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Донка Пашкулева

Certain Classes of Functions with Positive and Missing Coefficients

Donka Pashkouleva

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Abstract

In this paper we obtain coefficient bounds, distortion and closure theorems and radius of convexity for the class $\Omega^*(\alpha,\beta,\lambda,A,B)$.

1 Introduction

Let Ω be the class of functions of the form:

(1.1) $$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$$

which are analytic in $U^*=\{z:0<|z|<1\}$.

We denote by $\Omega(p)$ a class, consisting of functions of the form

(1.2) $$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n},\quad p\in N.$$

Let $\Omega^*(p)$ be the class of functions of the form

(1.3) $$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n},\quad a_{p+n}\geq 0,\quad p\in N.$$

Clearly, we have a relationship

$$\Omega^*(p)\subseteq\Omega(p)\subseteq\Omega.$$

We set

$$F(z)=(1-\lambda)f(z)+\lambda zf'(z),\quad f(z)\in\Omega(p),\quad 0\leq\lambda<\frac{1}{2},$$

that is

(1.4) $$F(z)=\frac{1-2\lambda}{z}+\sum_{n=0}^{\infty}\left[1+\lambda(p+n-1)\right]a_{p+n}z^{p+n}.$$

Given α $(0\leq\alpha<1)$, a function $f\in\Omega$ is said to be in the class of meromorphic starlike functions of order α, denoted by $\Omega^*(\alpha)$, if

(1.5) $$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad |z|<1.$$

Similarly, for α $(0\leq\alpha<1)$, a function $f\in\Omega$ is in the class of meromorphic convex function of order α, denote by $\Omega_k^*(\alpha)$ if

(1.6) $$-\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad |z|<1.$$

Definition

A function $f \in \Omega(p)$ is said to be in the class $\Omega(\alpha, \beta, \lambda, A, B)$ if it satisfies the condition

$$(1.7) \qquad \left| \frac{\frac{zF''(z)}{F'(z)} + 2}{B\left(1 + \frac{zF''(z)}{F'(z)}\right) + [B + (A-B)(1-\alpha)]} \right| < \beta,$$

where $F(z)$ defined in terms of $f(z)$ $(f \in \Omega(p))$ is given by (1.4) and

$$(1.8) \qquad 0 \le \alpha < 1,\ 0 < \beta, \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1.$$

Let us write

$$(1.9) \qquad \Omega^*(\alpha, \beta, \lambda, A, B) = \Omega^*(p) \cap \Omega(\alpha, \beta, \lambda, A, B).$$

We note that similar types of classes were studied rather extensively by Bajpai [1], Goel and Sohi [2], Joshi et al. [3] and Srivastava et al. [4].

In the present paper, we obtain coefficient inequalities and a distortion theorem for the class $\Omega^*(\alpha, \beta, \lambda, A, B)$. Further, it is shown that $\Omega^*(\alpha, \beta, \lambda, A, B)$ is closed under convex linear combination. Also we obtain the radius of convexity. We employ techniques, similar to those used by Uralegaddi and Ganigi [5].

2 Coefficient Inequalities

Theorem 2.1

Let the function $f(z)$ defined by (1.2) be analytic in U^*. If

$$(2.1) \qquad \begin{aligned} &\sum_{n=0}^{\infty} \{1 + (p+n-1)\lambda\} \left[(p+n+1) + \beta \{B(p+n) + (B-A)\alpha + A\}\right] (p+n) |a_{p+n}| \\ &\le \beta(B-A)(1-\alpha)(1-2\lambda), \end{aligned}$$

then $f(z) \in \Omega(\alpha, \beta, \lambda, A, B)$.

Proof.

Suppose that (2.1) holds true for all admissible values of α, β, A, B and λ.

