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Deformations of neutral almost Kähler manifolds

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DEFORMATIONS OF NEUTRAL ALMOST KAHLER MANIFOLDS [1 2]

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Abstract

Common deformations for neutral pseudoriemann manifolds with compatible almost complex structure and closed fundamental two-form are considered. An example of nontrivial common deformation on the cylindrical Heisenberg group with invariant almost complex structure and neutral pseudometric is given.

1 Preliminaries

Let M be a C^∞-smooth paracompact $2m$-dimensional manifold and J be an antiinvolutive automorphism of the tangent bundle on M. Then for every point $p \in M$ the restriction J_p of J on the fibre T_pM acts as an antiinvolutive automorphism too, i.e. $J_p : T_pM \to T_pM$ and $J^2 = -Id$. The couple (M, J) is called *an almost complex manifold* and J is called *an almost complex structure* on M.

Let ω be a closed nondegenerate differential two-form on M. Then the couple (M, ω) is called *a symplectic manifold* and the two-form ω is called *a symplectic form* on the manifold M.

Let g be a pseudoriemannian metric on M of signature (m, m) and let h be defined by the equality $h(X, Y) = 1/2\,(g(X, JY) + g(JX, Y))$ for each two vector fields X, Y defined on an open set in M. Then h is called *a neutral almost hermitian metric* on the almost complex manifold (M, J) and (M, J, h) is called *a neutral almost hermitian manifold.* The two-form Ω on a neutral almost hermitian manifold (M, J, h) defined by the equality $\Omega(X, Y) = h(JX, Y)$ where X, Y are vector fields defined on an open set in M is called *a fundamental form* of the neutral almost hermitian manifold. A neutral almost hermitian manifold is called *a neutral almost Kähler manifold*, if the fundamental form Ω is a closed two-form, i.e. if $d\Omega = 0$. Then Ω will be a symplectic form on M and (M, Ω) will be a symplectic manifold too. So each neutral almost Kähler manifold is a symplectic one.

Let us note that every almost complex manifold (M, J) and every almost symplectic manifold (M, ω) is an even dimensional orientable smooth differentiable manifold. Also the nondegenerate $2m$-form ω^n coinside with the volume form of the differentiable manifold M up to multiplication with a constant.

Let G be a linear Lie group, i.e. $G \subset GL(n, \mathbf{R})$, where $GL(n, \mathbf{R})$ is the principal real linear group (the group of nondegenerate $n \times n$ matrix with real coefficients). *A G-structure* on the real manifold M is called a reduction of the structure group $GL(n, \mathbf{R})$ of the principle tangent bundle of M to the subgroup G. Geometrically this reduction gives a principle vector bundle $G \to B_G \to M$, where B_G consist of all G-frames on M.

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The almost complex structure (M, J) is an example of a G-structure on M with G equals to the principle complex linear group $AH(m)$ of all $m \times m$ matrices with complex coefficients, considered as a subgroup of the group $GL(2m, \mathbf{R})$) by means of the embedding $(A+iB) \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. This group would be considered as a group $G \subset GL(2m, \mathbf{R})$ of matrices commuting with the "matrix of the standard complex structure" $S = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. Here E is the $m \times m$ unity matrix.

Other example of G-structure is that of neutral pseudoriemanian manifold (M, g). Here the group G is the group of all matrices commuting with the matrix $2m \times 2m$ matrix $N = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$, where E is a $m \times m$ unity matrix, (m, m) is the signature of the pseudoriemannian metric. In fact, this is the subgroup $G = O_m \times O_m$ of the orthogonal group O_{2m}, consisting of matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A, B \in O_m$.

Let us denote by $Sp(m, m)$ the group of symplectic $2m \times 2m$ matrices with index m. The following proposition is fulfilled /see [Dm] for a closed result/.

Proposition. *The following equalities holds* $AH(m) \cap O(2m, m) = O(2m, m) \cap Sp(m, m) = Sp(m, m) \cap AH(m)$.

Let M be a manifold with a G structure on it. Let $\{(U_j; x_j^1, \dots, x_j^n)\}_{j \in A}$ be an atlas on M and $\{\sigma_j\}$ be a reper for the principal bundle $G \to B_G \to M$ of the given G structure. Then on the intersection $U_j \cap U_k$ it is fulfilled $\sigma_k = \sigma_j g_{jk}$ for some matrix g_{jk} in the group G. Let in the natural frame of the chart $\frac{\partial}{\partial \sigma_j} = (\frac{\partial}{\partial x_j^1}, \dots, \frac{\partial}{\partial x_j^n})$ the reper σ_j is represented as $\sigma_j = \frac{\partial}{\partial x_j} \hat{\sigma}$ with some $n \times n$ matrix $\hat{\sigma}$.

