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Exact solutions of nonlocal Bitsadze - Samarskii problem and its generalizations

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Abstract

In the first part of this paper we find an explicit solution of Bitsadze - Samarskii problem for Laplace equation using operational calculus approach, based on two non-classical one-dimensional convolutions and a two-dimensional convolution. In fact, the explicit solution obtained is a way for effective summation of a solution obtained in the form of non-harmonic Fourier sine-expansion. This explicit solution is suitable for numerical calculation too. In the second part we consider "arbitrary" linear functionals Φ and Ψ on $C^1[0,a]$ and $C^1[0,b]$, respectively. The class of BVPs $u_{xx}+u_{yy}=F(x,y)$, $0<x<a$, $0<y<b$, $u(x,0)=0$, $u(0,y)=0$, $\Phi_\xi\{u(\xi,y)\}=g(y)$, $\Psi_\eta\{u(x,\eta)\}=f(x)$ is considered. An extension of Duhamel principle, known for evolution

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equations, is proposed. An operational calculus approach for explicit solution of these problems is developed. A classical example of such BVP is the Bitsadze - Samarskii problem.

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1 Exact solutions of nonlocal Bitsadze - Samarskii problem

In [2] it is posed the following nonlocal boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad -l<x<l, \quad 0<y<1,\\
&u(x,0)=0, \quad u(x,1)=f(x),\\
&u(-l,y)=g(y), \quad u(l,y)=u(0,y).
\end{aligned}$$

More elaborately, this problem is studied in A. Bitsadze's book [1] , p. 214 - 219. Some generalizations are proposed by A. Skubachevskii in [11]. In [4], p. 175 - 176 one of the authors proposed an explicit solution of the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad 0<x<1, \quad 0<y<1,\\
&u(x,0)=u(0,y)=0, \qquad\qquad (1.1)\\
&u(x,1)=f(x), \quad u(1,y)-u(\frac{1}{2},y)=0.
\end{aligned}$$

which is only a slight modification of Bitsadze - Samarskii's problem. This solution has the form

$$u(x,y)=-\int_{\frac{1}{2}}^{1}d\xi\{\int_{x}^{\xi}U(x+\xi-\eta,y)f^{(4)}(\eta)d\eta- \qquad (1.2)$$

$$-\int_{-x}^{\xi}U(\xi-x-\eta,y)f^{(4)}(|\eta|)sgn(|\eta|)d\eta\}$$

where

$$U(x,y) = \sum_{n=1}^{\infty} \frac{\sinh 4n\pi y \sin 4n\pi x}{32\pi^3 n^3 \sinh 4n\pi} + \sum_{n=1}^{\infty} \frac{9 \sinh \frac{2}{3}(2n-1)\pi y \sin \frac{2}{3}(2n-1)\pi x}{4\pi^3 (2n-1)^3 \cos \frac{2}{3}(1+n)n\pi \sinh \frac{2}{3}(2n-1)\pi} \tag{1.3}$$

is the solution of the same problem, but for the special choice $f(x) = \frac{x^3}{6} - \frac{7x}{24}$. It is a classical solution of (1.1) under the assumptions $f(0) = f''(0) = 0$, $f(1) - f(\frac{1}{2}) = f''(1) - f''(\frac{1}{2}) = 0$. Our aim here is to simplify (1.2) to

$$u(x,y) = \int_x^2 U_x(\frac{1}{2} + x - \xi, y) f''(\xi) d\xi - \int_{-x}^2 U_x(\frac{1}{2} - x - \xi, y) f''(|\xi|) sgn\xi d\xi - \int_x^1 U_x(1 + x - \xi, y) f''(\xi) d\xi + \int_{-x}^1 U_x(1 - x - \xi, y) f''(|\xi|) sgn\xi d\xi \tag{1.4}$$

where

$$U_x(x,y) = \sum_{n=1}^{\infty} \frac{\sinh 4n\pi y \cos 4n\pi x}{8\pi^2 n^2 \sinh 4n\pi} + \sum_{n=1}^{\infty} \frac{3 \sinh \frac{2}{3}(2n-1)\pi y \cos \frac{2}{3}(2n-1)\pi x}{2\pi^2 (2n-1)^2 \cos \frac{2}{3}(1+n)n\pi \sinh \frac{2}{3}(2n-1)\pi} \tag{1.5}$$

In a sense (1.4) is simpler than (1.2) since it uses only second derivatives of f instead of fourth ones and only simple integrals instead of repeated. The boundary value restrictions on f are also relaxed to $f(0) = f(1) - f(\frac{1}{2}) = 0$. Then (1.4) is a generalized solution of (1.1) in the following sense:

Definition 1.1 A function $u(x,y) \in C([0,1] \times [0,1])$ is said to be a generalised solution of Bitsadze - Samarskii problem (1.1), iff $u(x,y)$ satisfies the integral equation

$$L_x u + L_y u = L_x f(x).y \tag{1.6}$$

where

$$L_x u(x,y) = \int_0^x (x-\xi)u(\xi,y)d\xi - \tag{1.7}$$
$$-2x(\int_0^1 (1-\xi)u(\xi,y)d\xi - \int_0^{\frac{1}{2}} (\frac{1}{2}-\xi)u(\xi,y)d\xi)$$
$$L_y u(x,y) = \int_0^y (y-\eta)u(x,\eta)d\eta - y(\int_0^1 (1-\eta)u(x,\eta)d\eta)$$

The right inverse operators L_x and L_y of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ are defined in $C([0,1]\times[0,1])$ by

$$v = L_x u \ : \ \frac{\partial^2}{\partial x^2} v = u, \quad v(0,y) = v(1,y) - v(\frac{1}{2},y) = 0$$

and

$$w = L_y u \ : \ \frac{\partial^2}{\partial y^2} w = u, \quad w(x,0) = w(x,1) = 0,$$

correspondingly. Formally, (1.6) could be obtained from the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ applying to it the operator $L_x L_y$ and using the boundary value conditions.

