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Operational Calculus Approach to Nonlocal Cauchy Problems

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Operational Calculus Approach to Nonlocal Cauchy Problems

Ivan Dimovski and Margarita Spiridonova

Abstract. Let Φ be a linear functional on the space $C = C(\Delta)$ of continuous functions on an interval Δ. The nonlocal boundary problem for an arbitrary linear differential equation

$$P\left(\frac{d}{dt}\right) y = F(t)$$

with constant coefficients and with boundary value conditions of the form

$$\Phi\{y^{(k)}\} = \alpha_k, \;\; k = 0, 1, 2, \ldots, \deg P - 1$$

is said to be a nonlocal Cauchy boundary value problem.

For solution of such problems an operational calculus of Mikusiński's type, based on the non-classical convolution

$$(f * g)(t) = \Phi_\tau \{ \int_\tau^t f(t + \tau - \sigma)\, g(\sigma)\, d\sigma \},$$

is developed.

In the frames of this operational calculus the classical Heaviside algorithm extends nonlocal Cauchy problems. To such problem reduces the quest for periodic, antiperiodic and mean-periodic solutions of Linear Ordinary Differential Equations (LODE) with constant coefficients. Extensions of the Duhanel principle are proposed.

Remarks on the use of a computer algebra system on some steps of the algorithms are included.

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1. Introduction

Let it be given a linear functional Φ on the space $C = C(\Delta)$ of the continuous functions in an interval Δ of $\mathbb{R}$. The interval Δ may be chosen arbitrary: it may be finite, or infinite, closed, or open, but for some normalization, we assume that $0 \in \Delta$. For the functional Φ we assume that it is non-zero and continuous on $C(\Delta)$ with Stieltjes integral representation of the form

$$\Phi\{f\} = \int_\alpha^\beta f(t)\, d\gamma(t), \tag{1.1}$$

with $\alpha, \beta \in \Delta$ and $\gamma(t)$ being a function with bounded variation on $[\alpha, \beta]$.

According to the Riesz-Markov theorem (see Edwards [4], Ch. 4) representation (1) holds for arbitrary continuous linear functionals on $C(\Delta)$.

For our considerations it is important to introduce the notion of **exponential indicatrix** of the functional Φ.

Definition 1.1. The entire function of exponential type

$$E(\lambda) = \Phi_\tau\{e^{\lambda\tau}\} \tag{1.2}$$

is said to be the exponential indicatrix of Φ.

We consider the case when the support (carrier) of Φ contains at least two different points. We characterize this case as **nonlocal** one.

In the **nonlocal** case the indicatrix $E(\lambda)$ of the functional Φ has an infinite sequence of zeros

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n,\ \ldots$$

in $\mathbb{C}$ with corresponding multiplicities

$$\kappa_1,\ \kappa_2,\ \ldots,\ \kappa_n,\ \ldots,$$

i.e. $E(\lambda_n) = E'(\lambda_n) = \ldots = E^{(\kappa_n-1)}(\lambda_n) = 0$, but $E^{(\kappa_n)}(\lambda_n) \neq 0$ for $n \in \mathbb{N}$.

Further we consider in details the case when $\Phi\{1\} \neq 0$, i.e. the case, when $\lambda = 0$ is not a zero of the indicatrix $E(\lambda)$. The case when $E(0) = \Phi\{1\} = 0$ is also important, since such is the functional $\Phi\{f\} = f(T) - f(0)$, basic for determining of periodic solutions of LODEs with constant coefficients. On this example we will show how to reduce the case $E(0) = 0$ to the general one.

We normalize the functional Φ by the requirement

$$\Phi\{1\} = 1, \tag{1.3}$$

which is equivalent to $E(0) = 1$.

Definition 1.2. Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $a_0 \neq 0$ be a given n-th degree polynomial. The problem for determining of a solution of a LODE with constant coefficients

$$P\left(\frac{d}{dt}\right) y = F(t) \tag{1.4}$$

satisfying the boundary value conditions (BVCs)

$$\Phi\{y^{(k)}\} = \alpha_k,\ k = 0, 1, \ldots, n-1 \tag{1.5}$$

with given α_k is said to be a nonlocal Cauchy BVP, determined by the functional Φ.

