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Continuity and boundary behaviour of the Caratheodory metrics

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CONTINUITY AND BOUNDARY BEHAVIOUR OF THE CARATHEODORY METRICS

NIKOLAI NIKOLOV

ABSTRACT. The main purpose of this paper is to study the boundary behaviour of the higher order Caratheodory metrics, the singular Caratheodory metric and the Azukawa metric near an h-extendible boundary point of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$.

1. INTRODUCTION

In this paper we study the continuity and the boundary behaviour of the higher order Carahtheodory the metrics, the singular Caratheodory metric and the Azukawa metric of domains in $\mathbb{C}^n$.

It is well-known that the higher order Caratheodory metrics (except of the usual one) are upper semicontinuous but not continuous in general (cf. [4]). M. Jarnicki and P. Pflug [4] have proved that for bounded domains these metrics are continuous. In the present paper we generalize this result (Theorem 1) and prove the continuity of the higher order Caratheodory metrics of an arbitrary domain in $\mathbb{C}^n$ at the points where the usual Caratheodory metric is positive.

In Theorems 2 and 3 we establish stability and continuity results for the singular Caratheodory metric of strictly hyperconvex domains [7] and model domains [9], respectively. These theorems and a result of St. Nivoche [7] imply that the singular Caratheodory metric and the Azukawa metric coinside on such domains (Corollaries 2 and 3). Note that the continuity of the Azukawa metric for a class of domains, including the hyperconvex domain, has been recently proved by Wl. Zwonek [10].

Using Theorems 1 and 3, we obtain (Theorem 4) the precise nontangential limits of the higher order Caratheodury metrics, the singular Caratheodory metric and the Azukawa metric at an h-extendible boundary point of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. The case of the usual Caratheodory metric has been studied in [6].

2. Definitions and statements of the results

Let D be a domain in $\mathbb{C}^n$. Denote by $Hol(D,\Delta)$ the space of all holomorphic mappings from D into the unit disc Δ in $\mathbb{C}$. For each $k \in \mathbb{N}$, we define the k-th Caratheodory metric as follows (cf. [4]):

$$C_D^{(k)}(z,X) = \sup\{|f_{(k)}(z)X| : f \in Hol(D,\Delta), ord_z f \geq k\},$$

where $f_{(k)}(z)X = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f(z) X^\alpha$ and $ord_z f$ stands for the order of vanishing of f at z. Clearly $C_D^{(1)}(z,X)$ is just the usual Caratheodory metric. It is easy to see that

$$(C_D^{(k)}(z,X))^k (C_D^{(l)}(z,X))^l \leq (C_D^{(k+l)}(z,X))^{k+l} \ \forall k,l \in \mathbb{N}.$$

Thus the following limit exists:

$$\lim_{k\to\infty} C_D^{(k)}(z,X) =: C_D^\infty(z,X)$$

and it is called singular Caratheodory metric.

The Green function and the Azukawa metric are defined by

$g_D(z,w) = \sup\{u(w) : u$ is negative psh function on D such that $u(v) \leq \log|v-z| + O_u(1)\}$,

$$A_D(z,X) = \limsup_{\lambda\to 0, \lambda\neq 0} \frac{\exp g(z, z+\lambda X)}{|\lambda|}.$$

Let us note that these two functions are upper semicontinuous on $D\times D$ and $D\times\mathbb{C}^n$, respectively [5]. Moreover, the following inequalities hold:

$$C_D^{(k)}(z,X) \leq C_D^\infty(z,X) \leq A_D(z,X).$$

A domain D is said to be strictly hyperconvex if there exist another domain $D' \supset\supset D$ and a psh function φ on D' such that $\lim_{z\to\partial D}\varphi(z) = 0$ and $D = \{z \in D' : \varphi(z) < 0\}$.

Theorem 1. *Let D be a domain in $\mathbb{C}^n$ and let the point $z_0 \in D$ be such that $C_D^{(1)}(z_0,X) > 0\ \forall X \in \mathbb{C}^n \setminus \{0\}$. Then for each $k \in \mathbb{N}$ there exist a constant $C > 0$ and a neighbourhood $U \subset D$ of z_0 such that*

$$|C_D^{(k)}(z,X) - C_D^{(k)}(w,Y)| \leq C(||X-Y|| + (||X|| + ||Y||)||z-w||)$$

$\forall z,w \in U, \forall X,Y \in \mathbb{C}^n$.