Let us denote the unit disc in $\mathbf{R}$ by D and let us consider for the smooth manifold M the product $\mathcal{M} = M \times D$.

Definition 1. [Gr] *Deformation of G-structure on the smooth manifold M* is called a diffeomorphism $\varphi_j \times 1 : U_j \times D \to U_j \times D$ (with inverse diffeomorphism $\phi_j \times 1$) which satisfies one of the following two equivalent conditions:

(A) $\quad \varphi_{j*}(p, t)\sigma_j(p, 0) = \sigma_j(\varphi_j(p, t), t)g_j(p, t), \quad g_j(p, t) \in G, \quad p \in U_j, \quad t \in D.$

(B) $\quad \hat{\sigma}_j(y_j, t) = \hat{\sigma}_j(y_j, 0)g_j(y_j, t), \quad g_j(p, t) \in G, \quad y_j = \varphi_j(p), \quad t \in D.$

Now we can consider two kind of coordinates on $\mathcal{M}$, namely $\{U_j \times D; x_j^1, \dots, x_j^n, 1\}_{j \in A}^1$ and $\{U_j \times D; y_j^1; \dots, y_j^n, 1\}_{j \in A}$, where $(y_j^1, \dots, y_j^n) = \phi_j(x_j^1 \times 1, \dots, x_j^n \times 1)$, $(x_j^1, \dots, x_j^n) = \varphi_j(y_j^1, \dots, y_j^n, t)\}$. The first one is called Euler type coordinates and the second one - Lagrangue type coordinates.

Recall now the notion of the equivalence of deformations. Let $\sigma(t)$, $t \in D$, and $\sigma'(t)$, $t \in D$ be two two deformations of a G structure. They are called equivalent if there exist a family of bi-maps $\psi : M \times D \to M$ and $\tilde{\psi} : D \to D'$, such that

$$\psi(t)_{*,p}\sigma(t)(p) = \sigma'(\tilde{\psi}(t))(\psi(p, t)).$$

The deformation $\sigma = \{\sigma_j\}$ is called trivial, if it is equivalent to the constant deformation $\sigma_j(t) = \sigma_j(0)$.

We need one more definition to formulate the main result.

Definition 2. [DP] *Common deformation of some $(G_\alpha)_{\alpha \in A}$-structures on the smooth manifold M* is called a diffeomorphism $\varphi_j \times 1 : U_j \times D \to U_j \times D$ (with inverse diffeomorphism $\phi_j \times 1$) which satisfies one of the conditions (A) or (B) with $g_j(p,t) \in G_\alpha$ for each $\alpha \in A$.

2 Common deformation on almost pseudo-Kähler manifold

Here we follows the cheme of the paper [DP].

The common deformation of the almost pseudo-Kähler manifolds are deformations with G structures $O_m \times O_m$ and AH_{2m}. The coordinates y_j are the same for each of theses two G structures. Let us denote by Γ_{G_α} for $\alpha = 1, 2$, $G_1 = O_m \times O_m$ and $G_2 = AH_{2m}$ the sheaf of transition functions $\{f_{ij}\}$ of the coordinates y_j. It has a 1-cochain $\{f_{ij} \in C^1(\{U_i\}, \Gamma_{G_\alpha}[t])$, which belong to $Z^1(\{U_i\}\Gamma_\alpha[t])$ for $\alpha = 1, 2$. It determine an element of $\cap_{\alpha=1,2} H^1(M, \Gamma_{G_\alpha}[t])$. If $\{f_{ij} = 0\}$, then each deformation is trivial and conversally, if $\varphi_{ij} \in Z^1(\{U_i\}, \Gamma_{G_\alpha})$ for $\alpha = 1, 2$, then in (y)-coordinates setting $f_{ij}(y_j, t) = \varphi_{ij}(p, t)$ it is obtaned a common deformation of the G_1 and G_2 structures.

Proposition 1 (see [DP]). *If M is compact manifold, we have a bijective map between the set of common deformations of $\{\sigma_\alpha\}$ and the intersection $\cap_\alpha H^1(M, \Gamma_\alpha[t])$. In other word, the germes of common deformations may be identified with the element of the intersection written above.*

Let us denote by r the sheaf morphism $\Gamma_{G_\alpha}[t] \to \theta_{G_\alpha}$ which send the germs of $f(t)$ into the germs defined by $df(t)/dt|_{t=0}$. This morphism induces the the map f : $H^1(M, \Gamma_{G_\alpha}[t]) \to H^1(M, \theta_{G_\alpha})$. If $\{\varphi\} \in H^1(M, \Gamma_{G_\alpha}[t])$ is a one-cochain of deformation, then $\{r_\varphi\} \in H^1(M, \theta_{G_\alpha})$ is an infinitesimal common deformation corresponding to φ.