To such problems it reduces the determining of the periodic and antiperiodic solutions of LODEs with constant coefficients. In order to embrace a larger class of such problems, we introduce the notion of a mean-periodic solution, determined by a functional Φ.

Definition 1.3. A function $f \in \mathcal{C}(-\infty, \infty)$ is said to be a mean-periodic function with respect to the linear functional Φ if it satisfies the identity

$$\Phi_\tau\{f(t+\tau)\} = 0. \tag{1.6}$$

The class of the mean-periodic functions is introduced by J. Delsarte [3] and studied by L. Schwartz [7] et al.

It is easy to see that a necessary condition a LODE with constant coefficients (4) to have a Φ-mean-periodic solution is the right-hand side function $F(t)$ to be Φ-mean-periodic.

The next theorem shows that the problem for determining of the mean-periodic solutions of LODEs with constant coefficients is equivalent to a non-local Cauchy problem with homogenous BVCs.

Theorem 1.4. *Let $F(t) \in \mathcal{C}(-\infty, \infty)$ be a mean-periodic function with respect to a functional Φ. A solution $y(t)$ of $P\left(\frac{d}{dt}\right) y = F(t)$ is a Φ-mean-periodic function iff it is a solution of the nonlocal Cauchy problem* (1.4)–(1.5) *with homogenous BVCs i.e. with $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$.*

Proof. a) Let y be a mean-periodic solution of $P\left(\frac{d}{dt}\right) y = F(t)$, i.e.

$$\Phi_\tau\{y(t+\tau)\} = 0,\ -\infty < t < \infty$$

Then $\Phi_\tau\{y^{(k)}(t+\tau)\} = 0,\ k = 0, 1, \ldots, n-1$.

For $t = 0$ we obtain $\Phi_\tau\{y^{(k)}(\tau)\} = 0$, i.e. y is a solution of (1.4)–(1.5) with $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$.

b) Conversely, let $y(t)$ be a solution of (4)–(5) with $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$. Then $P\left(\frac{d}{dt}\right) y(t+\tau) = F(t+\tau)$ and hence

$$P\left(\frac{d}{dt}\right) \Phi_\tau\{y(t+\tau)\} = \Phi_\tau\{F(t+\tau)\} = 0,$$

i.e. $w(t) = \Phi_\tau\{y(t+\tau)\}$ is a solution of the homogenous LODE $P\left(\frac{d}{dt}\right) w = 0$ with

$$w^{(k)}(0) = \Phi_\tau\{y^{(k)}(\tau|\} = 0$$

Hence $w(t) \equiv 0$ or $\Phi_\tau\{y(t+\tau)\} = 0$. □

Example 1.1. Let $\Phi\{f\} = f(T) - f(0)$. The class of mean-periodic functions with respect to Φ is simply the class of T-periodic functions, i.e. the functions $f(t)$ with

$$f(T+t) = f(t).$$

Note, that Φ violates our restriction $\Phi\{1\} \neq 0$. Later, we will show that this restriction may be avoided by a modification of the corresponding nonlocal Cauchy problem.

Example 1.2. Let $\Phi\{f\} = f(T) + f(0)$, $T > 0$. The class of mean-periodic functions with respect to Φ coincides with the antiperiodic functions $f(t)$ with the antiperiod T, i.e. such that

$$f(T+t) = -f(t), -\infty < t < \infty.$$

Here $\Phi\{1\} = 1 + T \neq 0$. In order to normalize the functional, we are to take

$$\Phi\{f\} = \frac{f(0) + f(T)}{1+T}$$

instead of $f(0) + f(T)$.