Note that for an arbitrary domain D the function $C_D^k(z,\cdot)$ is continuous and log-psh on $\mathbb{C}^n$ but the function $C_D^k(\cdot,X)$ is not continuous ($k \geq 2$).

An immediate consequence of Theorem 1 is the following

Corollary 1. *Under the hypothesis of Theorem 1 the singular Caratheodory metric* $C_D^{\infty}(z,X)$ *is lower semicontinuous on* $U \times \mathbb{C}^n$.

Theorem 2. *Let* D *be a strictly hyperconvex domain in* $\mathbb{C}^n$. *Let* $\{D_j\}_{j=1}^{\infty}$ *be a sequence of domains such that* $\lim_{j\to\infty} h(\partial D_j, \partial D) = 0$, *where* h *denotes the Hausdorff distance. If* $D_j \times \mathbb{C}^n \ni (z_j, X_j) \to (z, X) \in D \times \mathbb{C}^n$, *then*

$$\lim_{j\to\infty} C_{D_j}^{\infty}(z_j, X_j) = C_D^{\infty}(z, X).$$

Corollary 2. *The singular Caratheodory metric of a stictly hyperconvex domain in* $\mathbb{C}^n$ *is continuous and coinsides with its Azukawa metric.*

Moreover, under the hypotheses of Theorem 2 we have

$$\lim_{j\to\infty} A_{D_j}(z_j, X_j) = A_D(z, X).$$

To state the next results, we first recall the notation of an h-extendible point [9] (or, a semiregular points in the terminology of [2]). Let p be a smooth, finite-type pseudoconvex boundary point of a domain D in $\mathbb{C}^n$ with Catlin's multitype $M = (m_1, m_2, \ldots, m_n)$ and D'Angelo's q-type Δ_q. Then p is said to be an h-extendible point if $\Delta_q = m_{n+1-q}$ for $1 \le q \le n$.

By [9] we may choose local holomorphic coordinates $z = (z_1, z')$ in which $p = 0$ and such that near p the domain D is defined by the equation $r < 0$ with

$$r(z) = Re z_1 + P(z') - R_1(z) + R_2(z).$$

Here P is a psh polynomial that is weighted homogeneous with respect to $(m_2, \ldots, m_n)$ and

$$0 \le R_1(z) \le C(Im z_1)^2, |R_2(z)| \le C((\sigma(z'))^{1+\gamma}$$

for $\sigma(z') = \sum_{j=2}^{n} |z_j|^{m_j}$ and some positive constants C and γ.

For each $\epsilon \in \mathbb{R}$ set

$$E_\epsilon = \{z \in \mathbb{C}^n : Re z_1 + P(z') + \epsilon\sigma(z') < 0\}.$$

The unbounded domain $E := E_0$ is said to be a model for D at p. It has been proved independently in [9] and [2] that a boundary point p of D is h-extendible iff the corresponding model E is a finite type domain (i.e. all its boundary points are of finite type).

Set $S_R = \{z \in \mathbb{C}^n : \sigma(z') := |z_1| + \sigma(z') < R\}$ for $R > 0$.

Theorem 3. *Let E be a finite type model domain and let $R(\epsilon)$: $\mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ be a function such that $\lim_{\epsilon\to 0} R(\epsilon) = +\infty$. Let $F_\epsilon = E_\epsilon \cap S_{R(\epsilon)}$, $(z_\epsilon, X_\epsilon) \in F_\epsilon \times \mathbb{C}^n$ and $\lim_{\epsilon\to 0}(z_\epsilon, X_\epsilon) = (z, X) \in E \times \mathbb{C}^n$. Then*

$$\lim_{\epsilon\to 0} C^{\infty}_{F_\epsilon}(z_\epsilon, X_\epsilon) = C^{\infty}_E(z, X).$$

Corollary 3. *The singular Caratheodory metric of a finite type model domain is continuous and coinsides with its Azukawa metric. Moreover, under the hypotheses of Theorem 3 we have*

$$\lim_{\epsilon\to 0} A_{F_\epsilon}(z_\epsilon, X_\epsilon) = A_E(z, X).$$

Theorem 4. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and let E be a model domain of D at an h-extendible boundary point $p \in \partial D$. Let S denotes any of the higher order Caratheodory metrics or the singular Caratheodory metric or the Azukawa metric. If X is a (1,0) vector field and Λ is a nontangential cone in D with vertex at p, then*

$$\lim_{\Lambda\ni z\to p} \frac{C^{(k)}_D(z, X_z)}{C^{(k)}_E(e, X(z))} = 1,$$

uniformly in X.

