c^i = O(h^2).$$

It is seen from here that

$$v^{Q,i}(x_i) \equiv a^i x_i^2 + b^i x_i + c^i = \frac{1}{2}\frac{f_i}{k_i}\frac{h^2}{4} + O(h^2) = O(h^2), \tag{12}$$

and the **Lemma 2** is proved.

2.2.3 An estimate for the difference between the discrete solution and the approximant

We prove here the following lemma:

Lemma 3 *Let the flux W be smooth enough within cells and on interfaces. Then $\|v-Y\| \le Mh^2$ in all grid points.*

Proof To prove this lemma, we will demonstrate that $L_h v_i = \Phi_i + O(h^2)$. It will follow from this that $L_h(v-Y) = O(h^2)$, and from the stability of the above operator equation with respect to the right hand side, the required estimate will follow. So, consider the difference operator from the scheme (3). Substitute $v(x)$ there and expand each value in a series around the nearest interface:

$$L_h v_i \equiv -\frac{1}{h}\left[k^H_{i+\frac{1}{2}}\frac{v^{i+1}_{i+1}-v^i_i}{h} - k^H_{i-\frac{1}{2}}\frac{v^i_i - v^{i-1}_{i-1}}{h}\right] =$$

$$= -k^H_{i+\frac{1}{2}}\frac{1}{h^2}\left[\left(v^{i+1}_{i+\frac{1}{2}+0} + \frac{h}{2}\frac{\partial v^{i+1}_{i+\frac{1}{2}+0}}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v^{i+1}_{i+\frac{1}{2}+0}}{\partial x^2}\right) - \left(v^i_{i+\frac{1}{2}-0} - \frac{h}{2}\frac{\partial v^i_{i+\frac{1}{2}-0}}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v^i_{i+\frac{1}{2}-0}}{\partial x^2}\right)\right] +$$

$$+k^H_{i-\frac{1}{2}}\frac{1}{h^2}\left[\left(v^i_{i-\frac{1}{2}+0} + \frac{h}{2}\frac{\partial v^i_{i-\frac{1}{2}+0}}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v^i_{i-\frac{1}{2}+0}}{\partial x^2}\right) - \left(v^{i-1}_{i-\frac{1}{2}-0} - \frac{h}{2}\frac{\partial v^{i-1}_{i-\frac{1}{2}-0}}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v^{i+1}_{i-\frac{1}{2}-0}}{\partial x^2}\right)\right]$$

Further, we use the fact that the approximant is continuous on the interface, and Lemma 1 which gives us an approximation for the flux of the approximant. We also use the fact that the second derivative of the approximant is known. Let us continue with the expression for discretization of the flux through $x_{i+\frac{1}{2}}$:

$$k^H_{i+\frac{1}{2}}\frac{v_{i+1}-v_i}{h} = \frac{2k_i k_{i+1}}{k_i+k_{i+1}}\left[\frac{1}{2}\frac{\partial v^{i+1}_{i+\frac{1}{2}+0}}{\partial x} + \frac{h}{8}\frac{\partial^2 v^{i+1}_{i+\frac{1}{2}+0}}{\partial x^2} + \frac{1}{2}\frac{\partial v^i_{i+\frac{1}{2}-0}}{\partial x} - \frac{h}{8}\frac{\partial^2 v^i_{i+\frac{1}{2}-0}}{\partial x^2}\right] =$$

$$\frac{1}{k_i+k_{i+1}}\left[k_i k_{i+1}\frac{\partial v^{i+1}_{i+\frac{1}{2}+0}}{\partial x} + k_{i+1}k_i\frac{\partial v^i_{i+\frac{1}{2}-0}}{\partial x} + \frac{h}{4}k_i k_{i+1}\frac{\partial^2 v^{i+1}_{i+\frac{1}{2}+0}}{\partial x^2} - \frac{h}{4}k_{i+1}k_i\frac{\partial^2 v^i_{i+\frac{1}{2}-0}}{\partial x^2}\right] =$$