Lemma 1.1 *If $u(x,y) \in C([0,1]\times[0,1])$ satisfies (1.6), then $u(x,y)$ satisfies the boundary value conditions:*

$$u(x,0) = u(0,y) = 0, \quad u(x,1) = f(x), \quad u(1,y) - u(\frac{1}{2},y) = 0$$

Proof. For $y=0$ from (1.6) we obtain $L_x u(x,0) = 0$. Applying the operator $\frac{\partial^2}{\partial x^2}$ to this equation we find $u(x,0) = 0$. In a similar way for $y=1$ we find $u(x,1) = f(x)$. In a similar way for $y=1$ we find $u(x,1) = f(x)$. Next, for $x=0$ from (1.6) we obtain $L_y u(0,y) = 0$. Applying the operator $\frac{\partial^2}{\partial y^2}$ to this equation we find $u(0,y) = 0$. Analogically, we find $u(1,y) - u(\frac{1}{2},y) = 0$. ⋄

Example. If $f(x) = \frac{x^3}{6} - \frac{7x}{24}$ then (1.3) is a generalized solution of boundary value problem (1.1) (see [4], p. 175).

Lemma 1.2 *If a function $u(x,y) \in C^2([0,1]\times[0,1])$ satisfy (1.6), then it is a classical solution of (1.1).*

Proof. We apply the operator $\frac{\partial^4}{\partial x^2 \partial y^2}$ to (1.6) and obtain $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. As for the boundary value conditions, they are satisfied by Lemma 1.1. ◇

In order to elucidate our approach for obtaining of an explicit solution, we will consider the following extension of Bitsadze - Samarskii problem (1.1):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y), \quad 0 < x < 1, \;\; 0 < y < 1, \tag{1.8}$$
$$u(x,0) = u(0,y) = 0,$$
$$u(x,1) = f(x), \quad u(1,y) - u(\frac{1}{2}, y) = g(y).$$

Where $f(x), g(y) \in C([0,1])$, $F(x,y) \in C([0,1] \times [0,1])$.

Definition 1.2 A function $u(x,y) \in C([0,1] \times [0,1])$ is said to be a generalized solution of problem (1.8), iff $u(x,y)$ satisfies the integral equation

$$L_x u + L_y u = L_x f(x).y + L_y g(y).x + L_x L_y F(x,y) \tag{1.9}$$

Formally, (1.9) could be obtained easily from the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$ applying the operator $L_x L_y$ to it and using the boundary value conditions.

Lemma 1.3 *If a function $u(x,y) \in C([0,1] \times [0,1])$ satisfy (1.9), then $u(x,y)$ fulfils the boundary value conditions:*

$$u(x,0) = u(0,y) = 0, \quad u(x,1) = f(x), \quad u(1,y) - u(\frac{1}{2}, y) = g(y).$$

Proof. Analogically to the proof of Lemma 1.1.◇

Lemma 1.4 *If a function $u(x,y) \in C^2([0,1] \times [0,1])$ satisfies (1.9), then it is a classical solution of (1.1).*

Proof. Applying the operator $\frac{\partial^4}{\partial x^2 \partial y^2}$ to (1.9), we obtain $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$. The boundary value conditions are satisfied by Lemma 1.3. ◇

In order to obtain an explicit solution of (1.1) or (1.9) we will outline an operational calculus approach to Bitsadze - Samarskii problem. To this end, we introduce three convolution algebras: $(C[0,1], \overset{x}{*})$, $(C[0,1], \overset{y}{*})$ and $(C([0,1] \times [0,1]), *)$.

Theorem 1.1 *The operation*

$$(f \overset{x}{*} g)(x) = \int_0^{\frac{1}{2}} h(x,\eta)d\eta - \int_0^1 h(x,\eta)d\eta, \qquad (1.10)$$

where

$$h(x,\eta) = \int_x^\eta f(x+\eta-\xi)g(\xi)d\xi - \int_{-x}^\eta f(|\eta-x-\xi|)g(|\xi|)sgn(\xi(\eta-x-\xi))d\xi$$

is a bilinear, commutative and associative operation on $C[0,1]$*, such that* $L_x f(x) = x \overset{x}{*} f$.

This a special case of a more general operation $(f \overset{x}{*} g) = -\frac{1}{2}\Phi_\xi\{\int_0^\xi h(x,\eta)d\eta\}$ in $C[0,a]$ where Φ is a linear functional in $C^1[0,a]$ for the special choice $\Phi\{f\} = 2(f(1) - f(\frac{1}{2}))$ and $a = 1$ (see [4], p. 119).

Theorem 1.2 *The operation*

$$(f \overset{y}{*} g)(y) = -\frac{1}{2}\left(\int_0^1 h(y,\eta)d\eta\right) \qquad (1.11)$$

where

$$h(y,\eta) = \int_y^\eta f(y+\eta-\xi)g(\xi)d\xi - \int_{-y}^\eta f(|\eta-y-\xi|)g(|\xi|)sgn(\xi(\eta-y-\xi))d\xi$$

is a bilinear, commutative and associative operation on C[0, 1], such that $L_y f = y \overset{y}{*} f$.

This again a special case of the above mentioned general operation for the special choice $a = 1$ and $\Phi\{f\} = f(1)$.

We may combine both one-dimensional convolutions into one two-dimensional convolution.

Theorem 1.3 *[4] The operation*

$$(f * g)(x,y) = \frac{1}{2}\int_0^1 \left(\int_0^1 h(x,y,\xi,\eta)d\xi - \int_0^{\frac{1}{2}} h(x,y,\xi,\eta)d\xi\right) d\eta \qquad (1.12)$$

where

$$h(x,y,\xi,\eta)=\int_x^{\xi}\int_y^{\eta} f(\xi+x-\sigma,\eta+y-\tau)g(\sigma,\tau)d\sigma\tau -$$

$$-\int_{-x}^{\xi}\int_y^{\eta} f(|\xi-x-\sigma|,\eta+y-\tau)g(|\sigma|,\tau)sgn(\xi-x-\sigma)\sigma d\sigma\tau -$$

$$-\int_x^{\xi}\int_{-y}^{\eta} f(\xi+x-\sigma,|\eta-y-\tau|)g(\sigma,|\tau|)sgn(\eta-y-\tau)\tau d\sigma\tau -$$

$$-\int_{-x}^{\xi}\int_{-y}^{\eta} f(|\xi-x-\sigma|,|\eta-y-\tau|)g(\sigma,|\tau|)sgn(|\xi-x-\sigma|)(\eta-y-\tau)\sigma\tau d\sigma\tau$$

is a bilinear, commutative and associative operation, in $C = C([0,1]\times[0,1])$ *such that the product* L_xL_y *has the representation*

$$L_xL_yu = \{xy\} * u. \tag{1.13}$$

Lemma 1.5

$$L_x\left\{\frac{\partial^2 u}{\partial x^2}\right\} = u(x,y)-u(0,y)-2x[u(1,y)-u(\frac{1}{2},y)] \tag{1.14}$$

and

$$L_x\left\{\frac{\partial^2 u}{\partial y^2}\right\} = u(x,y)+(y-1)u(x,0)-yu(x,1). \tag{1.15}$$

The proof is immediate.