Consider the expression

$$(2.2) \quad \Phi(F, F') = |zF''(z) + 2F'(z)| - \beta \left| B\{F'(z) + zF''(z)\} + [B + (A-B)(1-\alpha)] F'(z) \right|.$$

Replacing F and $F'$by their expansion in(2.2), we have for $0 <|z| = r < 1$,

$$\Phi(F, F') = \left| \sum_{n=0}^{\infty} \{1 + (p+n-1)\lambda\} (p+n)(p+n+1) a_{p+n} z^{p+n} \right| -$$

$$\beta \left| \frac{(1-2\lambda)(B-A)(1-\alpha)}{z^2} + \sum_{n=0}^{\infty} \{1 + (p+n-1)\lambda\} a_{p+n} [B(p+n) + (B-A)\alpha + A] z^{p+n-1} \right|$$

or

$$r^2\Phi(F,F') \le \sum_{n=0}^{\infty} \{1+(p+n-1)\lambda\}(p+n)(p+n+1)a_{p+n}z^{p+n+1} -$$

$$\beta\left\{(B-A)(1-\alpha)(1-2\lambda) - \sum_{n=0}^{\infty}[B((p+n)+(B-A)\alpha+A)](1+(p+n-1)\lambda)(p+n)|a_{p+n}|r^{p+n+1}\right\}$$

$$= \sum_{n=0}^{\infty}\{1+\lambda(p+n-1)\}(p+n+1)+\beta(B(p+n)+(B-A)\alpha+A)(p+n)|a_{p+n}|r^{p+n+1} -$$

$$\beta(B-A)(1-\alpha)(1-2\lambda)$$

Since the above inequality holds true for all r $(0<r<1)$, letting $r\to 1$, we have

$$\Phi(F,F') \le \sum_{n=0}^{\infty}\{(n+p+1)+\beta[B(p+n)+(B-A)\alpha+A]\}\{1+\lambda(p+n-1)\}(p+n)|a_{p+n}| -$$

$$\beta(B-A)(1-\alpha)(1-2\lambda) \le 0$$

by (2.1).

Hence, it follows that

$$\left|\frac{zF''(z)}{F'(z)}+2\right| \le \beta\left|B\left(1+\frac{zF''(z)}{F'(z)}\right)+[B+(A-B)(1-\alpha)]\right|^{1}$$

so that $f(z)\in\Omega(\alpha,\beta,\lambda,A,B)$. This completes the proof of the theorem.

Theorem 2.2

Let the function $f(z)$ defined by (1.3) be analytic in U^*, then $f(z)\in\Omega^*(\alpha,\beta,\lambda,A,B)$ if and only if (2.1) is satisfied.

Proof.

In view of Theorem 2.1, let us assume that function $f(z)$ defined by (1.3) is in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$. Then

$$\left|\frac{\frac{zF''(z)}{F(z)}+2}{B\left(1+\frac{zF''(z)}{F'(z)}\right)+[B+(A-B)(1-\alpha)]}\right| =$$

$$\left|\frac{-\sum_{n=0}^{\infty}(p+n)(p+n+1)a_{p+n}\{1+(p+n+1)\lambda\}z^{p+n-1}}{\frac{(B-A)(1-\alpha)(1-2\lambda)}{z^2}-\sum_{n=0}^{\infty}[B(p+n)+(B-A)\alpha+A](p+n)a_{p+n}z^{p+n-1}}\right| < \beta \; (z\in U^*).$$

By the fact that $\operatorname{Re} z<|z|$ for all z, we thus have

(2.3)

$$\operatorname{Re}\left\{\frac{(p+n)(p+n+1)a_{p+n}\{1+(p+n-1)\lambda\}z^{p+n-1}}{\frac{(B-A)(1-\alpha)(1-2\lambda)}{z^2}-\sum_{n=0}^{\infty}[B(n+p)+(B-A)\alpha+A](p+n)a_{p+n}z^{p+n+1}}\right\} < \beta.$$

Now choose the values of z on the real axis so that $\left(1+\frac{zF''(z)}{F'(z)}\right)$ is real, clearing denominator in (2.3) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=0}^{\infty}(n+p+1)(p+n)\{1+(p+n-1)\lambda\}\, a_{p+n} \leq$$

$$\leq \beta\left\{(B-A)(1-\alpha)(1-2\lambda) - \sum_{n=0}^{\infty}[B(n+p)+(B-A)\alpha + A]\,(p+n)\right\},$$

$$\sum_{n=0}^{\infty}(p+n+1)+\beta\,[B(n+p)+(B-A)\alpha+A]\,(p+n)\,\{1+(n+p-1)\lambda\}\, a_{p+n} \leq$$

$$(B-A)(1-\alpha)(1-2\lambda),$$

which proves Theorem 2.2.