$$= \frac{1}{k_i+k_{i+1}}\left[k_i\left(k_{i+1}\frac{\partial u_{i+\frac{1}{2}+0}}{\partial x}\right) - k_i\left(k_{i+1}\frac{h^2}{12}\frac{\partial^3 u_{i+\frac{1}{2}+0}}{\partial x^3}\right) + \right.$$

$$\left. +k_{i+1}\left(k_i\frac{\partial u_{i+\frac{1}{2}-0}}{\partial x}\right) - k_{i+1}\left(k_i\frac{h^2}{12}\frac{\partial^3 u_{i+\frac{1}{2}-0}}{\partial x^3}\right) + \frac{h}{4}k_i f_{i+1} - \frac{h}{4}k_{i+1}f_i\right] + O(h^3) =$$

$$= \frac{1}{k_i+k_{i+1}}\left[(k_i+k_{i+1})k_i\frac{\partial u_{i+\frac{1}{2}-0}}{\partial x} - (k_i+k_{i+1})\frac{h^2}{12}k_i\frac{\partial^3 u_{i+\frac{1}{2}-0}}{\partial x^3} + \frac{h}{4}(k_i f_{i+1} - k_{i+1}f_i)\right] + O(h^3) =$$

$$= k_i\frac{\partial u_{i+\frac{1}{2}-0}}{\partial x} - \frac{h^2}{12}k_i\frac{\partial^3 u_{i+\frac{1}{2}-0}}{\partial x^3} + \frac{h}{4}\frac{k_i f_{i+1} - k_{i+1}f_i}{k_i+k_{i+1}} + O(h^3)$$

The condition for the continuity of the flux through interfaces, as well as our assumption that the flux is smooth enough through interfaces, were essentially used in the manipulations above. More precisely, we assumed that the flux is twice continuously differentiable through interfaces, i.e. $\frac{\partial^2}{\partial x^2}\left(k_i\frac{\partial u_{i+\frac{1}{2}-0}}{\partial x}\right) = \frac{\partial^2}{\partial x^2}\left(k_{i+1}\frac{\partial u_{i+\frac{1}{2}+0}}{\partial x}\right)$. The assumption that the diffusivity coefficient is a constant within any cell, was also used.

With similar manipulations we obtain

$$k^H_{i-\frac{1}{2}} \frac{v_i - v_{i-1}}{h} = k_i \frac{\partial u_{i-\frac{1}{2}+0}}{\partial x} - \frac{h^2}{12} k_i \frac{\partial^3 u_{i-\frac{1}{2}+0}}{\partial x^3} - \frac{h}{4} \frac{k_i f_{i-1} - k_{i-1} f_i}{k_i + k_{i-1}} + O(h^3)$$

Now we are ready to finalize evaluating of $L_h v_i$. Using the obtained above expressions for the fluxes, and expanding them around the grid point x_i, we obtain:

$$L_h v_i = -k_i \frac{\partial^2 u_i}{\partial x^2} + O(h^2) + \frac{1}{4} \frac{k_i f_{i+1} - k_{i+1} f_i}{k_i + k_{i+1}} + \frac{1}{4} \frac{k_i f_{i-1} - k_{i-1} f_i}{k_i + k_{i-1}} + O(h^2) = \Phi_i + O(h^2)$$

So, we showed that $L_h(v - Y) = O(h^2)$. It follows from here that $||v - Y|| = O(h^2)$, and thus the **Lemma 3** is proved.

The following theorem is valid:

Theorem 1 *If the flux is sufficiently smooth within cells and on the interfaces, then the discrete solution approximates the exact one with the second order in grid points:*

$$||u - Y||_C \le M\ h^2.$$

Proof. The second order of accuracy in grid points follows from $||u - Y|| \le ||u - v|| + ||v - Y||$, and from Lemma 2 and Lemma 3.