In order to outline our operational calculus approach to the extended Bitsadze-Samarskii problem, we start with the general definition of a multiplier of convolutional algebra.

Definition 1.3 [10] A linear operator $M : C \to C$ is said to by a multiplier of the convolutional algebra $(C, *)$ if $M(u*v) = (Mu)*v$ for all $u, v \in C$.

We introduce some notations. The multipliers of the form $\{u(x,y)\}*$ will be denoted as $\{u\}$. Let $f = \{f(x)\}$ be a function of the variable x only and $g = \{g(y)\}$ be a function of the variable y only, but both considered as elements of C. The operators $[f]_y$ and $[g]_x$ defined by $[f]_y u = f \overset{x}{*} u$ and $[g]_x u = g \overset{y}{*} u$ are said to be partial numerical operators with respect to y and x correspondingly. In this notations we have $L_x = [x]_y$ and $L_y = [y]_x$.

The set of all the multipliers of the convolutional algebra $(C, *)$ is a commutative ring $\mathcal{M}$. The multiplicative set NN of the non-zero non-divisors of 0 in $\mathcal{M}$ is non-empty, since at least the operators $x \overset{x}{*} = [x]_y$ and $y \overset{y}{*} = [y]_x$ are non-divisors of 0.

Next we introduce the ring $\mathcal{M} = NN^{-1}\mathcal{M}$ of the multiplier fractions of the form $\frac{A}{B}$ where $A \in \mathcal{M}$ and $B \in NN$. The standard algebraic procedure named "localization" of constructing of this ring, is described, e.g. in Lang [9]. Most important for our considerations are the algebraic inverses $S_x = \frac{1}{L_x}$ and $S_y = \frac{1}{L_y}$ of the multipliers L_x and L_y correspondingly.

Lemma 1.6 *If* $u \in C^2([0, a] \times [0, b])$, *then*

$$u_{xx} = S_x u - S_x\{u(0, y)\} - 2[u(1, y) - u(\frac{1}{2}, y)]_x,$$

$$u_{yy} = S_y u - S_y\{(y - 1)u(x, 0)\} - [u(x, 1)]_y.$$

Proof. By multiplication of (1.14)and (1.15) by S_x and S_y , correspondingly.

Let us consider problem (1.1). Using boundary value conditions, the equation $u_{xx} + u_{yy} = 0$ together with the boundary conditions can be reduced to a single algebraic equation in $\mathcal{M}$. Indeed, then

$$u_{xx} = S_x u - [g(y)]_x, \qquad u_{yy} = S_y u - [f(x)]_y$$

and the BVP (1.8) takes the algebraic form:

$$(S_x + S_y)u = [f(x)]_y + [g(y)]_x + \{F(x, y)\}$$

If $S_x + S_y$ is a non-divisor of zero, then the last equation has a solution in $\mathcal{M}$:

$$u = \frac{1}{(S_x + S_y)}[f(x)]_y + \frac{1}{(S_x + S_y)}[g(y)]_x + \frac{1}{(S_x + S_y)}\{F(x, y)\}.$$

In order to show that the element $S_x + S_y$ is a non-divisor of zero in $\mathcal{M}$, we consider the following eigenvalue problem:

$$v''(y) + \mu^2 v(y) = 0, \quad y \in (0, 1), \quad v(0) = 0, \quad v(1) = 0 \tag{1.16}$$

The eigenvalues of (1.16) are $\mu_m = m\pi$, $m \in NN$, with corresponding eigenfunctions $sin m\pi x$.

Lemma 1.7 *The elements $S_x + S_y$ is a non-divisor of zero in $\mathcal{M}$.*

Proof. Assume the contrary, i.e. that there exists a non-zero multipliers fraction $\frac{A}{B} \neq 0$ with $(S_x + S_y)\frac{A}{B} = 0$. The last relation is equivalent to $(S_x + S_y)A = 0$. Since $A \neq 0$, then there exist a function $v \in C$ such that $Av = u \neq 0$. Then $(S_x+S_y)A = 0$ implies $(S_x+S_y)u = 0$ which is equivalent to

$$(L_x + L_y)u = 0 \tag{1.17}$$

We will show that the only solution of this equation is the trivial one, i.e. $u \equiv 0$, which would be a contradiction. To this end we multiply (1.17) by the eigenfunction $\varphi_n(y) = sinm\pi y$ of the eigenvalue problem (1.16) using the convolution product $f \overset{y}{*} g$, defined by (1.11). It easy to see that

$$u(x,y) \overset{y}{*} \sin m\pi y = \left\{ \gamma_m \int_0^1 u(x,\eta) \sin m\pi\eta d\eta \right\} \sin m\pi y$$

with a constant $\gamma_m \neq 0$, the exact value of which is unessential for us. The function

$$A_m(x) = \left\{ \gamma_m \int_0^1 u(x,\eta) \sin m\pi\eta d\eta \right\}$$

up to a non-zero constant is the m-th finite Fourier sine-transform of the function $u(x,y)$ with respect to y. From $(L_x + L_y)[u \overset{y}{*} \varphi_m(y) = 0$ we obtain

$$[L_x A_m(x)] \sin m\pi y + A_m(x) L_y \sin m\pi y = 0.$$

But $L_y \sin m\pi y = -\frac{1}{(m\pi)^2} \sin m\pi y$ and thus we obtain the following simple integral equation for $A_m(x)$:

$$L_x A_m(x) = -\frac{1}{(m\pi)^2} A_m(x)$$

It is equivalent to the BVP

$$A''_m(x) = (m\pi)^2 A_m(x), \quad A_m(0) = 0, \quad A_m(1) = 0. \tag{1.18}$$

The only solution of (1.18) is the trivial one: $A_m(x) \equiv 0$. Thus we proved that $\int_0^1 u(x,\eta) \sin m\pi\eta d\eta = 0$ for arbitrary $x \in [0,1]$ and $\bigvee n \in NN$. From a basic property of the Fourier sine-transform it follow $u(x,y) \equiv 0$ for arbitrary $x \in [0,1]$ and $y \in [0,1]$. This is a contradiction with the assumption $u(x,y) \neq$