Here $e = (-1, 0')$ and

$X(z) = ((-r(z))^{-1}(X_z)_1, (-r(z))^{-1/m_2}(X_z)_2, \dots, (-r(z))^{-1/m_n}(X_z)_n)$,

where $(X_z)_j$ is the j-th component of X_z in the local coordinates described above.

The limit in Theorem 4 makes sense, since the model domain E is hyperbolic with respect to the usual Caratheodory metric [6].

3. Proofs of the results

Proof of Theorem 1. Since $C^{(1)}_D(z, X)$ is a continuous function, which is homogeneous in the second variable, it follows from the hypotesis of the theorem that there exist a constant $C_1 > 0$ and a neighbourhood $U \subset\subset D$ of z_0 such that $C^{(1)}_D(z, X) \ge C_1||X||$ $\forall z \in U, \forall X \in \mathbb{C}^n$. Then $\forall z, w \in U, \forall X, Y \in \mathbb{C}^n$ we have

$$(1) \qquad |C^{(k)}_D(z, X) - C^{(k)}_D(w, Y)| \le |(C^{(k)}_D(z, X))^k - (C^{(k)}_D(w, Y))^k|(C_1 \max(||X||, ||Y||)^{1-k}.$$

Thus, to prove the theorem it suffices to estimate separately

$|(C^{(k)}_D(w, X))^k - (C^{(k)}_D(w, Y))^k|$ and $|(C^{(k)}_D(w, X))^k - (C^{(k)}_D(z, X))^k|$.

We may suppose that $C_D^{(k)}(w,X) \geq C_D^{(k)}(w,Y)$. By normal family arguments we can find an extremal function f for $C_D^{(k)}(w,X)$. Then by the Cauchy inequalities we obtain

$$(2)\quad 0 \leq (C_D^{(k)}(w,X))^k - (C_D^{(k)}(w,Y))^k \leq |f_{(k)}(w)X| - |f_{(k)}(w)Y| \leq dist^{-k}(U,\partial D)C_2\|X-Y\|(\|X\|+\|Y\|)^{k-1},$$

where $C_2 > 0$ is a constant which only depends on n and k.

Now we shall estimate $|(C_D^{(k)}(z,X))^k - (C_D^{(k)}(w,Y))^k|$.

Denote by $CV_D(z)$ the Caratheodory-Eisenman volume of D at a point $z \in D$, i.e. $CV_D(z) = \sup \det\ |JF(z)|$, where the supremum is taken over all holomorhic mappings F from D into the unit polydisc in $\mathbb{C}^n$ with $F(z) = 0$ and $JF = (\dfrac{\partial F_k}{\partial z_l})_{k,l}$ is the complex Jacobi matrix of F. As it is mentioned in [8] $CV_D(z) > 0$ iff $C_D^{(1)}(z,X) > 0\ \forall X \in \mathbb{C}^n \setminus \{0\}$. Since $CV_D(z)$ is a continuous function then $CV_D(z) \geq C_3 > 0\ \forall z \in U$.

Let $z \in U$ and F be an extremal mapping for $CV(z)$. For each $l \in \mathbb{N}, 1 \leq l \leq n$, set $(\lambda_{l,1},\dots,\lambda_{l,n}) = \underbrace{(0,\dots,0,1,0,\dots,0)}_{l-th\ place}(JF(z))^{-1}$ and $g_{z,l} = \sum_{j=1}^{n}\lambda_{l,j}F_j$. The function $g_{z,l}$ has the following properties:

$$g_{z,l}(z) = 0, \frac{\partial g_{z,l}}{\partial z_l}(z) = 1, \frac{\partial g_{z,l}}{z_j}(z) = 0, 1 \leq j \leq n, j \neq l.$$

Moreover, since $|\det JF(z)| \geq C_3 > 0$ and the components of $JF(z)$ are uniformly bounded on U (by the Cauchy inequalities), we have $\sup_D |g_{z,l}| \leq C_4\ \forall z \in U, 1 \leq l \leq n$.