3 Two - dimensional interface problems

3.1 Formulation of the problem

Consider now 2D elliptic equation:

$$Lu \equiv \text{div}\ \mathbf{W} = -\text{div}\,(k\ \text{grad}u)) = f(x_1, x_2), \quad (x_1, x_2) \in G, \tag{13}$$

$$u(x_1, x_2) = g(x_1, x_2), \quad (x_1, x_2) \in \partial G \tag{14}$$

where: $\mathbf{W}$ stands for the flux $\mathbf{W} = -k(x_1, x_2)\text{grad}u$, the unknown function is denoted by u, $k = k(x_1, x_2)$ stands for the discontinuous disffusivity coefficient. The domain G is assumed to be a rectangular. Conditions for continuity of the solution and the normal component of the flux through the interface are added:

$$[u] = 0, \quad \left[k \frac{\partial u}{\partial n}\right] = 0, \quad (x_1, x_2) \in \Gamma \tag{15}$$

The region under consideration is divided into $N_1 \times N_2$ finite volumes. Dimensions of a volume in x_1, x_2 are h_1, h_2, respectively. For simplicity, we suppose through this paper that $h_1 = h_2 = h$. The values of the unknown function are related to the volumes' centers. Additionally, values of the unknown function on the boundary are considered to account for the boundary conditions. Thus, after discretization of the equation and the boundary conditions we have $(N_1+2) \times (N_2+2)$ unknowns . The following notations are used below: $v_P = v_{i,j}$, $v_E = v_{i+1,j}$, $v_W = v_{i-1,j}$, $v_N = v_{i,j+1}$, $v_S = v_{i,j-1}$, $v_e = v_{i+\frac{1}{2},j}$, $v_w = v_{i-\frac{1}{2},j}$, $v_n = v_{i,j+\frac{1}{2}}$, $v_s = v_{i,j-\frac{1}{2}}$, $v_{ne} = v_{i+\frac{1}{2},j+\frac{1}{2}}$, $v_{nw} = v_{i-\frac{1}{2},j+\frac{1}{2}}$, $v_{se} = v_{i+\frac{1}{2},j-\frac{1}{2}}$, $v_{sw} = v_{i-\frac{1}{2},j-\frac{1}{2}}$.

In this paper we consider multidimensional problems that can be discretized in a coordinate-wise way. We consider only interfaces aligned with finite volumes surfaces. Thus discretization of a 2D problem is obtained as tensor product discretization of two 1D problems (like one investigated in the preceding section). A difference scheme approximating 2D interface problem, and converging pointwise with second order to its solution, is written as

$$-\frac{h_1}{h_2}k_s^h\, y_S - \frac{h_2}{h_1}k_w^h\, y_W + \left(\frac{h_1}{h_2}k_s^h + \frac{h_2}{h_1}k_w^h + \frac{h_2}{h_1}k_e^h + \frac{h_1}{h_2}k_n^h\right) y_P - \frac{h_2}{h_1}\bar{k}_e^h\, y_E - \frac{h_1}{h_2}\bar{k}_n^h\, y_N = h_1 h_2 \Phi_P, \quad (16)$$

where k_e^h stands for harmonic averaging of k_P and k_E, etc., and

$$\Phi_P = f_P + \frac{1}{4}\left(\frac{k_P f_E - k_E f_P}{k_P + k_E} + \frac{k_P f_W - k_W f_P}{k_P + k_W}\right) + \frac{1}{4}\left(\frac{k_P f_N - k_N f_P}{k_P + k_N} + \frac{k_P f_S - k_S f_P}{k_P + k_S}\right)$$

It is obvious that the finite difference operator corresponding to the above scheme is symmetric positive definite one. Difference scheme (16) is used in numerical experiments below.

3.2 Pointwise 2nd order convergence

We present here generalization to the 2D case of the proof from the preceding section.

3.2.1 Piecewise biquadratic approximant

We construct $v(x,y)$, a piecewise biquadratic approximant to the solution, in three stages.