0 and it proves the Lemma. Along with this, it is proven the uniqueness of the extended Bitsadze - Samarskii problem.◇

Let us consider Bitsadze - Samarskii problem (1.1) for $f(x) = \frac{x^3}{6} - \frac{7x}{24} = L_x = \frac{1}{S_x^2}$. In [4] a representation of the solution $U(x,y)$ of this problem by the series (1.3) is found. The same solution has the algebraic representation

$$U = \frac{1}{(S_x + S_y)}\left[\frac{x^3}{6} - \frac{7x}{24}\right]_y = \frac{1}{(S_x + S_y)} L_x\{x\} = \frac{1}{(S_x + S_y)} L_x^2 = \frac{1}{(S_x + S_y)S_x^2}$$

Then the solution of Bitsadze - Samarskii problem (1.1) for arbitrary f can be represented in the form:

$$u = \frac{1}{(S_x + S_y)}[f(x)]_y = S_x^2 \frac{1}{(S_x + S_y)S_x^2}[f(x)]_y = \frac{\partial^4}{\partial x^4}(U \overset{x}{*} fx). \qquad (1.19)$$

In [4] one of the authors had shown that for $f(x) \in C^4[0,1]$ which satisfies the conditions $f(0) = f(1) - f(\frac{1}{2}) = f''(1) - f''(\frac{1}{2}) = 0$ (1.19) is a representation of the classical solution of (1.1). Indeed, since $U(x,y)$ is a (generalised) solution of problem (1.1), we have $U(1,y) - U(\frac{1}{2},y) = 0$. Assuming that $f(x) \in C^2[0,1]$ with $f(0) = f(1) - f(\frac{1}{2}) = 0$ and using , we obtain

$$\begin{aligned} u(x,y) = {} & \frac{\partial^4}{\partial x^4}(U(x,y) \overset{x}{*} f(x)) = \qquad (1.20)\\ = {} & -\Big(\int_0^x (U_x(\xi + 1 - x, y) - U_x(x + 1 - \xi, y) - \\ & -U_x(\xi + \frac{1}{2} - x, y) + U_x(x + \frac{1}{2} - \xi, y))f''(\xi)d\xi + \\ & + \int_0^1 (U_x(x + 1 - \xi, y) - U_x(1 - x - \xi, y))f''(\xi)d\xi - \\ & - \int_0^{\frac{1}{2}} (U_x(x + \frac{1}{2} - \xi, y) - U_x(\frac{1}{2} - x - \xi, y))f''(\xi)d\xi \end{aligned}$$

with $U_x(x,y)$ given by (1.5). It is easy to see that this representation of the solution of (1.1) is equivalent to (1.4).

Theorem 1.4 *If $f(x) \in C^2[0,1]$, $f(0) = 0$, and $f(1) - f(\frac{1}{2}) = 0$, then (1.19) is a generalised solution of boundary value problem (1.1). If $f(x) \in C^4[0,1]$ and $f(0) = f''(0) = 0$, $f(1) - f(\frac{1}{2}) = f''(1) - f''(\frac{1}{2}) = 0$, then*

$$u(x,y)=\frac{\partial^4}{\partial x^4}(U(x,y)\overset{x}{*}fx)= \tag{1.21}$$

$$=-\int_x^{\frac{1}{2}}(U_x(\frac{1}{2}+x-\xi,y)f''(\xi)d\xi-\int_{-x}^{\frac{1}{2}}(U_x(\frac{1}{2}-x-\xi,y)f''(|\xi|)sgn\xi d\xi-$$

$$-\int_x^1(U_x(1+x-\xi,y)f''(\xi)d\xi-\int_{-x}^1(U_x(1-x-\xi,y)f''(|\xi|)sgn\xi d\xi$$

where

$$U_x(x,y)=\sum_{n=1}^{\infty}\frac{\sinh 4n\pi y\cos 4n\pi x}{8\pi^2n^2\sinh 4n\pi}+ \tag{1.22}$$

$$\sum_{n=1}^{\infty}\frac{3\sinh\frac{2}{3}(2n-1)\pi y\cos\frac{2}{3}(2n-1)\pi x}{2\pi^2(2n-1)^2\cos\frac{2}{3}(1+n)n\pi\sinh\frac{2}{3}(2n-1)\pi}$$

is a classical solution of (1.1).

The proof the first part is a matter of a direct check. The second is proven in [4]. $\diamond$

2 Generalization of nonlocal Bitsadze - Samarskii problem.

2.1 Introductions.

Let Φ be a linear functional on $C^1[0,a]$ and Ψ be a linear functional on $C^1[0,b]$. Then they have Stieltjes type representations:

$$\Phi\{f\}=Af(a)+\int_0^a f'(t)d\alpha(t),\quad f\in C^1[0,a] \tag{2.1}$$

and

$$\Psi\{f\}=Bg(b)+\int_0^b g'(t)d\beta(t),\quad g\in C^1[0,b] \tag{2.2}$$

where α and β are function with bounded variation, A and B being constant. We consider the potential equation

$$u_{xx}+u_{yy}=F(x,y) \tag{2.3}$$

on the rectangle $G = \{(x,y) : 0 < x < a, 0 < y < b\}$ with local BV conditions

$$u(x,0) = \varphi(x) \quad and \quad u(0,y) = \psi(x) \tag{2.4}$$

and nonlocal BV conditions

$$\Phi_\xi\{u(\xi,y)\} = g(y), \quad \Psi_\eta\{u(x,\eta)\} = f(x) \tag{2.5}$$

with some mild smoothness requirements for the given functions F, φ, ψ, f and g. The only restrictions on the functionals Φ and Ψ are the requirements $\Phi \neq 0$ and $\Psi \neq 0$. They are connected with the approach chosen and may be ousted by means of some technical involvements. For the sake of some normalization of the functionals Φ and Ψ, we assume

$$\Phi_\xi\{\xi\} = 1, \quad \Psi_\eta\{\eta\} = 1 \tag{2.6}$$

We consider the space $C(G)$ and $C^1(G)$ of the continuous and smooth functions on $G = [0,a] \times [0,b]$, respectively.