Let now $z,w \in U, X \in \mathbb{C}^n, C_D^k(w,X) \geq C_D^k(z,X)$ and let f be an extremal function for $C_D^k(w,X)$. For each multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ set $h_{z,\alpha} = \dfrac{1}{\alpha!}\sum_{l=1}^{n}(f_{z,l})^{\alpha_l}$. Define the following sequence of holomorphic functions on D:

$$f_0 = f, f_{j+1} = f_j - \sum_{|\alpha|=j}\frac{1}{\alpha!}h_{z,\alpha}.D^\alpha f_j(z).$$

Since $D^\alpha h_{z,\alpha}(z) = 1$ and $D^\beta h_{z,\alpha}(z) = 0\ \forall\beta \neq \alpha, |\beta| \leq \alpha$, it follows by induction that $ord_z f_j \geq j$. On the other hand, $|D^\alpha f(z) - D^\alpha f(w)| \leq C_5\|z-w\|\ \forall z,w \in U$ in view of the Rolle theorem and the Cauchy inequalities. We also have $D^\alpha f(w) = 0$ and $|h_{z,\alpha}| \leq C_6\ \forall|\alpha| < k, \forall z \in U$. Then, using again the Rolle theorem, the Cauchy inequalities and inductive arguments, $\forall z,w \in U, j \leq k$ we get $|D^\alpha f_j(z)| \leq C_7\|z -$

$w|| \forall |\alpha| < k, |D^\alpha f_j(z) - D^\alpha f(w)| \leq C_7||z-w|| \forall |\alpha| = k$ and $\sup_D |f_j| \leq 1 + C_7||z-w||$.

Hence, according to the definition of $C_D^k(w, X)$, we have

$$(1+C_6||z-w||)(C_D^k(z,X))^k \geq |(f_{(k)}(z)X| \geq |f_{(k)}(w)X| - C_7||X||^k||z-w||.$$

Therefore, since $C_D^{(k)}(z,X) \leq ||X||dist^{-1}(z,\partial D)$ and f is extremal for $C_D^{(k)}(w,X)$, we obtain that

$$(3) \qquad 0 \leq (C_D^{(k)}(w,X))^k - (C_D^{(k)}(z,X))^k \leq C_8||X||^k||z-w||.$$

The inequalities (1), (2) and (3) complete the proof of Theorem 1.

Proof of Theorem 2. Using a nonsingular linear mapping (if it is necessary), we may suppose that the components of the vector X are not equal to 0 and hence the components of the vectors $X_j \to X$ are not equal to 0, too. Consider the linear mappings $\Psi_j(w) = \frac{(w - z_j)X}{X_j} + z$ and set $G_j = \Psi_j(D_j)$. We have

$$(4) \qquad C_{D_j}^\infty(z_j, X_j) = C_{G_j}^\infty(z, X).$$

Since D is a strictly hyperconvex domain, there exist another domain $D' \supset\supset D$ and a psh function φ on D' such that $\lim_{z\to\partial D} \varphi(z) = 0$ and $D = \{z \in D' : \varphi(z) < 0\}$. Let $\epsilon > 0$ be such that the open set $\{w \in D' : \varphi(w) < -\epsilon\}$ contains the point z. Denote by D_ϵ the connected component of this set, which contains z. Let $R > 0$ be such that the ball $B(0,R) \supset\supset D$ and let $D_{-\epsilon}$ be the connected component of the open set $B(z,R) \cap \{w \in D' : \varphi(w) < \epsilon\}$, which contains D. Since $D_\epsilon \subset\subset D \subset\subset D_{-\epsilon}$, $\forall j \gg 1$ we may find domains G_ϵ and $G_{-\epsilon}$ such that $D_\epsilon \subset\subset G_\epsilon \subset G_j \subset G_{-\epsilon} \subset\subset D_{-\epsilon}$ and $G_\epsilon \subset D \subset G_{-\epsilon}$. Hence

$$(5) \qquad C_{G_{-\epsilon}}^\infty(z,X) \leq C_{G_j}^\infty(z,X) \leq C_{G_\epsilon}^\infty(z,X),$$

$$(6) \qquad C_{G_{-\epsilon}}^\infty(z,X) \leq C_D^\infty(z,X) \leq C_{G_\epsilon}^\infty(z,X).$$

We shall prove that

$$(7), \qquad \lim_{\epsilon\to 0}(C_{G_\epsilon}^\infty(z,X) - C_{G_{-\epsilon}}^\infty(z,X)) = 0.$$

which together with (4), (5) and (6) will complete the proof of Theorem 2.