A. Constructing piecewise bilinear interpolant.
In 2D case we do this in three substages.

A1. First, we construct an auxliliary 1D linear interpolant $v^{LX,e}(x)$, satisfying the interface conditions at point x_e, and the following interpolation conditions:

$$v_P^{LX,e} = u_P, \qquad v_E^{LX,e} = u_E$$

As in 1D case, it is obvious that

$$v_e^{LX,e} = u_e + O(h^2). \quad (17)$$

In a similar way we construct $v^{LX,w}(x)$, $v^{LY,n}(y)$, $v^{LY,s}(y)$, and in all cases the interpolants approximate the exact solution with second order on the interface, as it is in (17). This procedure is repeated for all grid cells.

A2. Second, having values $v_e^{LX,e}$, $v_n^{LX,n}$, and similar values in negbouring cells, we interpolate v_{ne}, a value in the corner of the grid cell. We repeat this for all corners.

A3. Finally, we construct a bilenar interpolant in any quadrant of any cell. Let us denote by $v^{L,++}(x,y)$ the bilinear interpolant in the top right quadrant of the cell around the node P. It is constructed by the following condition:

$$v_P^{L,++} = u_P, \; v_e^{L,++} = v_e^{LX,e}, \; v_n^{L,++} = v_n^{LY,n}, \; v_{ne}^{L,++} = v_{ne}.$$

The piecewise bilinear interpolant $v^L(x,y)$ over all computational domain is obtained by above described pieces. We will need further the fact that

$$v_{e-0}^L = u_{e-0} + O(h^2), \qquad v_{n-0}^L = u_{n-0} + O(h^2), \; etc. \quad (18)$$

B. Constructing an auxiliary piecewise bi-quadratic interpolant.

Denote $R(x,y)=u(x,y)-v^L(x,y)$. Further we construct $v^Q(x,y), x\in G$ under the conditions that any its piece $v^{Q,P}(x,y),(x,y)\in V_P$, satisfies the following relations:

$$v^{Q,P}_{w+0}=R_w=u_w-v^L_w,\qquad v^{Q,P}_{e-0}=R_e=u_e-v^L_e \tag{19}$$

$$-\frac{\partial}{\partial x}\left(k_P\frac{\partial v^{Q,P}_P}{\partial x}\right)=-\frac{\partial}{\partial x}\left(k_P\frac{\partial u_P}{\partial x}\right)=f_P+\frac{\partial}{\partial y}\left(k_P\frac{\partial u_P}{\partial y}\right) \tag{20}$$

$$v^{Q,P}_{n+0}=R_n=u_n-v^L_n,\qquad v^{Q,P}_{s-0}=R_s=u_s-v^L_s \tag{21}$$

$$-\frac{\partial}{\partial y}\left(k_P\frac{\partial v^{Q,P}_P}{\partial y}\right)=-\frac{\partial}{\partial y}\left(k_P\frac{\partial u_P}{\partial y}\right)=f_P+\frac{\partial}{\partial x}\left(k_P\frac{\partial u_P}{\partial x}\right) \tag{22}$$

C. Constructing piecewise bi-quadratic approximant.

Now we consider piecewise bi-quadratic approximant, $v(x,y)$,

$$v(x,y)=v^L(x,y)+v^Q(x,y) \tag{23}$$

By constructing, it satisfies the following relations:

$$v_e=v^L_e+v^Q_e=v^L_e+u_e-v^L_e=u_e,\quad v_w=u_w,\quad v_n=u_n,\quad v_s=u_s \tag{24}$$

$$-\frac{\partial}{\partial x}\left(k_P\frac{\partial v_P}{\partial x}\right)=-\frac{\partial}{\partial x}\left(k_P\frac{\partial u_P}{\partial x}\right)=f_P+\frac{\partial}{\partial y}\left(k_P\frac{\partial u_P}{\partial y}\right) \tag{25}$$

$$-\frac{\partial}{\partial y}\left(k_P\frac{\partial v_P}{\partial y}\right)=-\frac{\partial}{\partial y}\left(k_P\frac{\partial u_P}{\partial y}\right)=f_P+\frac{\partial}{\partial x}\left(k_P\frac{\partial u_P}{\partial x}\right) \tag{26}$$

It follows from here that the approximant is continuous in the midpoints of interfaces.

