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Донка Пашкулева

Certain Classes of Functions with Negative Coefficients

Donka Pashkouleva

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Abstract

The aim of this paper is to obtain coefficient estimates, distortion theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the subclass $TS_\lambda(u,\alpha,\beta)$ with negative coefficients.

1 Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let S^* and C be subclasses of S that are, respectively, starlike and convex.

A function

$$f(z) \in \widetilde{C} \Leftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \qquad z \in U. \tag{1.2}$$

Let S_p be a class of starlike functions related to $\widetilde{C}$ defined as

$$f(z) \in S_p \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad z \in U. \tag{1.3}$$

Note that

$$f \in \widetilde{C} \Leftrightarrow zf'(z) \in S_p. \tag{1.4}$$

A function f of the form (1.1) is in $S_p(\alpha)$ if it satisfies the analytic characterization:

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq\left|\frac{zf'(z)}{f(z)}-1\right|,\ -1\leq\alpha<1,\ z\in U. \tag{1.5}$$

The function $f\in\widetilde{C}(\alpha)$ if and only if $zf'(z)\in S_p(\alpha)$.

By $\widetilde{C}_\beta$, $0\leq\beta<\infty$ we denote the class of all β-convex functions introduced by Kanas and Wisniowska [1]. It is known that [1] that $f\in\widetilde{C}_\beta$ if and only if it satisfies the following condition:

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|\frac{zf'(z)}{f'(z)}\right|,\ z\in U,\ \beta\geq 0. \tag{1.6}$$

We consider the class S^*_β, $0\leq\beta<\infty$, of β-starlike functions [2], which are associated with the class $\widetilde{C}_\beta$ by the relation

$$f\in C^*_\beta\Leftrightarrow zf'(z)\in S^*_\beta. \tag{1.7}$$

Thus, the class S^*_p is the subclass of S, consisting of functions that satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|,\ z\in U,\ \beta\geq 0. \tag{1.8}$$

For a function $f\in S$, we define

$$\begin{aligned}&D^0_\lambda f(z)=f(z)\\&D^1_\lambda f(z)=(1-\lambda)f(z)+\lambda zf'(z)=D_\lambda f(z)\\&D^n_\lambda f(z)=D_\lambda(D^{n-1}_\lambda f(z))\\&\lambda>0,\ n\in N=\{1,2,\ldots\}\end{aligned} \tag{1.9}$$

This operator was introduced by Al-Oboudi [3], and when $\lambda=1$, we get the Salagean operator [4].

It can be easily seen that

$$D^n_\lambda f(z)=z+\sum_2^\infty[1+\lambda(k-1)]^n a_k z^k\ \ (n\in N_0=N\cup\{0\}). \tag{1.10}$$

For $p\geq 0$, $-1\leq\alpha<1$, $n\in N_0$ and $\lambda>0$ we let $S_\lambda(n,\alpha,\beta)$ denote the subclass of S consisting of functions f of the form (1.1) and satisfying the analytic condition

$$\Re\left\{\frac{z\left(D^n_\lambda f(z)\right)'}{D^n_\lambda f(z)}-\alpha\right\}>\beta\left|\frac{z\left(D^n_\lambda f(z)\right)'}{D^n_\lambda f(z)}-1\right|. \tag{1.11}$$

We denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; a_k \geq 0. \tag{1.12}$$

Further, we define the class $TS_\lambda(n, \alpha, \beta)$ by

$$TS_\lambda(n, \alpha, \beta) = S_\lambda(n, \alpha, \beta) \cap T. \tag{1.13}$$

We note that $TS_1(n, \alpha, \beta) = TS(n, \alpha, \beta)$ [5], $TS_0(0, \alpha, 1) = TS^0(\alpha, 1)$ and $TS_1(1, \alpha, 1) = \widetilde{C}(\alpha)$ $(0 \leq \alpha < 1)$ [6].

2 Coefficient estimates

Theorem 1 *A necessary and sufficient condition for the function $f(z)$ of the form (1.12) to be in the class $TS_\lambda(n, \alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)][1+\lambda(k-1)]^n a_k \leq 1 - \alpha \tag{2.1}$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda \geq 0$ and $n \in N_0$.