Corollary 1

Let function $f(z)$ defined by (1.3) be in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$. Then

$$|a_{n+p}| \leq \frac{(B-A)\beta(1-\alpha)(1-2\lambda)}{(n+p)\,\{(n+p+1)+\beta\,[B(n+p)+(B-A)\alpha+A]\}\,\{1+(p+n-1)\lambda\}} z^{p+n},$$

where equality holds for the function

$$f_{p+n}(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)(1-2\lambda)}{(n+p)\,\{(n+p+1)+\beta\,[B(n+p)+(B-A)\alpha+A]\}\,\{1+(p+n-1)\lambda\}} z^{p+n}. \tag{2.4}$$

3 A Distortion Theorem

Theorem 3.1

Let the function $f(z)$ defined by (1.3) be in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$. Then for $0<|z|=r<1$

$$\frac{1}{r} - \frac{(B-A)\beta(1-\alpha)(1-2\lambda)r^p}{p\,\{(p+1)+\beta\,[Bp+A+(B-A)\alpha]\}\,\{1+(p-1)\lambda\}} \leq$$
$$\leq |f(z)| \leq \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)(1-2\lambda)r^p}{p\,\{(p+1)+\beta\,[Bp+A+(B-A)\alpha]\}\,\{1+(p-1)\lambda\}}, \tag{3.1}$$

where equality holds for the function

$$f_p(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)(1-2\lambda)}{p\,\{(p+1)+\beta\,[Bp+(B-A)\alpha+A]\}\,\{1+(p-1)\lambda\}} z^p \tag{3.2}$$

and

$$\frac{1}{r^2} - \frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{\{1+(p-1)\lambda\}\,\{(p+1)+\beta\,[Bp+A+(B-A)\alpha+A]\}} r^{p-1} \leq$$
$$\leq |f'(z)| \leq \frac{1}{r^2} + \frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{\{1+(p-1)\lambda\}\,\{(p+1)+\beta\,[Bp+(B-A)\alpha+A]\}} r^{p-1}, \tag{3.3}$$

where equality holds for the function $f_p(z)$ given by (3.2) at $z=\pm r$.

Proof.

In view of Theorem 2.2 we have

$$\sum_{n=0}^{\infty} |a_{p+n}| \leq \frac{(B-A)\beta(1-\alpha)(1-2\lambda)}{p\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}\{1+(p-1)\lambda\}}. \tag{3.4}$$

Thus, for $0 < |z| = r < 1$

$$\begin{aligned}
|f(z)| &\leq \frac{1}{r} + \sum_{n=0}^{\infty} |a_{p+n}| r^{p+n} \\
&\leq \frac{1}{r} + r^p \sum_{n=0}^{\infty} |a_{p+n}| \leq \\
&\leq \frac{1}{r} + r^p \frac{(B-A)\beta(1-\alpha)(1-2\lambda)}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}\{1+(p-1)\lambda\}}.
\end{aligned}$$

$$\begin{aligned}
f(z) &\geq \frac{1}{r} - \sum_{n=0}^{\infty} |a_{p+n}| r^{p+n} \\
|f(z)| &\geq \frac{1}{r} - r^p \sum_{n=0}^{\infty} |a_{p+n}| \\
&\geq \frac{1}{r} - \frac{r^p(B-A)\beta(1-\alpha)(1-2\lambda)}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}\{1+(p-1)\lambda\}},
\end{aligned}$$

which together yield (3.1).

It follows from Theorem 2.2 that

$$\sum_{n=0}^{\infty} (n+p)|a_{n+p}| \leq \frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{\{1+(p-1)\lambda\}\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}.$$

Hence,

$$\begin{aligned}
|f'(z)| &\leq \frac{1}{r^2} + \sum_{n=0}^{\infty} (p+n)|a_{p+n}| r^{p+n-1} \leq \\
&\leq \frac{1}{r^2} + r^{p-1} \sum_{n=0}^{\infty} (p+n)|a_{p+n}| \leq \\
&\leq \frac{1}{r^2} + \frac{\beta(B-A)(1-\alpha)(1-2\lambda)r^{p-1}}{\{1+(p-1)\lambda\}\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}
\end{aligned}$$

$$\begin{aligned}
|f'(z)| &\geq \frac{1}{r^2} - \sum_{n=0}^{\infty} (p+n)|a_{p+n}| r^{p+n-1} \\
&\geq \frac{1}{r^2} - r^{p-1} \sum_{n=0}^{\infty} (p+n)|a_{p+n}| \\
&\geq \frac{1}{r^2} - \frac{\beta(B-A)(1-\alpha)(1-2\lambda)r^{p-1}}{\{1+(p-1)\lambda\}\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}},
\end{aligned}$$

which together yield (3.3).