Lemma 4 *Let the solution $u(x,y)$ be sufficiently smooth within any cell V_P. Then the following relations hold*

$$k_P\frac{\partial v_{e-0}}{\partial x}=k_P\frac{\partial u_{e-0}}{\partial x}-\frac{h^2}{12}k_P\frac{\partial^3 u_{e-0}}{\partial x^3}+O(h^3),\ \ k_P\frac{\partial v_{w+0}}{\partial x}=k_P\frac{\partial u_{w+0}}{\partial x}-\frac{h^2}{12}k_P\frac{\partial^3 u_{w+0}}{\partial x^3}+O(h^3) \tag{27}$$

$$k_P\frac{\partial v_{n-0}}{\partial y}=k_P\frac{\partial u_{n-0}}{\partial y}-\frac{h^2}{12}k_P\frac{\partial^3 u_{n-0}}{\partial y^3}+O(h^3),\ \ k_P\frac{\partial v_{s+0}}{\partial y}=k_P\frac{\partial u_{s+0}}{\partial y}-\frac{h^2}{12}k_P\frac{\partial^3 u_{s+0}}{\partial y^3}+O(h^3) \tag{28}$$

Proof. The proof of the above relation is based on direct calculation of coefficients of the piecewise bi-quadratic approximant from the equations (24),(25),(26). This is done in two stages: in $x-$ direction, and in $y-$ direction, and calculations in any direction repeat the calculations for the 1D case, presented in the preceding section.

3.2.2 The estimate for the error of the approximant

In this subsection we prove the following Lemma:

Lemma 5 *$v(x, y)$ approximates the solution in point P with second order.*

Proof. To prove this, let us consider

$$|u_P - v_P| = |u_P - v_P^L - v_P^Q| = |u_P - u_P - v_P^Q| = |-v_P^Q|$$

That is, we have to prove that $v_P^Q = O(h^2)$. Let write the piece of the auxiliary approximant $v^Q(x)$ on V_P in a general form:

$$v^{Q,P}(x) = a^P x^2 + b^P xy + c^P y^2 + d^P x + e^P y + g^P, \qquad x \in V_P$$

Now consider

$$v^{Q,P}(x_e) \equiv a^P(x_i + \frac{h}{2})^2 + b^P(x_i + \frac{h}{2})y_j + c^i y_j^2 + d^P(x_i + \frac{h}{2}) + e^P y_j + g^P = R(x_e) = O(h^2)$$

$$v^{Q,P}(x_w) \equiv a^P(x_i - \frac{h}{2})^2 + b^P(x_i - \frac{h}{2})y_j + c^i y_j^2 + d^P(x_i - \frac{h}{2}) + e^P y_j + g^P = R(x_w) = O(h^2)$$

$$v^{Q,P}(x_n) \equiv a^P x_i^2 + b^P x_i(y_j + \frac{h}{2}) + c^i(y_j + \frac{h}{2})^2 + d^P x_i + e^P(y_j + \frac{h}{2}) + g^P = R(x_n) = O(h^2)$$

$$v^{Q,P}(x_s) \equiv a^P x_i^2 + b^P x_i(y_j - \frac{h}{2}) + c^i(y_j - \frac{h}{2})^2 + d^P x_i + e^P(y_j - \frac{h}{2}) + g^P = R(x_n) = O(h^2)$$

$$-2k_P(a^P + c^P) = f_P$$

From the last equation above we have

$$a^P + c^P = -\frac{f_P}{2k_P} \tag{29}$$

After summation of the first, second, third, and fourth equations we obtain

$$v^{Q,P}(x_P) = \frac{1}{2}\frac{f_P}{k_P}\frac{h^2}{4} + O(h^2) = O(h^2), \tag{30}$$

and the **Lemma 5** is proved.