Denote by $g_{-\epsilon}$ the Green function of $D_{-\epsilon}$. Let χ be a C^∞-smooth function such that $\chi \equiv 1$ on D_ϵ and $supp\ \chi \subset G_\epsilon$. Let $m \in \mathbb{N}$ and f_m be an extremal function for $C_{G_\epsilon}^{(m)}(z,X)$. Since $g_{-\epsilon}(z,\cdot)$ is a psh function and $g_{-\epsilon}(z,w) > -\infty$, $z \neq w$ (because $D_{-\epsilon}$ is bounded) we may solve the $\bar\partial$-problem $\bar\partial h_m = \bar\partial(\chi f_m)$ on the bounded pseudoconvex domain $D_{-\epsilon}$ $(0 < \epsilon \ll 1)$ with the L^2 estimate [3]:

$$\int_{D_{-\epsilon}} |h_m|^2 \frac{\exp(-2(m+n)g_{-\epsilon}(z,w))}{(1+||w||^2)^2} dV(w) \le \tag{8}$$

$$2\int_{D_{-\epsilon}} |\bar\partial(\chi f_m)|^2 \exp(-2(m+n)g_{-\epsilon}(z,w)) dV(w).$$

It is clear that $t_m = \chi f_m - h_m$ is a holomorphic function on $D_{-\epsilon}$. The convergence of the first integral and the inequality $g_{-\epsilon}(z,w) \le \log \dfrac{||w-z||}{dist(z,\partial D_{-\epsilon})}$ [4] imply that $ord_z h_m \ge m+1$. Then, using that $\chi \equiv 1$ on $D_\epsilon \ni z$, we get

$$(t_m)_{(m)}(z)X = (f_m)_{(m)}(z)X. \tag{9}$$

Now we shall estimate $\sup_{G_{-\epsilon}} |t_m|$. By the mean value inequality we deduce that

$$VolB(O,1)(dist(G_{-\epsilon}, \partial D_{-\epsilon}))^{2n} \sup_{G_{-\epsilon}} |t_m|^2 \le \int_{D_{-\epsilon}} |t_m|^2 \le \tag{10}$$

$$2\int_{D_{-\epsilon}} (|\chi f_m|^2 + |h_m|^2) \le 2(VolG_\epsilon + \int_{D_{-\epsilon}} |h_m|^2).$$

As $g_{-\epsilon}$ is negative on $D_{-\epsilon}$ we obtain from (8) that

$$\int_{D_{-\epsilon}} |h_m|^2 \le 2(1+R^2)^2 \max_{w\in supp\bar\partial\chi} \exp(-2(m+n)g_{-\epsilon}(z,w)) \int |\bar\partial\chi|^2. \tag{11}$$

On the other hand, since $D_{-\epsilon} \subset B(0,R)$, we have that (cf. [5]) $g_{-\epsilon}(z,w) \ge C \sup_{k\in\mathbb{N}}(k(\varphi(w)-\epsilon) + \dfrac{||w||^2 - R^2}{k})$ for w out of a neighbourhood U of z, where the constant C depends only on $diam\ U$ and R. Since $supp\ \bar\partial\chi \subset G_\epsilon \setminus D_\epsilon \subset D_{-\epsilon} \setminus D_\epsilon$, it follows that

$$\max_{w\in supp\bar\partial\chi} \exp(-2(m+n)g_{-\epsilon}(z,w)) \le b(\epsilon, m),$$

where $b(\epsilon,m) = \inf_{k\in\mathbb{N}} \exp(2C(m+n)(2k\epsilon + \dfrac{R^2}{k}))$. This together with (10) and (11) shows that

$$\sup_{G_{-\epsilon}} |t_m| \le C_\epsilon(1 + b(\epsilon,m)) \tag{12}$$

for some constant C_ϵ, which does not depend on $m \in \mathbb{N}$.

Since f_m is an extremal function for $C^m_{G_\epsilon}(z,X)$, we conlude, in view of the definition of $C^m_{G_{-\epsilon}}(z,X)$, (9) and (12), that

$$C^m_{G_{-\epsilon}}(z,X) \ge C^m_{G_\epsilon}(z,X)(C_\epsilon(1+b(\epsilon,m)))^{\frac{1}{m}}.$$

Letting consequently $m \to \infty, \epsilon \to 0+, k \to \infty$ and using (6) we get (7).

