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# Мероморфни функции върху двумерния комплексен тор с предварително зададени стойности върху два пораждащи тора

Лилия Ник. Апостолова

# Meromorphic functions on the two-dimensional complex torus with prescribed value on two generating torus

L. N. Apostolova

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# Meromorphic functions on the two dimensional complex torus with prescribed value on two generating torus*†

L. N. Apostolova

**Abstract**

A system of partial differential equations with constant complex coefficients on $\mathbf{C}^2$ is considered. D'Alembert solution is given. Existence of meromorphic solutions on the two-dimentional complex torus with prescribed value on two generating torus is proved. Some examples are given.

## 1 The problem

The following problem will be solved here: to find a meromorphic solution $f(z,w)$ of a system of partial differential homogeneous equations with constant coefficients on $\mathbf{C}^2$ satisfying the conditions $f(z,0) = \varphi(z)$ and $f(0,w) = \varphi(w)$, where $\varphi(z)$, $\psi(w)$ are meromorphic functions on the parallelepipedon in $\mathbf{C}$, on the streap in $\mathbf{C}$, or on the whole complex plane $\mathbf{C}$.

More precisely we will consider the following problem: to solve the system

$$-\frac{\partial^2 f}{\partial z^2} - i\frac{\partial^2 f}{\partial w^2} = 0, \qquad \frac{\partial f}{\partial \overline{z}} = 0, \qquad \frac{\partial f}{\partial \overline{w}} = 0 \tag{1}$$

with conditions on the solution $f(z,w)$ as follows

$$f(z,w)|_{w=0} = \varphi(z), \qquad f(z,w)|_{z=0} = \psi(w) \quad \text{on} \quad \mathbf{C}^2, \tag{2}$$

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where $\varphi(z)$ and $\psi(w)$ are meromorphic functions on $\mathbf{C}$.

The differential operators of second order with constant complex coefficients, called *even and odd double-complex Laplace operators*,

$$\Delta_{\pm} = \frac{\partial^2}{\partial z^2} \pm i \frac{\partial^2}{\partial w^2}$$

arise naturally (see [1], [3]). In [1] is consider a function theory for such operators, in some sense analogous to the function theory in [4].

## 2 D'Alembert solution of the problem on $\mathbf{C}^2$

**Theorem.** For each two meromorphic functions $\varphi(z)$ and $\psi(w)$ the following meromorphic in $z$ and $w$ function $f(z,w)$ gives a solution of the system (1) with conditions (2):

$$f(z,w) = \frac{1+i}{2\sqrt{2}}\varphi\left(z + \frac{1+i}{\sqrt{2}}w\right) + \frac{1+i}{2\sqrt{2}}\varphi\left(z - \frac{1+i}{\sqrt{2}}w\right)$$
$$+\frac{1-i}{\sqrt{2}}\psi\left(w + \frac{1+i}{\sqrt{2}}z\right) + \frac{1-i}{2\sqrt{2}}\psi\left(w - \frac{1+i}{\sqrt{2}}z\right). \qquad (3)$$

This solution will be called *a d'Alembert solution* of the system (1) on the complex space $\mathbf{C}^2$. This solution is a modification of the following fundamental for the operator $\Delta_+$ d'Alembert type solution

$$\frac{1}{2}\left(\varphi_0\left(z + \frac{1+i}{\sqrt{2}}w\right) + \varphi_0\left(z - \frac{1+i}{\sqrt{2}}w\right)\right)$$
$$+\frac{1+i}{2\sqrt{2}}\left(\varphi_1\left(z + \frac{1+i}{\sqrt{2}}w\right) + \varphi_1\left(z - \frac{1+i}{\sqrt{2}}w\right)\right)$$
$$+\frac{1}{\sqrt{2}(1+i)}\int_{z-\frac{1+i}{\sqrt{2}}w}^{z+\frac{1+i}{\sqrt{2}}w} \psi_0(t)\,dt + \frac{1}{2\sqrt{2}}\int_{z-\frac{1+i}{\sqrt{2}}w}^{z+\frac{1+i}{\sqrt{2}}w} \psi_1(t)\,dt,$$

where $\varphi_0$, $\varphi_1$, $\psi_0$, $\psi_1$ are complex functions of complex variable. See for instance [5].