Thus the following proposition of the paper [DP] is obtained:

Proposition 2 (see [DP]). *If M is compact, we have that*

$$\{rf_{ij}\} \in \cap_{\alpha=1,2} H^1(M, \theta_{G_\alpha})$$

is an infenithesimal common deformation determined by the transition functions $\{f_{ij}\}$ of the (y_j)-coordinates of the common deformations σ_α, $\alpha = 1, 2$.

Remark. Each element of $H^1(M, \Gamma_1[t]) \cap H^1(M, \Gamma_2[t])$ is a germ of common deformation of the almost pseudo-Kähler structure on the compact manifold M, but not this the case for $\cap_{\alpha=1,2} H^1(M, \theta_{G_\alpha})$. In [DP] is posed the following question: If $H^2(M, \theta_J) \cap H^2(M, \theta_g) = 0$, does every element of $\cap_{\alpha=1,2} H^1(M, \theta_{G_\alpha})$ define a germ of common deformation on M?

Let $\tilde{\omega} : M \times D \to D$ be a deformation of the given almost pseudo-Kähler manifold as almost complex and pseudoriemanian structure. Then a family (M_t, g_t, J_t) is given, $t \in D$, and a family of fundamental two-forms $\Omega_t(X, Y) = g_t(J_t X, Y)$ appears. If there exist deformation of the considered type, then by the condition (B) is obtained

$$\Omega_j(y_j, t) = \Omega_j(y_j, 0) \text{ on each chart } U_j \subset M.$$

That means that each common deformation of an almost pseudo-Kähler manifold is an almost pseudo-Kähler manifold. So we obtain the following resulting proposition.

Proposition 3. *The fundamental forms of the common deformation manifolds of a given almost pseudo-Kähler manifold as pseudoriemanian ($(O_{n-2} \times O_2)$-structure) and almost complex (AH-structure) i.e. as almost complex and pseudoriemanian manifold are closed and the common resulting deformation is an almost pseudo-Kähler manifold.*

3 Example - a common deformation of the cylindrical Heisenberg group with neutral pseudometric and invariant almost complex structure

3.1 Cylindrical Heisenberg group

Let H be the Heisenberg group, i.e. the group of matrices

$$m_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y and z are real numbers. This group is a closed subgroup of the Lie group $GL(3, \mathbf{R})$. Having in mind the coordinate system $m_{x,y,z} \to (x, y, z) \in \mathbf{R}^3$, we obtain that H can be considered as $\mathbf{R}^3$ equipped with the following "multiplication"

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

Let us denote by H_x the subgroup of H defined by the equalities $y = z = 0$, i.e.

$$(x_1, 0, 0)(x_2, 0, 0) = (x_1 + x_2, 0, 0).$$

We have also $(x, 0, 0)^{-1} = (-x, 0, 0)$, which means that H_x is isomorphic to the additive group of real numbers $\mathbf{R}$. Similarly, the subgroup H_y, $(x = z = 0)$ is defined as follows:

$$(0, y_1, 0)(0, y_2, 0) = (0, y_1 + y_2, 0), \ (0, y, 0)^{-1} = (0, -y, 0).$$

A cylindrical Heisenberg group is by definition the cartesian product of the Heisenberg group H and the circle S^1, i.e. $H \times S^1$. As a matrix group it is the group of the following matrices

$$m_{x,y,z} = \begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i t} \end{pmatrix}, x, y, z, t \in \mathbf{R}.$$

So this is a closed subgroup of $GL(4, \mathbf{C})$.

On the other hand we have that $H \times S^1$ can be represented as $\mathbf{R}^3 \times S^1$ equipped with the following "multiplication"

$$(x_1, y_1, z_1, t_1) \cdot (x_2, y_2, z_2, t_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2, t_1 + t_2 (mod 1)).$$

The cartesian product $H_x \times S^1$ is a subgroup of $H \times S^1$ isomorphic to the cylinder $\mathbf{R} \times S^1$ and so is $H_y \times S^1$.