Proof of Corollary 2. By a result of St. Nivoche [7], for each point $z \in D$, we have $C_D^\infty(z, X) = A_D(z, X)$ for almost all $X \in \mathbb{C}^n$. On the other hand, $A_D(z, \cdot)$ is a log-psh function on $\mathbb{C}^n$ [1]. The contiuous function $C_D^\infty(z, \cdot)$ is log-psh, too, as the supremum of the sequence of log-psh functions $\{C_D^k(z, \cdot)\}_{k=1}^\infty$. It is well-known that if two psh functions coinsides almosr everywhere, then they coinside everywhere. Therefore $C_D^\infty(z, X) = A_D(z, X)$ on $D \times \mathbb{C}^n$.

From this and the inequality $A_D(z_j, X_j) \geq C_{D_j}^\infty(z_j, X_j)$ it follows that

$$\liminf_{j\to\infty} A_D(z_j, X_j) \geq A_D(z, X). \tag{13}$$

Let the sequence of domains $\{G_j\}_{j=1}^\infty$ be such that $G_j \subset G_{j+1} \cap D_j$ and $\cup_{j=1}^\infty G_j = D$. Since the Azukawa metric is upper semicontinuous [5], we get

$$\limsup_{j\to\infty} A_{D_j}(z_j, X_j) \leq \limsup_{j\to\infty} A_{G_l}(z_j, X_j) \leq A_{G_l}(z, X) \tag{14}$$

$\forall l \in \mathbb{N}$. On the other hand, $\lim\limits_{l\to\infty} A_{G_l}(z, X) = A_D(z, X)$ [1], which together with (13) and (14) shows $\lim\limits_{j\to\infty} A_D(z_j, X_j) = A_D(z, X)$.

Proof of Theorem 3. The proof of Theorem 3 is similar to that of Theorem 2.

By results in [9, 2], the model E is a finite type domain iff there exists a smooth positive weighted homogeneous (with the same weights as for P) function a on $\mathbb{C}^{n-1}$ such that $P - \epsilon a$ is strictly plurisubharmonic on $\mathbb{C}^{n-1}$ for $0 < \epsilon \leq 2$. Let $\epsilon \leq 1$. Set

$$G_{-\epsilon} = \{z \in \mathbb{C}^n : r_{-\epsilon}(z) := Re z_1 + P(z') - \epsilon(1 + a(z')) < 0\}. \tag{15}$$

Since $F_\epsilon = E_\epsilon \cap S_{R(\epsilon)}$ $(\epsilon \leq 1)$ is a strictly hyperconvex domain, it suffices to prove, in view of Theorem 2, that

$$\lim_{\epsilon\to 0+} (C_{G_{-\epsilon/2}}^\infty(z, X) - C_{F_\epsilon}^\infty(z, X)) = 0 \tag{16}$$

uniformly on the compact subsets of $E \times \mathbb{C}^n$.

Denote by $g_{-\epsilon}$ the Green function of $G_{-\epsilon}$. Since the algebra of bounded holomorphic functions on $G_{-\epsilon}$ separates the points of $G_{-\epsilon}$ [9], we have

$$g_{-\epsilon}(z, w) > -\infty, \ z \neq w. \tag{17}$$

On the other hand, $C_{G_{-1}}^{(1)}(z, X) > 0 \ \forall z \in G_{-1}, X \in \mathbb{C}^n$ [6]. Hence, if f is an extremal function for $C_{G_{-1}}^{(1)}(z, X)$, then the Levi form of $|f|^2$ is positive near (z,X). It follows by compact arguments that for each compact subset K of G_{-1} there is a smooth bounded psh function on G_{-1}, which is strictly psh on K.

Let K be a compact subset of E and $H_\epsilon = E_{2\epsilon} \cap S_{R(\epsilon)/2}$. For $0 < \epsilon \ll 1$ we have $K \subset\subset H_\epsilon \subset\subset F_\epsilon \subset\subset G_{-\epsilon/2} \subset\subset G_{-\epsilon}$. Let χ be a smooth function such that $\chi \equiv 1$ on H_ϵ and $supp\ \chi \subset F_\epsilon$. Let $(z, X) \in K \times \mathbb{C}^n$, $m \in \mathbb{N}$, and let f_m be an extremal function for $C_{F_\epsilon}^{(m)}(z, X)$. As we already know there is a smooth psh function s on G_{-1}, which is strictly psh on $supp\ \chi$. and $-1 \le s \le 0$. Denote by c the minimum of the eigenvalues of the Levi form of s on $\{supp\ \chi\}$. In view of (17), we may solve $\bar\partial$-problem $\bar\partial h_m = \bar\partial(\chi f_m)$ on the pseudoconvex domain $G_{-\epsilon}$ with the L^2 estimate [3]:

$$\int_{G_{-\epsilon}} |h_m|^2 \exp(-2(m+n)g_{-\epsilon}(z,w) - s)dV(w) \le$$

$$c^{-1} \int_{G_{-\epsilon}} |\bar\partial(\chi f_m)|^2 \exp(-2(m+n)g_{-\epsilon}(z,w) - s)dV(w).$$