Further, we introduce the right inverse operators L_x and L_y of and on $C([0,a] \times [0,b])$ as the solutions $v(x,y) = L_x u(x,y)$ and $w(x,y) = L_y u(x,y)$ of the elementary BVPs

$$\frac{\partial^2 v}{\partial x^2} = u(x,y), \quad v(0,y) = 0, \quad \Phi_\xi\{v(\xi,y)\} = 0 \tag{2.7}$$

and

$$\frac{\partial^2 w}{\partial y^2} = u(x,y), \quad w(x,0) = 0, \quad \Psi_\eta\{w(x,\eta)\} = 0 \tag{2.8}$$

The operators L_x and L_y have the explicit representations:

$$L_x\{u(x,y\} = \int_0^x (x-\xi)u(\xi,y)d\xi - x\Phi_\xi\{\int_0^\xi (\xi-\eta)u(\eta,y)d\eta\}, \tag{2.9}$$

$$L_y\{u(x,y\} = \int_0^y (y-\eta)u(x,\eta)d\eta - y\Psi_\eta\{\int_0^\eta (\eta-\zeta)u(x,\zeta)d\zeta\}. \tag{2.10}$$

2.2 Convolutions.

One of the authors had found a convolution $(f_1 \overset{x}{*} f_2)(x)$ in $C[0,a]$ and a convolution $(g_1 \overset{y}{*} g_2)(y)$ in $C[0,b]$ such that the operators L_x and L_y are the convolution operator $\{x\} \overset{x}{*}$ and $\{y\} \overset{y}{*}$, correspondingly.

Theorem 2.1 *[6] The operations*

$$(f_1 \overset{x}{*} f_2)(x) = -\frac{1}{2}\Phi_\xi \left\{ \int_0^\xi h(x,\eta)d\eta \right\} \tag{2.11}$$

$$(g_1 \overset{gy}{*} g_2)(y) = -\frac{1}{2}\Psi_\eta \left\{ \int_0^\eta k(y,\zeta)d\zeta \right\} \tag{2.12}$$

where

$$h(x,\eta) = \int_x^\eta f_1(\eta+x-\zeta)f_2(\zeta)d\zeta - \int_{-x}^\eta f_1(|\eta-x-\zeta)f_2(|\zeta|)sgn(\zeta(\eta-x-\zeta))d\zeta \tag{2.13}$$

$$k(y,\eta) = \int_y^\eta g_1(\eta+y-\zeta)g_2(\zeta)d\zeta - \int_{-y}^\eta g_1(|\eta-y-\zeta)g_2(|\zeta|)sgn(\zeta(\eta-y-\zeta))d\zeta \tag{2.14}$$

are bilinear, commutative and associative operations on $C([0,a])$ and $C([0,b])$, respectively, such that it hold the representations

$$L_x f(x) = \{x\} \overset{x}{*} f(x) \tag{2.15}$$

and

$$L_y g(y) = \{y\} \overset{y}{*} g(y). \tag{2.16}$$

For a proof see [6].

By means of (2.11) and (2.12) a two-dimensional convolution in $C([0,a] \times [0,b])$ can be defined.

Theorem 2.2 *[8] The operation*

$$(u * v)(x,y) = \frac{1}{4}\tilde{\Phi}_\xi \tilde{\Psi}_\eta \{h(x,y,\xi,\eta)\}, \tag{2.17}$$

where

$$\tilde{\Phi}_\xi\{f(\xi)\} = \Phi_\xi\left\{\int_0^\xi f(\sigma)d\sigma\right\}, \qquad \tilde{\Psi}_\eta\{g(\eta)\} = \Psi_\eta\left\{\int_0^\eta g(\tau)d\tau\right\}$$

with

$$h(x,y,\xi,\eta) = \int_x^\xi\int_y^\eta f(\xi+x-\sigma,\eta+y-\tau)g(\sigma,\tau)d\sigma\tau -$$

$$-\int_{-x}^\xi\int_y^\eta f(|\xi-x-\sigma|,\eta+y-\tau)g(|\sigma|,\tau)sgn(\xi-x-\sigma)\sigma d\sigma\tau -$$

$$-\int_x^\xi\int_{-y}^\eta f(\xi+x-\sigma,|\eta-y-\tau|)g(\sigma,|\tau|)sgn(\eta-y-\tau)\tau d\sigma\tau -$$

$$-\int_{-x}^\xi\int_{-y}^\eta f(|\xi-x-\sigma|,|\eta-y-\tau|)g(\sigma,|\tau|)sgn(|\xi-x-\sigma|)(\eta-y-\tau)\sigma\tau d\sigma\tau$$

is a bilinear, commutative and associative operation in $C(G)$ such that

$$L_x\{u(x,y)\} = \{x\} \overset{x}{*} \{u(x,y)\}, \quad L_y\{u(x,y)\} = \{x\} \overset{y}{*} \{u(x,y)\} \tag{2.18}$$

$$L_xL_y\{u(x,y)\} = \{xy\} * \{u(x,y)\}. \tag{2.19}$$

The linear space $C = C(G)$ equipped with the multiplication (2.17) is a commutative Banach algebra $(C,*)$.

Further, we introduce the algebra MM of the multipliers of $(C,*)$. Let us remind the definition of a multiplier of $(C,*)$.

Definition 2.1 (See [10]) A mapping $M : C \to C$ is said to by a multiplier of the convolutional algebra $(C,*)$ iff the relation

$$M(u*v) = (Mu)*v \tag{2.20}$$

holds for all $u, v \in C$.

As it is shown in Larsen [5] each such mapping for our convolution (2.17) is automatically linear and continuous. That's why, further we consider each multiplier of $(C,*)$ as a continuous linear operator.