2. The basic nonlocal Cauchy problem

The simplest nonlocal Cauchy problem is the case $n = 1$. We write it in the form

$$\frac{dy}{dt} - \lambda y = f(t),\ \Phi\{y\} = 0 \tag{2.1}$$

Let us remind that for the sake of normalization, we assumed $\Phi\{1\} = 1$. It is easy to see that (2.1) has an explicit solution $y(t) = R_\lambda f(t)$ given by

$$R_\lambda f(t) = \frac{1}{E(\lambda)} \int_0^t e^{\lambda(t-\tau)} f(\tau) d\tau - \frac{e^{\lambda t}}{E(\lambda)} \Phi_\tau \left\{ \int_0^\tau e^{\lambda(\tau-\sigma)} f(\sigma) d\sigma \right\} \tag{2.2}$$

In fact, this is the resolvent operator of the differentiation operator $\dfrac{d}{dt}$ with the nonlocal boundary condition $\Phi\{y\} = 0$.

Tacitely, we assumed that $0 \in \Delta$, which is not an essential restriction.

Basic for our next considerations is a non-classical convolution operation $f * g$ in $C(\Delta)$, such that the resolvent operator R_λ is the multiplier

$$R_\lambda = \left\{ \frac{e^{\lambda t}}{E(\lambda)} \right\} *$$

of the convolution algebra $(C(\Delta), *)$.

Theorem 2.1. (Dimovski [1], pp. 52–53) *The operation*

$$(f * g)(t) = \Phi_\tau \{ \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma \} \tag{2.3}$$

is a bilinear, commutative and associative operation in $C = C(\Delta)$*, such that*

$$R_\lambda f(t) = \left\{ \frac{e^{\lambda t}}{E(\lambda)} \right\} * f(t). \tag{2.4}$$

Proof. The commutativity of (2.3) is almost evident. As for the associativity, it needs some care. At first, we prove the special associativity relation

$$(e^{\lambda t} * e^{\mu t}) * e^{\nu t} = e^{\lambda t} * (e^{\mu t} * e^{\nu t}). \tag{2.5}$$

Indeed, it is easy to find

$$e^{pt} * e^{qt} = \Phi_\tau \left\{ \int_\tau^t e^{p(t+\tau-\sigma)} e^{q\sigma} d\sigma \right\} =$$

$$= e^{pt}\Phi_\tau \left\{ e^{p\tau} \int_\tau^t e^{(q-p)\sigma} d\sigma \right\} = e^{pt}\Phi_\tau \left\{ e^{p\tau} \frac{e^{(q-p)t} - e^{(q-p)\tau}}{q-p} \right\} =$$

$$= \frac{e^{qt}\Phi_\tau\{e^{p\tau}\} - e^{pt}\Phi_\tau\{e^{q\tau}\}}{q-p} = \frac{e^{qt}E(p) - e^{pt}E(q)}{q-p}$$

Then the verification of above associativity relation (2.5) is a matter of a simple check.

Let l, m and n be arbitrary non-negative integers. We differentiate (2.5) l times to λ, m times to μ and n times to ν, thus obtaining

$$(\{t^l e^{\lambda t}\} * \{t^m e^{\mu t}\}) * \{t^n e^{\nu t}\} = \{t^l e^{\lambda t}\} * (\{t^m e^{\mu t}\} * \{t^n e^{\nu t}\})$$

Letting $\lambda \to 0$, $\mu \to 0$ and $\nu \to 0$, we get

$$(t^l * t^m) * t^n = t^l * (t^m * t^n)$$

for $l, m, n \in \mathbb{N}_0$.

□

From the bilinearity of (2.3) it follows the validity of the associativity relation

$$(f * g) * h = f * (g * h)$$

for arbitrary polynomials f, g and h. It remains to use the Weierstrass approximation theorem and the continuity of (2.3) to prove it for arbitrary functions of $\mathcal{C}$.

The proof of identity (2.4) is a matter of a simple check, using (2.2).

For the applications of convolution (2.3) it is essential to know when the product $f * g$ is a differentiable function. Now we may be sure only that if $f, g \in \mathcal{C}$, then $f * g \in \mathcal{C}$. But, there is a general class of functionals Φ, for which $f, g \in \mathcal{C}$ implies $f * g \in \mathcal{C}^1$.