P r o o f:

It is clear, that $-\varphi'' - i\left(\pm\frac{1+i}{\sqrt{2}}\right)^2\varphi'' = 0$ and $-\psi'' - i\left(\pm\frac{1+i}{\sqrt{2}}\right)^2\psi'' = 0$, as $\varphi$ and $\psi$ are functions of one complex variable.

# 3 Meromorphic solutions on the two-dimensional complex torus

Let us choose the function $\varphi(z)$ to be a double-periodic function with mean periods 1 and $\frac{1-i}{\sqrt{2}}$ and $\psi(w)$ to be a double-periodic function with mean periods 1 and $\frac{1+i}{\sqrt{2}}$. Then the solution $f(z,w)$ will be a double-periodic function of the variables $z$ and $w$ with mean periods $\omega^{(1)} = (1, \frac{1-i}{\sqrt{2}})$, $\omega^{(2)} = (1, \frac{1+i}{\sqrt{2}})$. In such a way the solution $f(z,w)$ factorizes to a function $F(M,N)$ on the factor torus $T^2/T_1 \times T_2$, obtained by factorization of $T^2$ trough the lattice

$$\Omega = (\omega^{(1)}, \omega^{(2)}) = \left( \begin{array}{cc} 1 & \frac{1-i}{\sqrt{2}} \\ 1 & \frac{1+i}{\sqrt{2}}) \end{array} \right) \tag{4}$$

in $\mathbf{C}^2$, $\det \Omega = \sqrt{2}i$. So we obtain a class of solutions on the two dimensional complex torus, factorizing by the product of two one dimensional complex torus with values on $T^{(1)}$ and $T^{(2)}$ equals to the elliptic functions $\varphi(z)$, $\psi(w)$.

Following [2] we decompose the $2 \times 2$ matrix $\Omega$ as a sum of two $2 \times 2$ matrices as follows

$$\Omega = (\omega^{(1)}, \omega^{(2)}) = \left( \begin{array}{cc} 1 & \frac{1-i}{\sqrt{2}} \\ 1 & \frac{1+i}{\sqrt{2}} \end{array} \right) = \left( \begin{array}{cc} 1 & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} \end{array} \right) + i \left( \begin{array}{cc} 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{array} \right) = \Sigma + iT.$$

# 4 Two examples

## 4.1 A class of meromorphic solutions on a complex plane

Let us consider the boundary functions $\varphi(z), \psi(w)$ and the holomorphic functions of complex variables $e^{inz}$, $n = 0, \pm 1, \pm 2, \ldots$.

Every meromorphic two-periodic on the two-dimensional torus $\mathbf{C} \times \mathbf{C}/_{\Gamma_1 \times \Gamma_2}$ function $F(z)$ generate a formal Fourier series

$$F(z) = \sum_{-\infty}^{\infty} c_n e^{inz} \tag{5}$$

with respect to the exponential functions. We have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(z) e^{-inz} \, dz.$$

The partial sums of the corresponding Fourier series gives a class of solutions of the problem (1). For more information see [6].

On the one dimensional complex linear space

$$L : \{(z, w) \in \mathbf{C}^2 : \Im\left(z \pm \frac{1+i}{\sqrt{2}} w\right) = 0\} \tag{6}$$

all the obtained in (3) solutions $f(z, w)$ will be double periodic meromorphic functions.

Now let us consider the complex plane $2\Im\left(z \pm \frac{1+i}{\sqrt{2}} w\right) = 2y \pm (u\sqrt{2} - v\sqrt{2})$, which is one-dimensional complex linear space in in $\mathbf{C}^2$. We obtain for $L$:

$$L : y = 0, v = -u.$$

So $L : \{(z = x, w = u(1 - i)\}$ is a complex one dimensional linear space with complex coefficients in $\mathbf{C}^2$. Making the factor $\mathbf{C}^2/L$ and bringing the partial sums of the series (5) we obtain a class of meromorphic solutions on the obtained in such a way complex plane. See Fig. 1.

### 4.2 The problem on $\mathbf{C}^2$ with quadratic boundary data

Now we shall consider the particular solution of the problem with quadratic boundary data. Let us take for the boundary data of the problem (1) the following functions:

$$\varphi(z) = az^2 + a_1 z + a_2 \quad \text{and} \quad \psi(w) = bw^2 + b_1 w + b_2.$$

Obviously, the constants $a_1, a_2, b_1, b_2$ can be chosen to be each complex numbers. The functions $az^2$ and $bw^2$ are holomorphic functions. We must to assure the validity of the first equation in the system (1). We check $(az^2)'' + i(bw^2)'' = a + ib = 0$. So the functions

$$a(z^2 - iw^2) + a_1 z + b_1 w + a_2 + b_2$$

will be solutions of the problem (1) for every choice of the constants (see also [3]).