If $f \in C[0,a]$ and $g \in C[0,b]$, then the convolutional operators $f \overset{x}{*}$ and $g \overset{y}{*}$ defined in C by

$$(f \overset{x}{*})u = f \overset{x}{*} u, \quad (g \overset{y}{*})u = g \overset{y}{*} u$$

are multipliers of $(C,*)$ (See Dimovski and Spiridonova [8]). Of course, the operator $\{F(x,y)\}$ is also multiplier of $(C,*)$.

Further, we use the notations

$$[f]_y = \{f(x)\} \overset{x}{*}, \quad [g]_x = \{g(y)\} \overset{y}{*} \tag{2.21}$$

2.3 A two-dimensional operational calculus.

In $\mathcal{M}$ there are elements which a non-divisors of 0. Indeed, such elements are the multipliers $\{x\} \overset{x}{*}$ and $\{y\} \overset{y}{*}$, i.e. the operators L_x and L_y.

Denote by $\mathcal{N}$ the set of the non-zero non-divisors of zero on $\mathcal{M}$. The set $\mathcal{N}$ is a multiplicative subset on $\mathcal{M}$, i.e. such that $p, q \in \mathcal{N}$ implies $pq \in \mathcal{N}$.

Further, we consider multipliers fractions of the form $\frac{M}{N}$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$. They are introduced in a standard manner, using the well-known method of "localisation" from the general algebra [9].

Denote by $\mathscr{M}$ the set $\mathcal{N}^{-1}\mathcal{M}$ of multipliers fractions. We consider it as a commutative ring containing the basic field ($\mathbb{R}$ or $\mathbb{C}$), the algebras $(C[0,a], \overset{x}{*})$, $(C[0,b], \overset{y}{*})$, $(C,*)$ and $\mathcal{M}$, due to the embeddings

$$R \hookrightarrow \mathscr{M} \ \ or\ \ R \hookrightarrow \mathscr{M}: \quad \alpha \mapsto \frac{\alpha L_x}{L_x}$$

$$(C[0,a], \overset{x}{*}) \hookrightarrow \mathscr{M}: \quad f \mapsto \frac{(L_x f) \overset{x}{*}}{L_x}$$

$$(C[0,b], \overset{y}{*}) \hookrightarrow \mathscr{M}: \quad g \mapsto \frac{(L_y g) \overset{y}{*}}{L_y}$$

$$(C([0,a] \times [0,b]), *) \hookrightarrow \mathscr{M}: \quad u \mapsto \frac{(L_x L_y u)*}{L_x L_y}$$

Further, we consider all numbers, functions, multiplier and multipliers fractions as elements of *a single algebraic system*: the ring $\mathscr{M}$ of the multipliers fractions.

2.4 Explicit solution of nonlocal BVPs for the potential equation.

Further, we consider following boundary value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y), \quad 0 < x < a, \quad 0 < y < b, \\ &u(x,0) = u(0,y) = 0, \\ &\Phi_\xi\{u(\xi,y)\} = g(y), \quad \Psi_\eta\{u(x,\eta)\} = f(x). \end{aligned} \tag{2.22}$$

Definition 2.2 A function $u(x,y) \in C^1([0,a] \times [0,b])$ is said to be a generalised solution of (2.22) iff $u(x,y)$ satisfies the integral relation

$$L_x u + L_y u = L_x f(x).y + L_y g(y).x + L_x L_y F(x,y). \tag{2.23}$$

Formally, (2.23) could be obtained from the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$ applying to it the operator $L_x L_y$ and taking into account the boundary value conditions.

Lemma 2.1 *If $u(x,y) \in C1([0,a]\times[0,b])$ satisfy (2.23), then $u(x,y)$ satisfies the boundary value conditions:*

$$u(x,0) = u(0,y) = 0,$$

$$\Phi_\xi\{u(\xi,y\} = g(y), \quad \Psi_\eta,\{u(x,\eta)\} = f(x).$$

Proof. Let us consider (2.23). For $y = 0$ we find $L_x u(x,0) = 0$. Next we apply the operator $\frac{\partial^2}{\partial x^2}$ and find $u(x,0) = 0$. For $x = 0$ we find $L_y u(0,y) = 0$. Applying $\frac{\partial^2}{\partial y^2}$, we get $u(0,y) = 0$. If apply Ψ to (2.23), we obtain $L_x \Psi_\eta\{u(x,\eta)\} = L_x f(x)$. Then applying $\frac{\partial^2}{\partial x^2}$ we obtain $\Psi_\eta\{u(x,\eta)\} = f(x)$. At last, applying Φ to (2.23), we get $L_y \Phi_\xi\{u(\xi,y)\} = L_y g(y)$ and hence $\Phi_\xi\{u(\xi,y)\} = g(y)$.$\diamond$

Lemma 2.2 *If $u(x,y) \in C^2([0,a] \times [0,b])$ satisfy (2.23) then it is a classical solution of (2.22).*

Proof. Applying the operator $\frac{\partial^4}{\partial x^2 \partial y^2}$ to (2.23), we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$. The fulfilment of the boundary value conditions follows from Lemma 2.1.$\diamond$

Lemma 2.3 *If $u(x,y) \in C^2(G)$, then it holds:*

$$L_x\left\{\frac{\partial^2 u}{\partial x^2}\right\} = u(x,y) + (x\Phi_\xi\{1\} - 1)u(0,y) - x\Phi_\xi\{u(\xi,y)\} \tag{2.24}$$

and

$$L_y\left\{\frac{\partial^2 u}{\partial y^2}\right\} = u(x,y) + (y\Psi_\eta\{1\} - 1)u(x,0) - y\Phi_\eta\{u(x,\eta)\} \tag{2.25}$$

For a proof, see [5].

Most important for our considerations are the algebraic inverses $S_x = \frac{1}{L_x}$ and $S_y = \frac{1}{L_y}$ of the multipliers L_x and L_y, correspondingly.