Then $t_m = \chi f_m - h_m$ is a holomorphic function on $G_{-\epsilon}$. It is not difficult to see that $|P(z'+\delta) - P(z')| \le C|\delta|(1+\sigma(z'))\ \forall\delta, |\delta| \le 1$, which ipmlies that $dist(G_{-\epsilon/2}, \partial G_{-\epsilon}) > 0$.

As in the proof of Theorem 2 we get

$$(t_m)_{(m)}(z)X = (f_m)_{(m)}(z)X$$

and

$$\sup_{G_{-\epsilon/2}} |t_m| \le C_\epsilon(1 + \sup_{w\in F_\epsilon\setminus H_\epsilon} \exp(-2(m+n)g_{-\epsilon}(z,w)))$$

So, to obtain (16) it suffices to prove that

$$\lim_{\epsilon\to 0+} \inf_{w\in F_\epsilon\setminus H_\epsilon} (\exp g_{-\epsilon}(z,w))^{-2(m+n)} = 0 \tag{18}$$

uniformly in $z \in K$.

Since there is a psh function q on $G_{-1} \supset G_{-\epsilon}$ which is strictly psh on K and $-1 \le q \le 0$, we have (cf. [4]) $g_{-\epsilon}(z,w) \ge C \sup_{k\in\mathbb{N}}(kr_{-\epsilon}(w) + \frac{q(w)}{k})$. for w out of an neighbourhood of $K \ni z$. By the definitions of E (cf. Section 2) and $G_{-\epsilon}$ (cf. (15)) we obtain

(19)

$$g_{-\epsilon}(z,w) \ge -C \inf_{k\in\mathbb{N}}(k\epsilon(1+a(w')) + \frac{1}{k})\ \forall z \in K \subset E, w \in G_{-\epsilon}\setminus H_\epsilon \supset F_\epsilon \setminus H_\epsilon.$$

Let g denotes the Green function of the domain $G = \{z \in \mathbb{C}^n : Re z_1 + P(z') - a(z') < 0\}$. Since $g_{-\epsilon}(z,w) \ge g(z-1,w)$, the equality (18) will be a consequence of (19) if

$$\lim_{w\in G; w\to\infty} g(z,w) = 0 \tag{20}$$

holds uniformly in $z \in G \cap S(R)$, where $R > 0$ is an arbitrary number.

To prove (20) we shall use that there is a pickfunction of G at the boundary point 0 [9], i.e. a holomorphic function p on G, which is continuos on $\overline{G}$ and such that $p(0) = 1$ and $|p(\zeta)| < 1$ on $\overline{G} \setminus \{0\}$. Set $\pi_t(\zeta) = (t^{-1}\zeta_1, t^{-1/m_2}\zeta_2, \dots, t^{-1/m_n}\zeta_n)$, $t > 0$, $v_w(\zeta) = p(\pi_{\sigma(w)}(\zeta))$, $u_w(\zeta) = \log|\dfrac{v_w(z) - v_w(\zeta)}{1 - \overline{v}_w(z)v_w(\zeta)}|$. Since π_t is an automorphism of G and p is a pickfunction of G at 0, u_w is a negative psh function on G with logarithmic pole at w. By the definition of S_R (cf. Section 2) it follows that $\lim\limits_{\sigma(z)<R; w\to\infty} u(z, w) = 0$. This implies (20), which completes the proof of Theorem 3.

Corollary 3 is a consequence of Theorem 3, Corollary 2 and the fact that the domains $E \cap S_R$ are strictly hyperconvex.

Proof of Theorem 4. The case of the usual Caratheodory metric is proved in [6].

Since the model domain E is hyperbolic with respect to the usual Caratheodory metric [6], it follows from Theorem 1 that the higher order Caratheodory metrics are continuous on $E \times \mathbb{C}^n$. Then the same arguments as in [6] give Theorem 4 for any of these metrics.