Proof. Let (2.1) holds true, then we have

$$\beta \left| \frac{z\,(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right| - \Re \left\{ \frac{z\,(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right\} \leq (1+\beta) \left| \frac{z\,(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right| \leq$$

$$\leq \frac{(1+\beta) \sum_{k=2}^{\infty} (k-1)[1+\lambda(k-1)]^n |a_k|}{1 - \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n |a_k|} \leq 1 - \alpha.$$

Then $f(z) \in TS_\lambda(n, \alpha, \beta)$.

Conversely, let $f(z) \in TS_\lambda(n, \alpha, \beta)$ and z be real, then

$$\frac{1 - \sum_{k=2}^{\infty} k[1+\lambda(k-1)]^n a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n a_k z^{k-1}} - \alpha \geq$$

$$\geq \beta \left| \frac{\sum_{k=2}^{\infty} (k-1)[1+\lambda(k-1)]^n a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n a_k z^{k-1}} \right|.$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n a_k \leq 1-\alpha.$$

Remark 1 *If $f(z) \in S_\lambda(n, \alpha, \beta)$ the condition (2.1) is only sufficient.*

Corollary 1 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n, \alpha, \beta)$. Then*

$$a_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n},\ k \geq 2. \tag{2.2}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{[k(1+\alpha)-(\alpha+\beta)][1+\lambda(k-1)]^n} z^k. \tag{2.3}$$

3 Growth and distortion theorem

Theorem 2 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n, \alpha, \beta)$. Then*

$$\left|D_\lambda^i f(z)\right| \geq |z| - \frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}|z|^2 \tag{3.1}$$

and

$$\left|D_\lambda^i f(z)\right| \leq |z| + \frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}|z|^2 \tag{3.1}$$

for $z \in U$, where $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^n} z^2\ (z \in U). \tag{3.3}$$

Proof. Note that $f(z) \in TS_\lambda(n, \alpha, \beta)$ if and only if $D_\lambda^i f \in TS_\lambda(n, \alpha, \beta)$ and that

$$D_\lambda^i f(z) = z - \sum_{k=2}^{\infty}[1+\lambda(k-1)]^i a_k z^k \tag{3.4}$$

Using Theorem 1, we know that

$$(3.5)\qquad (2-\alpha+\beta)(1+\lambda)^{n-i}\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{i}a_k \le 1-\alpha,$$

that is, that

$$(3.6)\qquad \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{i}a_k \le \frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}.$$

It follows from (3.4) and (3.6) that

$$(3.7)\qquad \begin{aligned} |D_\lambda^i f(z)| &\ge |z|-|z|^2\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{i}a_k \ge \\ &\ge |z|-\frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}|z|^2 \end{aligned}$$

and

$$(3.8)\qquad \begin{aligned} |D_\lambda^i f(z)| &\le |z|+|z|^2\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{i}a_k \le \\ &\le |z|+\frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}|z|^2 \end{aligned}$$

Finally, we note that the bounds in (3.1) are attained for the function $f(z)$ defined by

$$(3.9)\qquad D_\lambda^i f(z) = z-\frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n-i}}z^2,\ z\in U.$$

This completes the proof of Theorem 2.

Corollary 2 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n,\alpha,\beta)$. Then*

$$(3.10)\qquad \begin{aligned} &|z|-\frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n}}|z|^2 \le |f(z)| \\ &\le |z|+\frac{1-\alpha}{(2-\alpha+\beta)(1+\lambda)^{n}}|z|^2. \end{aligned}$$

The equalities in (3.10) are attained for the function $f(z)$ given by (3.3).

Proof. Taking $i=0$ in Theorem 2, we immediately obtain (3.10).

4 Extreme points

From Theorem 1, we see that $TS_1(n,\alpha,\beta)$ is closed under convex linear combination which enables us to determine the extreme points for this class.