Lemma 2.4 *If $u \in C^2([0,a] \times ([0,b])$, then*

$$u_{xx} = S_x u(x,y) + S_x\{(x\Phi_\xi\{1\} - 1)u(0,y)\} - [\Phi_\xi\{u(\xi,y)\}]_x, \tag{2.26}$$

and

$$u_{yy} = S_y u(x,y) + S_y\{(y\Psi_\eta\{1\} - 1)u(x,0)\} - [\Phi_\eta\{u(x,\eta)\}]_y. \tag{2.27}$$

Proof. By multiplication of (2.24) and (2.25) by S_x and S_y, correspondingly.◇

Using the boundary value conditions of (2.22), the equation $u_{xx} + u_{yy} = F(x,y)$ can be reduced to a single algebraic equation in $\mathcal{M}$. Indeed, by (2.26) and (2.27) we find

$$u_{xx} = S_x u - [g(y)]_x. \tag{2.28}$$

$$u_{yy} = S_y u - [f(x)]_y. \tag{2.29}$$

and the equation $u_{xx} + u_{yy} = F(x,y)$ takes the algebraic form:

$$(S_x + S_y)u = F(x,y) + [g(y)]_x + [f(x)]_y.$$

If $S_x + S_y$ is non-divisor of zero, then the last equation has the following formal solution in $\mathcal{M}$:

$$u = \frac{1}{(S_x + S_y)}\{F(x,y)\} + \frac{1}{(S_x + S_y)}[f(x)]_y + \frac{1}{(S_x + S_y)}[g(y)]_x.$$

The requirement $S_x + S_y$ to be a non-divisor of 0 in $\mathcal{M}$ is equivalent to a theorem for uniqueness of the solution of (2.22). Therefore, our next task is to study the uniqueness for problem (2.22). In the direct algebraic approach we a following, this problem reduces to the purely algebraic requirement the elements $S_x + S_y$ of $\mathcal{M}$ to be a non-divisor of zero in $\mathcal{M}$.

To this end we consider the following two eigenvalue problems:

$$u''(x) + \lambda^2 u(x) = 0, \quad x \in (0,a), \quad u(0) = 0, \quad \Phi_\xi\{u(\xi)\} = 0 \; in \; C[0,a], \tag{2.30}$$

$$v''(v) + \mu^2 v(y) = 0, \quad y \in (0,b), \quad v(0) = 0, \quad \Psi_\eta\{u(\eta)\} = 0 \; in \; C[0,b]. \tag{2.31}$$

Let λ_n and μ_m be the eigenvalues of (2.30) and (2.31) for $n, m \in N$, correspondingly.

Lemma 2.5 *If there exists a dispersion relation of the form $\lambda_n^2 + \mu_m^2 = 0$ for some $n, m \in N$, then $S + S$ is a divisor of zero in $\mathcal{M}$.*

Proof. Let for some $n, m \in N$ we have $\lambda_n^2 + \mu_m^2 = 0$. Then

$$(S_x + S_y) \sin \lambda_n x \sin \mu_m y = -(\lambda_n^2 + \mu_m^2) \sin \lambda_n x \sin \mu_m y = 0. \quad \diamond$$

Theorem 2.3 *Let $a \in supp\Phi$. If $\lambda_n^2 + \mu_m^2 \neq 0$ for all $n, m \in N$, then $S_x + S_y$ is a non-divisor of zero in $\mathcal{M}$.*

Proof. Assume the contrary. It is easy to see, that $S_x + S_y$ is a divisor of zero in $\mathcal{M}$ iff there is a function $u \in C^2(G)$, $u \neq 0$, such that $(S_x + S_y)u = 0$. This relation is equivalent to

$$(L_x + L_y)u = 0. \tag{2.32}$$

Let λ_n is an arbitrary eigenvalue of (2.30). Then λ_n is a zero of the sine-indicatrix $E(\lambda) = \Phi_\xi\left\{\frac{\sin \lambda\xi}{\lambda}\right\}$ of the functional Φ. Let $\varkappa_n$ be the multiplicity of λ_n as a zero of $E(\lambda)$. To λ_n it corresponds the finite sequence of an eigenfunction $\sin \lambda_n x$ and $\varkappa_n - 1$ associated eigenfunctions

$$\varphi_{n,s}(x) = \left(L_x + \frac{1}{\lambda_n^2}\right)^s, \quad 0 \leq s \leq \varkappa_n - 1,$$

where

$$\varphi_{n,0}(x) = \frac{1}{\pi i} \int_{\Gamma_n} \frac{\sin \lambda x}{\lambda E(\lambda)} d\lambda$$

(see Dimovski and Petrova [7], p.94)

Note that $\varphi_{n,\varkappa_n-1}(x) = \alpha_n \sin \lambda_n x$ with some $\alpha_n \neq 0$. The corresponding $\varkappa_n$- dimensional eigenspace is

$$E_{\lambda_n}^{(\varkappa_n)} = span\{\varphi_{n,s}(x), s = 0, 1, ..., \varkappa_n - 1\}.$$

The spectral projector $P_{\lambda_n} : C \to E_{\lambda_n}^{(\varkappa_n)}$ is given by $P_{\lambda_n}\{f\} = f * \varphi_n$. According to a theorem of N. Bozhinov [3] in the case $a \in supp\Phi$, the projectors P_{λ_n} form a total system, i.e. a system for which $P_{\lambda_n}\{f\} = 0$, $\bigvee n \in N$ implies $f \equiv 0$. For a simple proof of Bozhinov's theorem for our case, see [7] p. 97-98.

Denote $u_n(x, y) = u(x, y) \overset{x}{*} \varphi_n(x)$. From $(L_x + L_y)u = 0$ it follows

$$(L_x + L_y)u_n = 0. \tag{2.33}$$

We will show that (2.33) has only the trivial solution $u_n = 0$ in $E_{\lambda_n}^{(\varkappa_n)}$. Assume that there exists a nonzero solution u_n of (2.33), i.e. of the form

$$u_n(x, y) = A_{n,k}(y)\varphi_{n,k}(x) + A_{n,k+1}(y)\varphi_{n,k+1}(x) + ... + A_{n,\varkappa_n-1}(y)\varphi_{n,\varkappa_n-1}(x) \tag{2.34}$$

with $A_{n,k}(y) \neq 0$ for some k , $0 \leq k \leq \varkappa_n - 1$. We apply the operator $\left(L_x + \frac{1}{\lambda_n^2}\right)^{\varkappa_n-k-1}$ to (2.33) and obtain

$$(L_x + L_y)A_{n,\varkappa_n-1}(y)\varphi_{n,\varkappa_n-1}(x) = 0,$$

since $\left(L_x + \frac{1}{\lambda_n^2}\right)^s \varphi_{n,0}(x) = 0$, for $s \geq \varkappa_n$.