Hence

$$\liminf_{\Lambda \ni z \to p} \frac{C_D^\infty(z, X_z)}{C_E^\infty(e, X(z))} \geq 1.$$

Indeed, suppose the contrary. Then we may find a vector field X and a sequence of points $\Lambda \ni \{z_j\}_{j=1}^\infty \to p$ such that $\lim\limits_{\Lambda \ni z_j \to p} \dfrac{X(z_j)}{||X(z_j)||} = Y$ and

$$\liminf_{\Lambda \ni z_j \to p} \frac{C_D^\infty(z_j, X_{z_j})}{C_E^\infty(e, X(z_j))} < 1,$$

Since C_E^∞ is a continuous function (Corollary 3), we conclude that

$$\liminf_{\Lambda \ni z_j \to p} \frac{C^\infty{}_D(z_j, X_{z_j})}{||X(z_j)||} < C_E^\infty(e, Y)$$

Hence

$$\liminf_{\Lambda \ni z_j \to p} \frac{C_D^{(k)}(z_j, X_{z_j})}{||X(z_j)||} < C_E^{(k)}(e, Y)$$

$\forall k \gg 1$, which is contradiction.

Since $C_D^\infty \leq A_D$ and $C_E^\infty \equiv A_E$ (Corollary 3), to complete the proof of Theorem 4, it suffices to show the inequality

$$\limsup_{\Lambda \ni z \to p} \frac{A_D(z, X_z)}{A_E(e, X(z))} \leq 1. \tag{21}$$

Let $\epsilon > 0$ be arbitrary. By the definition of E_ϵ and S_δ (cf. Section 2) there is a number $\delta = \delta(\epsilon) > 0$ such that $E_\epsilon \cap S_\delta \subset D$. Since $\pi_{-r(z)}$ is an automorphism of E and $X(z) = \pi_{-r(z)}(X_z)$, it follows that

$$A_D(z, X_z) \leq A_{F_{z,\epsilon}}(e(z), X(z)), \tag{22}$$

where $e(z) = \pi_z(z)$ and $F_{z,\epsilon} = E_\epsilon \cap U_{-\delta/r(z)}$. It is clear that $\lim\limits_{z\to p} -\dfrac{\delta}{r(z)} = +\infty$ and $\lim\limits_{\Lambda \ni z \to p} e(z) = (-1, 0') = e$. Then Theorem 3 gives

$$\lim_{\Lambda \ni z \to p} \frac{A_{F_{z,\epsilon}}(e(z), X(z))}{A_{E_\epsilon}(e, X(z))} = 1, \tag{23}$$

$$\lim_{\epsilon \to 0} \frac{A_{E_\epsilon}(e(z), X(z))}{A_E(e, X(z))} = 1. \tag{24}$$

Now (21) follows by (22), (23) and (24).

References

[1] K. Azukawa, *The invariant pseudo-metric related to negative plurisubharmonic functions*, Kodai Math. J. 10 (1987), 83-92.

[2] K. Diederich, G. Herbort, *Pseudoconvex domains of semiregular type*, Contributions to complex analysis and analytic geometry (H. Skoda and J. M. Trepreau, eds.), Aspects of Math. E26, pp.127-162, Vieweg, Braunschweig, 1994

[3] L. Hörmander, *L^2 estimates and existence theorems for $\bar\partial$ operator*, Acta Math. 113 (1965), 89-152.

[4] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, De Gruyter, Berlin, New York, 1993.

[5] M. Jarnicki, P. Pflug, *The pluricomplex Green function*, Indiana Univ. Math. J. 44 (1995), 535-543.

[6] N. Nikolov, *Nontangential weighted limit of the infinitesimal Caratheodory metric in an h−extendible boundary point of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$*, Acta Math. Hung. 82 (1999).

[7] St. Nivoche, *The pluricomplex Green function, capacitative notions, and approximation problems in $\mathbb{C}^n$*, Indiana Univ. Math. J. 44 (1995), 489-510.

[8] E. A. Poletskiĭ, B. V. Shabat *Invariant metrics* in "Several Complex Variables III", G. M. Khenkin (ed.), Springer Verlag, 1989, pp. 63-112.

[9] J. Yu, *Peak functions on weakly pseudoconvex domains*, Indiana Univ. Math. J. 43 (1994), 1271-1295.

[10] Wlodzimierz Zwonek, *Regularity properties of the Azukawa metric*, preprint.

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