It can be easily seen that the function $f_p(z)$ defined by (3.2) is extremal for Theorem 3.1.

4 A Pair of Closure Theorems

Theorem 4.1

The class $\Omega^*(\alpha,\beta,\lambda,A,B)$ is closed under convex linear combinations.

Proof.

Let each of the functions

$$|f_{p+j}(z)| = \frac{1}{z} + \sum_{n=0}^{\infty} |a_{n+p,j}|\, z^{p+n} \quad (j=1,2)$$

be in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$. It is sufficient to prove that the function defined by

$$h(z) = (1-\mu)f_1(z) + \mu f_2(z) \quad (0 \le \mu \le 1)$$

is also in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$.

Since

$$h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} [(1-\mu)\,|a_{p+n,1}| + \mu\,|a_{p+n,2}|]\, z^{p+n}, \quad 0 \le \mu \le 1,$$

using Theorem 2.2, we have

$$\sum_{n=0}^{\infty}(p+n)\{1+(p+n-1)\lambda\}\{(p+n+1)+\beta[(p+n)B+(B-A)\alpha+A]\} \times$$
$$\{(1-\mu)\,|a_{p+n,1}| + \mu\,|a_{p+n,2}|\}$$
$$= (1-\mu)\sum_{n=0}^{\infty}(p+n)\{1+(p+n-1)\lambda\}\{(p+n+1)+\beta[(p+n)B+(B-A)\alpha+A]\}\,|a_{p+n,1}| +$$
$$+\mu\sum_{n=0}^{\infty}(p+n)\{1+(p+n-1)\lambda\}\{(p+n+1)+\beta[(p+n)B+(B-A)\alpha+A]\}\,|a_{p+n,2}|$$
$$\le (1-\mu)(B-A)\beta(1-\alpha)(1-2\lambda) + \mu(B-A)\beta(1-\alpha)(1-2\lambda) = (B-A)(1-\alpha)(1-2\lambda)\beta,$$

which indicates that $h(z) \in \Omega^*(\alpha,\beta,\lambda,A,B)$.

This completes the proof of Theorem 4.1.

Theorem 4.2

Let

$$f_0(z) = \frac{1}{z},$$

$$f_{p+n}(z) = \frac{1}{z} + \frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{(p+n)\{1+(p+n-1)\lambda\}\{(p+n+1)+\beta[(p+n)B+(B-A)\beta+A]\}} z^{p+n}.$$

Then $f(z) \in \Omega^*(\alpha,\beta,\lambda,A,B)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z),$$

where

$$\sum_{n=0}^{\infty} \mu_n = 1, \qquad \mu_n \ge 0$$

and $\mu_n = 0$ for $n = 1,2,\ldots,p$ if $p \ge 2$ $(p < n)$.

Proof.

Suppose that

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)=$$

$$=\frac{1}{z}+\sum_{n=0}^{\infty}\mu_n\left(\frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{(p+n)\left\{1+(p+n-1)\lambda\right\}\left\{(p+n+1)+\beta\left[(p+n)B+(B-A)\alpha+A\right]\right\}}z^n\right).$$

Since

$$\sum_{n=0}^{\infty}\frac{(n+p)\left\{(n+p-1)+\beta\left[(p+n)B+(B-A)\alpha+A\right]\right\}\left\{1+(n+p-1)\lambda\right\}}{(B-A)(1-\alpha)(1-2\lambda)\beta}\mu_n\times$$

$$\frac{\beta(B-A)(1-\alpha)(1-2\lambda)}{(p+n)\left\{1+(p+n-1)\lambda\right\}\left\{(p+n+1)+\beta\left[(p+n)B+(B-A)\alpha+A\right]\right\}}$$

$$=\sum_{n=1}^{\infty}\mu_n=1-\mu_0\leq 1, \tag{4.1}$$

which shows that $f(z)\in\Omega^*(\alpha,\beta,\lambda,A,B)$ with reference to Theorem 2.2.