3.2.3 An estimate for the difference between the discrete solution and the approximant

We prove here the following lemma:

Lemma 6 . *Let the normal components $W^{(x)}, W^{(y)}$ of the flux $\mathbf{W} = -gradu$ be smooth enough within cells and on interfaces. Then $||v - Y|| \leq Mh^2$ in grid points.*

Proof. To prove this lemma, we will demonstrate that $L_h v_P = \Phi_P + O(h^2)$. It will follow from this that $L_h(v - Y) = O(h^2)$, and from stability of the above operator equation with respect to the right hand side, the required estimate will follow. So, consider the difference operator from the scheme (16), substitute $v(x_1, x_2)$ there and expand it arround interfaces:

$$L_h v_P \equiv -\frac{1}{h}\left[-k_e^H \frac{v_E^E - v_P^P}{h} + k_w^H \frac{v_P^P - v_W^W}{h}\right] - \frac{1}{h}\left[-k_n^H \frac{v_N^N - v_P^P}{h} + k_s^H \frac{v_P^P - v_S^S}{h}\right] =$$

$$= -k_e^H \frac{1}{h^2}\left[(v_{e+0}^E + \frac{h}{2}\frac{\partial v_{e+0}^E}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{e+0}^E}{\partial x^2}) - (v_{e-0}^P - \frac{h}{2}\frac{\partial v_{e-0}^P}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{e-0}^P}{\partial x^2})\right] +$$

$$-k_w^H \frac{1}{h^2}\left[(v_{w+0}^P + \frac{h}{2}\frac{\partial v_{w+0}^P}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{w+0}^P}{\partial x^2}) - (v_{w-0}^W - \frac{h}{2}\frac{\partial v_{w-0}^W}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{w-0}^W}{\partial x^2})\right] -$$

$$-k_n^H \frac{1}{h^2}\left[(v_{n+0}^N + \frac{h}{2}\frac{\partial v_{n+0}^N}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{n+0}^N}{\partial x^2}) - (v_{n-0}^P - \frac{h}{2}\frac{\partial v_{n-0}^P}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{e-0}^P}{\partial x^2})\right] +$$

$$+k_s^H \frac{1}{h^2}\left[(v_{s+0}^P + \frac{h}{2}\frac{\partial v_{s+0}^P}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{s+0}^P}{\partial x^2}) - (v_{s-0}^S - \frac{h}{2}\frac{\partial v_{s-0}^S}{\partial x} + \frac{h^2}{8}\frac{\partial^2 v_{s-0}^S}{\partial x^2})\right]$$

Further, we use the fact that the approximant is continuous on the interface, and Lemma 4 which gives us an approximation for the normal components of the flux of the approximant. We also use the fact that the normal derivatives of the normal components of the flux of the approximant are known from (25),(26). So, let us continue with the expression for the discretization of the flux through x_e:

$$\frac{2k_P k_E}{k_P + k_E}\frac{v_E^E - v_P^P}{h} = \frac{2k_P k_E}{k_P + k_E}\left[\frac{1}{2}\frac{\partial v_{e+0}^E}{\partial x} + \frac{h}{8}\frac{\partial^2 v_{e+0}^E}{\partial x^2} + \frac{1}{2}\frac{\partial v_{e-0}^P}{\partial x} - \frac{h}{8}\frac{\partial^2 v_{e-0}^P}{\partial x^2}\right] =$$

$$\frac{1}{k_P + k_E}\left[k_P k_E \frac{\partial v_{e+0}^E}{\partial x} + k_E k_P \frac{\partial v_{e-0}^P}{\partial x} + \frac{h}{4} k_P k_E \frac{\partial^2 v_{e+0}^E}{\partial x^2} - \frac{h}{4} k_E k_P \frac{\partial^2 v_{e-0}^P}{\partial x^2}\right] =$$

$$= \frac{1}{k_P + k_E}\left[k_P k_E \frac{\partial u_{e+0}}{\partial x} - k_P k_E \frac{h^2}{12}\frac{\partial^3 u_{e+0}}{\partial x^3} + k_E k_P \frac{\partial u_{e-0}}{\partial x} - k_E k_P \frac{h^2}{12}\frac{\partial^3 u_{e-0}}{\partial x^3}\right.$$