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*L. N. Apostolova*
*Institute of Mathematics and Informatics,*
*Bulgarian Acad. of Sci., ul. "Acad. G. Bonchev" 8,*
*Sofia-1113, Bulgaria,*
*E-mail:* liliana@math.bas.bg

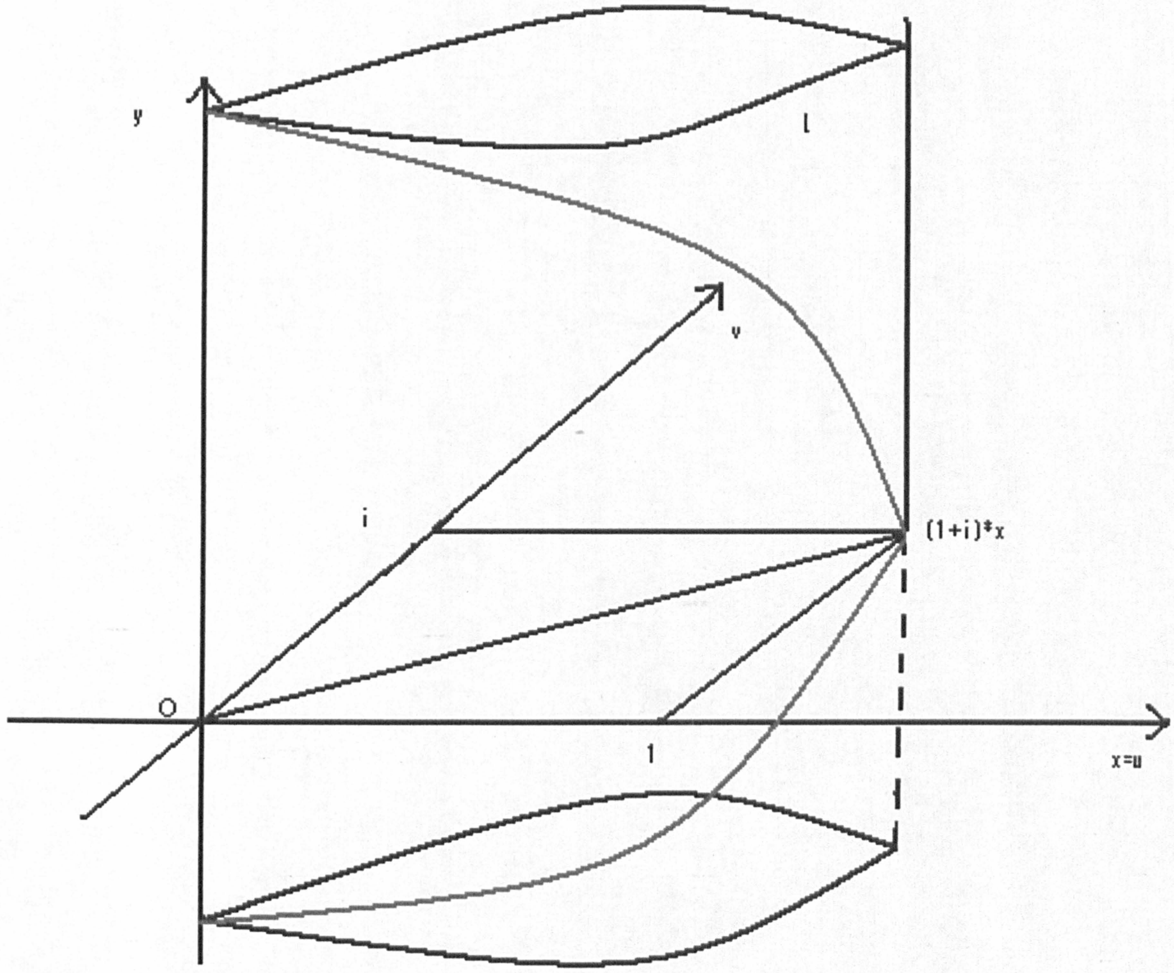


Fig. 1.