Theorem 3 *Let*

$$f_1(z) = z \tag{4.1}$$

and

$$f_k(z) = z - \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n} z^k,\ k \geq 2. \tag{4.2}$$

Then $f(z) \in TS_\lambda(n,\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{4.3}$$

where $\mu_k \geq 0$ $(k \geq 1)$ *and* $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \mu_k f_k(z) \\ &= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n} \mu_k z_k. \end{aligned} \tag{4.4}$$

Then it follows that

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{1-\alpha} \times \\ &\times \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n} \mu_k = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned} \tag{4.5}$$

Therefore, by Theorem 1, $f(z) \in TS_\lambda(n,\alpha,\beta)$.

Conversely, assume that the function $f(z)$ defined by (1.12) belongs to the class $TS_\lambda(n,\alpha,\beta)$. Then

$$a_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n},\ k \geq 2. \tag{4.6}$$

Setting

$$\mu_k = \frac{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{1-\alpha}, \; k \geq 2 \tag{4.7}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k \tag{4.8}$$

we see that $f(z)$ can be expressed in the form (4.3). This completes the proof of Theorem 3.

Corollary 3 *The extreme points of the class $TS_\lambda(n, \alpha, \beta)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n} z^k, \; k \geq 2.$$

5 Radii of close-to-convexity, starlikeness and convexity

A function $f(z) \in T$ is said to be close-to-convex of order ρ if it satisfies

$$\Re f'(z) > \rho, \; 0 \leq \rho < 1, \; z \in U. \tag{5.1}$$

Theorem 4 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n, \alpha, \beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$ where*

$$\begin{aligned} r_1 &= r_1(n, \alpha, \beta, \lambda, \rho) = \\ &= \inf_k \left\{ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{k(1-\alpha)} \right\}^{\frac{1}{k-1}}. \end{aligned} \tag{5.2}$$

The result is sharp, with extremal $f(z)$ given by (2.3).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \text{ for } |z| < r_1(n, \alpha, \beta, \lambda, \rho)$$

where $r_1(n, \alpha, \beta, \lambda, \rho)$ is given by (5.2). Indeed we find from (1.12) that

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z) - 1| \leq 1 - \rho$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.3}$$

But, by Theorem 1, (5.3) will be true if

$$\left(\frac{k}{1-\rho}\right) |z|^{k-1} \leq \frac{k(1+\beta) - (\alpha+\beta)][1+\lambda(k-1)]^n}{1-\alpha},$$

that is, if

$$|z| \leq \left\{\frac{(1-\rho)[k(1+\beta) - (\alpha+\beta)][1-\lambda(k-1)^n}{k(1-\alpha}\right\}^{\frac{1}{k-1}}, \quad k \geq 2. \tag{5.4}$$

Theorem 4 follows easily from (5.4).

Theorem 5 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n, \alpha, \beta)$. Then the function $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$\begin{aligned} r_2 &= r_2(n, \alpha, \beta, \lambda, \rho) \\ &= \inf_k \left\{\frac{(1-\rho)[k(1+\beta) - (\alpha+\beta)][1+\lambda(k-1)]^n}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}, \\ k &\geq 2. \end{aligned} \tag{5.5}$$

The result is sharp, with the extreme function $f(z)$ given by (2.3).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho \text{ for } |z| < r_2(n,, \alpha, \beta, \lambda, \rho)$$

where $r_2(n, \alpha, \beta, \lambda, \rho)$ is given by (5.5). Indeed we find again from (1.12) that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\rho}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.6}$$

But, by Theorem 1, (5.6) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right) |z|^{k-1} \leq \frac{[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{1-\alpha},$$

that is, if

$$|z| \leq \left\{\frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.7}$$

Theorem 5 follows easily from (5.7).

Corollary 4 *Let the function $f(z)$ defined by (1.12) be in the class $TS_\lambda(n, \alpha, \beta)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$\begin{aligned} r_3 &= r_3(n, \alpha, \beta, \lambda, \rho) \\ &= \inf_k \left\{\frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]^n}{k(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}, \\ k &\geq 2. \end{aligned} \tag{5.8}$$

The result is sharp with extremal function $f(z)$ given by (2.3).

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