3.2 Left invariant vector fields and invariant almost complex structure on the cylindrical Heisenberg group

The cylindrical Heisenberg group has an atlas of two charts (local coordinate systems): for instance

$$\varphi(m_{xyzt}) = (x, y, z, t'), \text{ where } x, y, z \in \mathbf{R}, \ -1/2 < t' < 1/2, \ t' = t(mod\ 1),$$

$$\psi(m_{xyzt}) = (x, y, z, t''), \quad \text{where } x, y, z \in \mathbf{R},\ 0 < t'' < 1,\ t'' = t(mod\ 1),$$

Here m_{xyzt} is an element of $H \times S^1$ (sometimes we will write m instead of m_{xyzt}).

Recall that a left invariant vector field X on the Lie group G is by definition a vector field X on G such that for each $m \in G$ the equality $de_m \circ X = X \circ e_m$ is verified, e_m is a left translation by step m on G and de_m - the differential of e_m. It is not difficult to calculate $d_e e_m$, (the differential at the unit e of $H \times S^1$) when $G = H \times S^1$. Let us denote by $A(m)$ the Jacobian of $\varphi \circ e_m \circ \varphi^{-1}$, i.e.

$$A(m) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x = \varphi^1(m)$. Then

$$d_e e_m : (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \to (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \cdot A(m),$$

if we use a matrix notations.

Now let us denote by $Y(G)$ the Lie algebra of all left invariant vector fields on G.

Proposition. *The following vector fields*

$$e_1 = \partial/\partial\varphi^1,\ e_2 = \partial/\partial\varphi^2 + x\partial/\partial\varphi^3,\ e_3 = \partial/\partial\varphi^3,\ e_4 = \partial/\partial\varphi^4$$

form a base of $Y(H \times S^1)$ *as a vector space.*

P r o o f. We have by definition $d_e e_m \cdot X(e) = X(e_m(e))$, i.e. $d_e e_m \cdot X(e) = X(m)$, for every $m \in H \times S^1$. Then

$$(\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \cdot A(m) \cdot \mathbf{X}(e) = X(m),$$

where

$$X(e) = (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_e \cdot \mathbf{X}(e).$$

Since

$$(e_1(m), e_2(m), e_3(m), e_4(m)) = (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \cdot A(m)$$

we obtain

$$(e_1(m), e_2(m), e_3(m), e_4(m)) \cdot \mathbf{X}(m) = X(m),$$

which means that (e_1, e_2, e_3, e_4) is a base of $Y(H \times S^1)$.

Otherwise, we could take as the Lie algebra $Y(G)$ the tangent space T_eG at the identity supplied with Lie algebra structure induced by requiring that the vector space isomorphism of $Y(G)$ with T_eG to be an isomorphism between Lie algebras. With this in mind we have that all left invariant almost complex structures on the even-dimensional Lie group G can be obtained from complex structures on T_eG. However we prefer to formulate the following definition.

Definition 3 (see [ADP]). The almost complex structure J on the Lie group G is a left invariant almost complex structure on G if and only if each left invariant translation e_m, $m \in G$ is an almost holomorphic map from G to G.

That means that for every $m \in G$ we have $d_e e_m \circ J_e = J_m \circ d_e e_m$. As corollary we have that J_m is obtained by J_e as follows

$$J_m = d_e e_m \circ J_e \circ (d_e e_m)^{-1}.$$

If

$$J_e : (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4) \to (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_e \cdot {}^t\Delta,$$

where $\Delta = \|\lambda_{jn}\|$, $\Delta^2 = -E_4$ (${}^t\Delta$ is the transposed matrix of the matrix Δ), we obtain immediately the matrix representation of J_m

$$\mathbf{J}(m) = \|J_q^p(m)\| =$$

$$= \begin{pmatrix} \lambda_{11} & \lambda_{21} - \lambda_{31}x & \lambda_{31} & \lambda_{41} \\ \lambda_{12} & \lambda_{22} - \lambda_{32}x & \lambda_{32} & \lambda_{42} \\ \lambda_{13} + \lambda_{12}x & \lambda_{23} + (\lambda_{22} - \lambda_{33})x - \lambda_{32}x^2 & \lambda_{33} + \lambda_{32}x & \lambda_{43} + \lambda_{42}x \\ \lambda_{14} & \lambda_{24} - \lambda_{34}x & \lambda_{34} & \lambda_{44} \end{pmatrix},$$

(here $x = \varphi^1(m)$). Thus we have

$$J_m : (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \to (\partial/\partial\varphi^1, \partial/\partial\varphi^2, \partial/\partial\varphi^3, \partial/\partial\varphi^4)_m \mathbf{J}(m)$$

and

$$J_m : ((e_1(m), e_2(m), e_3(m), e_4(m)) \to ((e_1(m), e_2(m), e_3(m), e_4(m)) \cdot {}^t\Delta(m),$$

which means that in the base (e_1, e_2, e_3, e_4) the operator J has a constant matrix. This property can be used as a definition of the left invariant almost complex structure too.