$$\left. + \frac{h}{4} k_P \left(f_E - \frac{\partial}{\partial y}\left(k_E \frac{\partial u_E}{\partial y}\right)\right) - \frac{h}{4} k_E \left(f_P - \frac{\partial}{\partial y}\left(k_P \frac{\partial u_P}{\partial y}\right)\right)\right] + O(h^3) =$$

$$= \frac{1}{k_P + k_E}\left[(k_P + k_E) k_P \frac{\partial u_{e-0}}{\partial x} - (k_P + k_E)\frac{h^2}{12} k_P \frac{\partial^3 u_{e-0}}{\partial x^3} + \frac{h}{4}(k_P f_E - k_E f_P) - \right.$$

$$\left. -\frac{h}{4}\left(k_P \frac{\partial}{\partial y}\left(k_E \frac{\partial u_E}{\partial y}\right) - k_E \frac{\partial}{\partial y}\left(k_P \frac{\partial u_P}{\partial y}\right)\right)\right] + O(h^3) =$$

$$= k_P \frac{\partial u_{e-0}}{\partial x} - \frac{h^2}{12} k_P \frac{\partial^3 u_{e+0}}{\partial x^3} + \frac{h}{4}\frac{k_P f_E - k_E f_P}{k_P + k_E} + O(h^3)$$

$$-\frac{h}{4}\frac{1}{k_P + k_E}\left[k_P \frac{\partial}{\partial y}\left(k_E \frac{\partial u_{e+0}}{\partial y}\right) + \frac{h}{2} k_P \frac{\partial^2}{\partial x \partial y}\left(k_E \frac{\partial u_{e+0}}{\partial y}\right)\right.$$

$$\left. -k_E \frac{\partial}{\partial y}\left(k_P \frac{\partial u_{e-0}}{\partial y}\right) + \frac{h}{2} k_E \frac{\partial^2}{\partial x \partial y}\left(k_P \frac{\partial u_{e-0}}{\partial y}\right)\right]$$

The above expression can be further simplified if the solution is twice continuously differentiable within any cell, and if the x_1 component of the flux is twice continuously differentiable in x_2

direction within any cell. The condition for continuity of solution on the interface, $u(x_{i+\frac{1}{2}+0}, x_2) = u(x_{i+\frac{1}{2}-0}, x_2)$ will imply under the above assumptions that

$$k_P k_E \frac{\partial}{\partial x_2}\left(\frac{\partial u_{e+0}}{\partial x_2}\right) = k_E k_P \frac{\partial}{\partial x_2}\left(\frac{\partial u_{e-0}}{\partial x_2}\right)$$

and the condition for continuity of the normal component of the flux $k_E \frac{\partial u_{e+0}}{\partial x_1} = k_P \frac{\partial u_{e-0}}{\partial x_1}$ will imply

$$\frac{\partial^2}{\partial x_2^2}\left(k_E \frac{\partial u_{e+0}}{\partial x_1}\right) = \frac{\partial^2}{\partial x_2^2}\left(k_P \frac{\partial u_{e-0}}{\partial x_1}\right)$$

Thus, we obtain

$$\frac{2k_P k_E}{k_P + k_E}\frac{v_E^E - v_P^P}{h} = k_P \frac{\partial u_{e-0}}{\partial x} + \frac{h}{4}\frac{k_P f_E - k_E f_P}{k_P + k_E} - \frac{h^2}{12} k_P \frac{\partial^3 u_{e-0}}{\partial x^3} - \frac{h^2}{8} k_P \frac{\partial^3 u_{e-0}}{\partial y^2 \partial x} + O(h^3)$$

With similar manipulations we obtain

$$k_w^H \frac{v_P - v_W}{h} = k_P \frac{\partial u_{w+0}}{\partial x} - \frac{h}{4}\frac{k_P f_W - k_W f_P}{k_P + k_W} - \frac{h^2}{12} k_P \frac{\partial^3 u_{w+0}}{\partial x^3} - \frac{h^2}{8} k_P \frac{\partial^3 u_{w+0}}{\partial y^2 \partial x} + O(h^3)$$

Similar expressions can be obtained for normal components of the fluxes through nord and south boundaries of the cells. Substituting the expressions for normal components of the fluxes in the difference scheme, we obtain

$$L_h v_P = \Phi_P + O(h^2)$$

So, we showed that $L_h(v - Y) = O(h^2)$. It follows from here that $||v - Y|| = O(h^2)$, and thus the **Lemma 6** is proved.