Conversely, suppose that the function $f(z)$ defined by (1.3) belongs to the class $\Omega^*(\alpha,\beta,\lambda,A,B)$, then

$$|a_{p+n}|\leq\frac{(B-A)(1-\alpha)(1-2\lambda)\beta}{(n+p)\left\{1+(n+p-1)\lambda\right\}\left\{(n+p+1)+\beta\left[(n+p)B+(B-A)\alpha+A\right]\right\}}$$

by Corollary 1. Setting

$$\mu_{p+n}=\frac{(n+p)\left\{1+(n+p-1)\lambda\right\}\left\{(n+p+1)+\beta\left[(n+p)B+(B-A)\alpha+A\right]\right\}}{(B-A)\beta(1-\alpha)(1-2\lambda)}|a_{p+n}|,$$

$\mu_j=0$ $(j=1,2,\ldots,p-1)$ if $p\geq 2$ and $\mu_0=1-\sum_{n=0}^{\infty}\mu_n\geq 0.$

It follows that

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z).$$

This completes the proof of Theorem 4.2.

Theorem 4.3 (Radius of Convexity)

If $f(z)$ given by (1.3) is in the class $\Omega^*(\alpha,\beta,\lambda,A,B)$ then $f(z)$ is convex in the disk

$$0<|z|=r=r(\alpha,\beta,\lambda,A,B)=$$

$$\inf_n\left(\frac{\left\{(n+p+1)+\beta\left[B(n+p)+(B-A)\alpha+A\right]\right\}\left\{1+(n+p-1)\lambda\right\}}{(B-A)\beta(1-\alpha)(p+n+2)(1-2\lambda)}\right)^{\frac{1}{p+n+1}}.$$

The result is sharp for the function given by (2.4).

Proof.

It is sufficient to prove the $n \in \{0, N\}$

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < 1 \text{ for } 0 < |z| \leq r(\alpha, \beta, \lambda, A, B).$$

We have

$$\begin{aligned}
\left|\frac{zf''(z)}{f'(z)} + 2\right| &= \left|\frac{zf''(z) + 2f'(z)}{f'(z)}\right| = \\
&= \left|\frac{\sum_{n=0}^{\infty}(p+n)(p+n-1)a_{p+n}z^{p+n-2} + \sum_{n=0}^{\infty}2(p+n)a_{p+n}z^{p+n-1}}{-\frac{1}{z^2} + \sum_{n=0}^{\infty}(p+n)a_{p+n}z^{p+n-1}}\right| \\
&= \left|\frac{\sum_{n=0}^{\infty}(p+n)(p+n-1)a_{p+n}z^{p+n-1}}{-\frac{1}{z^2} + \sum_{n=0}^{\infty}(p+n)a_{p+n}z^{p+n-1}}\right| \\
&= \left|\frac{\sum_{n=0}^{\infty}(p+n)(p+n -1)\ a_{p+n}z^{p+n-1}}{1 - \sum_{n=0}^{\infty}(p+n)a_{p+n}z^{p+n-1}}\right| \\
&= \left|\sum_{n=0}^{\infty}(p+n)(p+n+2)a_{p+n}z^{p+n-1}\right| \leq 1
\end{aligned}$$

by Theorem 2.2.

$$\sum_{n=0}^{\infty}\frac{(n+p)\{(n+p+1) + \beta[B(n+p) + (B-A)\alpha + A]\}\{1 + (n+p-1)\lambda\}}{\beta(1-\alpha)(1-2\lambda)(B-A)}a_{n+p} \leq 1.$$

But (4.1) holds if and only if

$$\sum_{n=0}^{\infty}(p+n)(p+n+2)|z|^{p+n+1} \leq$$

$$\frac{[1 + (p+n-1)\lambda]\,(n+p)\,\{(n+p+1) + \beta\,[B(n+p) + (B-A)\alpha + A]\}}{(B-A)(1-\alpha)(1-2\lambda)\beta},$$

$$|z|^{p+n+1} \leq \frac{[(n+p+1) + \beta\,\{B(n+p) + (B-A)\alpha + A\}]\,[1 + (p+n-1)\lambda]}{(p+n+2)(B-A)(1-\alpha)(1-2\lambda)\beta},$$

$$|z| \leq \left(\frac{[(n+p+1) + \beta\,\{B(n+p) + (B-A)\alpha + A\}]\,[1 + (p+n-1)\lambda]}{(p+n+2)(1-\alpha)(1-2\lambda)(B-A)\beta}\right)^{\frac{1}{p+n+1}}.$$

This completes the proof of Theorem 4.3.

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