3.3 Neutral pseudometric on the cylindrical Heisenberg group and fundamental two-form

Let us consider the pseudoriemannian metric defined in base (e^1, e^2, e^3, e^4) as follows:

$$(g_{ij})_{i,j=1}^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This is a left invariant metric on the cylindrical Heisenberg group - its matrix in the orthogonal base of one-forms (e^1, e^2, e^3, e^4) for the left invariant base of vector fields (e_1, e_2, e_3, e_4) has constant coefficients. So it is fulfilled $g = (e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2$. This is neutral metric for the cylindrical Heisenberg group. In coordinates x, y, x, t this metric has the following matrix representation: $g = dx^2 + (dy + xdz)^2 - dz^2 - dt^2 = dx^2 + dy^2 + x(dydz + dzdy) + x^2dz^2 - dz^2 - dt^2 = dx^2 + dy^2 + 2xdydz + (x^2 - 1)dz^2 - dt^2$, i.e. the matrix of the metric g in coordinates (x, y, z, t) is the following one:

$$(g_{ij})_{i,j=1}^4 = \begin{pmatrix} 1 & x & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & x^2 - 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now let us consider the fundamental two-form of the neutral cylindrical Heisenberg group endowed with the almost complex structure J defined by $Je_1 = e_2$, $Je_2 = -e_1$ $Je_3 = e_4$, $Je_4 = -e_3$. First of all we must see that the metric g is an hermitian one, i.e. the equalities $g(JX, JY) = g(X, Y)$ holds for all vector fields X, Y on $H \times S^1$. Indeed, $g(JX, JY)(X, Y) = ((e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2)(JX, JY) = ((J^*e^1)^2 + (J^*e^2)^2 - (J^*e^3)^2 - (J^*e^4)^2)(X, Y) = ((e^1)^2 + (-e^2)^2 - (e^3)^2 - (-e^4)^2)(X, Y) = ((e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2)(X, Y) = g(X, Y)$. Then the fundamental two-form is defined by the equality $\Phi(X, Y) = g(JX, Y) = J^*e^1 \wedge e^1 + J^*e^2 \wedge e^2 - J^*e^3 \wedge e^3 - J^*e^4 \wedge e^4 = 2(-e^1 \wedge e^2 + e^3 \wedge e^4)$ for all vector fields X, Y on the manifold. In coordenate (x, y, z, t) the two-form Φ is the following one $\Phi = -e^1 \wedge e^2 + e^3 \wedge e^4 = dx \wedge (dy + xdz) + dz \wedge dt = dx \wedge dy + xdx \wedge dz + dz \wedge dt$. Let us compute its diferential $d\Phi$, $d\Phi = d^2x \wedge dy - dx \wedge d^2y + dx \wedge dx \wedge dz + xd^2x \wedge dz + xdx \wedge d^2z + d^2z \wedge dt - dz \wedge d^2t = 0$. So the fundamental form is a nondegenerate closed two form and so the cylindrical Heisenberg group with almost complex structure J and neutral metric g is an almost pseudo-Kähler manifold.

3.4 Nontrivial common deformation of the neutral almost Kähler cylindrical Heisenberg group

Now let us consider the following mapping on the $H \times S^1 \times D$ (D is the unit disc in $\mathbf{R}$) on the chart $(U_j, x_j^1, x_j^2, x_j^3, x_j^4, t)$, $j = 1, 2$ of $H \times S^1 \times D$ (see §1) by the formula

$$F^{\pm 1}(x_1, x_2, x_3, x_4, t) = (x_j^1 \mp tx_j^4, (1-t)^{\pm 1} x_j^2, (1-t)^{\pm 1} x_j^3, x_j^4, t) = (y_j^1, y_j^2, y_j^3, y_j^4, t).$$

This is an invertible mapping, so it is a nondegenerate one and it inverse is given by the similar formula (with sign - instead of +).

It would be checked that this is a well defined mapping on $H \times S^1 \times D$ and it fulfilled the conditions of Definition 1. So this is a nontrivial deformation of the neutral almost pseudo-Kähler manifold $H \times S^1$ endowed with the almost complex structure J and with pseudo-Kähler metric g. According to Proposition 3 this deformation is an almost pseudo-Kähler manifold all times (for every $t \in U$). As the mapping F is non-degenerate this give an example of a non-trivial almost pseudo-Kähler deformation of the cylindrical Heisenberg group.

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