The following theorem is valid:

Theorem 2 *If the normal components of the flux are sufficiently smooth within cells and on the interfaces, then*

$$||u - Y||_C \leq M\ h^2.$$

Proof. The second order of accuracy in grid points follows from $||u - Y|| \leq ||u - v|| + ||v - Y||$, and from Lemma 5 and Lemma 6.

4 Numerical experiments

A 2-D interface problem with different coefficient in 4 subregions and with known analytical solution is solved. Results from computations are presented in Tables 1 and 2 for two different sets of diffusivity coefficients, respectively. Relative C norm stands for $\frac{max|u_{i,j} - Y_{i,j}|}{max|u_{i,j}|}$, where max is taken over all grid nodes. Relative L_2 norm has similar meaning.

Table 1. Relative C norm and relative L_2 norm of the error $u - Y$, and their ratios. Harmonic averaging (HA) based scheme and modified HA scheme (MHA). 4 subregions.
$u^{ex} = \frac{1}{k} sin\left(\frac{\pi x}{2}\right)\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)(1 + x^2 + y^2), \quad k = \{10^1, 10^{-1}, 10^{+3}, 1\}$

	Relative C norm				Relative L_2 norm			
Nodes	HA scheme		MHA scheme		HA scheme		MHA scheme	
12x12	2.32d-3	-	4.79d-4	-	4.52d-4	-	1.09d-4	-
22x22	7.72d-4	*3.00*	7.38d-5	*6.49*	1.26d-4	*3.59*	1.85d-5	*5.89*
42x42	2.33d-4	*3.31*	1.87d-5	*3.93*	3.31d-5	*3.81*	4.56d-6	*4.05*
82x82	6.50d-5	*3.58*	4.83d-6	*3.66*	8.44d-6	*3.92*	1.27d-6	*3.59*
162x162	1.74d-5	*3.74*	1.34d-6	*3.83*	2.13d-6	*3.96*	3.39d-7	*3.75*

Table 2. Relative C norm and relative L_2 norm of the error $u - Y$, and their ratios. Harmonic averaging (HA) based scheme and modified HA scheme (MHA). 4 subregions.
$u^{ex} = \frac{1}{k} sin\left(\frac{\pi x}{2}\right)\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)(1 + x^2 + y^2), \quad k = \{10^{-2}, 1, 10^{-4}, 10^{+6}\}$

	Relative C norm				Relative L_2 norm			
Nodes	HA scheme		MHA scheme		HA scheme		MHA scheme	
12x12	1.75d-2	-	1.74d-3	-	3.07d-3	-	4.12d-4	-
22x22	5.97d-3	*2.93*	2.56d-4	*6.80*	8.38d-4	*3.82*	6.03d-5	*6.80*
42x42	1.80d-3	*3.32*	3.54d-5	*7.23*	2.14d-4	*3.92*	9.13d-6	*7.23*
82x82	5.03d-4	*3.58*	5.49d-6	*6.45*	5.40d-5	*3.96*	1.63d-6	*6.45*
162x162	1.35d-4	*3.73*	1.33d-6	*4.13*	1.35d-5	*4.00*	3.42d-7	*4.13*

The numerical results confirm the proved second order pointwise convergence for considered here scheme. They also demonstrate the advantage of new scheme with respect to harmonic averaging based scheme.

Acknowledgements

The author is in debt to his colleagues N. Kol'kovska, R.Lazarov, and P.Vabishchevich for valuable discussions.

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