But $\varphi_{n,\varkappa_n-1}(x) = \alpha_n$ with $\alpha_n \neq 0$. Denote $A_{n,\varkappa_n-1}(y) = A_n(y)$. Consider $(L_x + L_y)A_n(y) \sin \lambda_n x = 0$ as an equation for $A_n(y)$. It is equivalent to the BVP

$$\frac{\partial^2}{\partial x^2}(A_n(y) \sin \lambda_n x) + \frac{\partial^2}{\partial y^2}(A_n(y) \sin \lambda_n x) = 0$$

$$A_n(0) = 0 \;\; and \;\; \Psi_\eta\{A_n(\eta)\} = 0,$$

which reduces to

$$A_y''(y) - \lambda_n^2 A_n(y) = 0, \;\; A_n(0) = 0 \;\; and \;\; \Psi_\eta\{A_n(\eta)\} = 0.$$

From this equation it follows that $-\lambda_n^2$ is an eigenvalue $-\mu_n^2$ of problem (2.31). Hence $\lambda_n^2 + \mu_n^2 = 0$ which is a contradiction. Hence $u_n(x, y) \equiv 0$ for all $n \in N$. By N. Bozhinov's theorem it follows that $u_n(x, y) \equiv 0$. Thus we proved, that $S_x + S_y$ is a non-divisor of 0 in $\mathcal{M}$.

2.4.1

Let us consider BVP (2.22) for $f(x) = L_x\{x\} = \frac{1}{S_x^2}$ and $g(y) = F(x,y) \equiv 0$. We assume that there exists a generalized solution of this problem and denote it by $U(x,y)$. It has the following algebraic representation:

$$U = \frac{1}{(S_x + S_y)} L_x\{x\} = \frac{1}{(S_x + S_y)} L_x^2 = \frac{1}{(S_x + S_y)S_x^2}$$

Then there exists also the solution of problem (2.22) for arbitrary $f(x)$, $g(y)$ and $F(x,y)$ and it can by represented in the form:

$$u = \frac{1}{(S_x + S_y)}\{F(x,y)\} + \frac{1}{(S_x + S_y)}[f(x)]_y + \frac{1}{(S_x + S_y)}[g(y)]_x =$$

$$= S_x^2 \left[\frac{1}{(S_x + S_y)S_x^2} F(x,y) + \frac{1}{(S_x + S_y)S_x^2}[f(x)]_y + \frac{1}{(S_x + S_y)S_x^2}[g(y)]_x\right]$$

$$u = \frac{\partial^4}{\partial x^4}\left[U * F(x,y) + U \overset{x}{*} f(x) + U \overset{y}{*} g(y)\right]$$

provided the denoted derivative exists.

2.4.2

Let us consider BVP (2.22) for $F(x,y) = xy = L_x L_y = \frac{1}{S_x S_y}$ and $g(y) = f(x) \equiv 0$. We denote the solution of this problem by $W(x,y)$. Then we have an algebraic representation of this solution:

$$W = \frac{1}{(S_x + S_y)} L_x L_y = \frac{1}{(S_x + S_y) S_x S_y}$$

The solution of problem (2.22) for arbitrary $f(x), g(y)$ and $F(x,y)$ can by represented in the form:

$$u = S_x S_y \left[\frac{1}{S_x S_y (S_x + S_y)}[f(x)]_y + \frac{1}{S_x S_y (S_x + S_y)}[g(y)]_x + \frac{1}{S_x S_y (S_x + S_y)}\{F(x,y)\}\right]$$

$$u = \frac{\partial^4}{\partial x^2 \partial y^2}\left[W \overset{x}{*} f(x) + W \overset{y}{*} g(y) + W * F(x,y)\right]$$

but we will illustrate these conditions on the example of Bitsadze - Samarskii's problem.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad 0<x<1, \quad 0<y<1,\\
&u(x,0)=u(0,y)=0, \quad u(x,1)=f(x),\\
&u(1,y)-u(\frac{1}{2},y)=0.
\end{aligned} \tag{2.35}$$

This is the special case of boundary value problem (2.2) when $\Phi_\xi\{u(\xi,y)\} = 2(u(1,y)-u(\frac{1}{2},y))$ and $\Psi_\eta\{u(x,\eta\} = u(x,1)$. Following the approach outlined above, we can find ([9], p 175) that the solution $U(x,y)$ of (2.35) for $f(x) = L_x\{x\} = \frac{x^3}{6} - \frac{7x}{24}$ is

$$\begin{aligned}
U(x,y) \;&=\; \sum_{n=1}^{\infty}\frac{\sinh 4n\pi y \sin 4n\pi x}{32\pi^3 n^3 \sinh 4n\pi}+\\
&\sum_{n=1}^{\infty}\frac{9\sinh\frac{2}{3}(2n-1)\pi y\sin\frac{2}{3}(2n-1)\pi x}{4\pi^3(2n-1)^3\cos\frac{2}{3}(1+n)n\pi\sinh\frac{2}{3}(2n-1)\pi}
\end{aligned}$$

Then, for $f \in C^2[0,1]$, with $f(0) = f(1) - f(\frac{1}{2}) = 0$, we obtain

$$\begin{aligned}
&u(x,y)=\int_x^2 U_x(\frac{1}{2}+x-\xi,y)f''(\xi)d\xi-\\
&-\int_{-x}^2 U_x(\frac{1}{2}-x-\xi,y)f''(|\xi|)sgn\xi d\xi-\\
&-\int_x^1 U_x(1+x-\xi,y)f''(\xi)d\xi+\int_{-x}^1 U_x(1-x-\xi,y)f''(|\xi|)sgn\xi d\xi
\end{aligned}$$

as a generalized solution of (2.35). It can be shown that it is a classical solution too, if $f \in C^4[0,1]$ and additionally, $f''(0) = f''(1) - f''(\frac{1}{2}) = 0$ (Cf. Theorem 1.4).

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