Theorem 2.2. *Let Φ be a linear functional of the form*

$$\Phi\{f\} = \Psi_\tau \left\{ \int_0^\tau f(\sigma)\, d\sigma, \right\}$$

where Ψ is a linear functional on C. Then $f * g \in C^1$ for arbitrary $f, g \in C$ and

$$(f * g)'(t) = \Psi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma \right\} + \Phi\{f\}\, g(t) + \Phi\{g\}\, f(t) - \\ -\Psi\{1\} \int_0^t f(t-\tau) g(\tau)\, d\tau. \tag{2.6}$$

Proof. First, we will prove (2.6) for $f \in C^1$, $g \in C$. Then

$$(f * g)'(t) = \Phi_\tau \left\{ \frac{\partial}{\partial t} \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma \right\}$$

But

$$\frac{\partial}{\partial t} \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma = \int_\tau^t f'(t+\tau-\sigma)\, g(\sigma)\, d\sigma + f(\tau)\, g(t)$$

and

$$\frac{\partial}{\partial \tau} \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma = \int_\tau^t f'(t+\tau-\sigma)\, g(\sigma)\, d\sigma - f(t)\, g(\tau)$$

Hence

$$\frac{\partial}{\partial t} \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma = \frac{\partial}{\partial \tau} \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma - \\ + f(\tau)\, g(t) + f(t)\, g(\tau)$$

and (2.6) follows easily.

Next, we apply (2.6) for $Lf \in C^1$ and $g \in C$ and obtain

$$(f * g)(t) = \frac{d}{dt}(Lf * g) = \Psi_\tau \left\{ \int_\tau^t (Lf)(t+\tau-\sigma)\, g(\sigma)\, d\sigma \right\} + \\ + \Phi\{g\}\, Lf(t) - \Psi\{1\} \int_0^t f(t-\tau) g(\tau)\, d\tau.$$

since $\Phi\{Lf\} = 0$.

Differentiating this expression, we get (2.6) in the general case $f, g \in C$.

□

Another important property of the convolution product $(f * g)(t)$ is the fact that it always satisfies the boundary condition $\Phi\{f * g\} = 0$.

Theorem 2.3. *Let $f, g \in C$. Then $\Phi\{f * g\} = 0$.*

Proof. From (2.3) it follows

$$\Phi_t\{(f*g)(t)\} = \Phi_t\Phi_\tau\{\int_\tau^t f(t+\tau-\sigma)\,g(\sigma)\,d\sigma\}$$

From the Fubini property $\Phi_t\Phi_\tau\{h(t,\tau)\} = \Phi_\tau\Phi_t\{h(t,\tau)\}$ of the functional Φ it follows

$$\Phi\{f*g\} = \Phi_t\Phi_\tau\{h(t,\tau)\} = \Phi_\tau\Phi_t\{h(t,\tau)\} = -\Phi_\tau\Phi_t\{h(\tau,t)\} = -\Phi\{f*g\},$$

since $h(t,\tau) = -h(\tau,t)$. Hence $\Phi\{f*g\} = 0$. □

Theorem 2.4. *The subspace C_Φ of the mean-periodic functions in $C(\mathbb{R})$ with respect to the functional Φ is an ideal in the convolution algebra $(C,*)$.*

Proof. We are to prove that $f \in C_\Phi$ and $g \in C$ imply $f*g \in C_\Phi$. First, we prove this for $g = \{1\}$, i.e. that if $f \in C_\Phi$, then $Lf \in C_\Phi$. Denote

$$\varphi(t) = \Phi_\tau\{(Lf)(t+\tau)\}$$

We have $\varphi'(t) = \Phi_\tau\{(Lf)'(t+\tau)\} = \Phi_\tau\{f(t+\tau)\} = 0$

Therefore $\varphi(t) = C = \text{const}$. But $\varphi(0) = \Phi_\tau\{(Lf)(\tau)\} = 0$ and hence $\varphi(t) \equiv 0$.

Thus we proved that $L^n f \in C_\Phi$ for $n = 1, 2, \ldots$. But $L^n f = (L^{n-1}\{1\}) * f = (A_{n-1} * f)(t)$, where A_{n-1} is a polynomial exactly of $(n-1)$–th degree.

Hence $f,\, g \in C_\Phi$ implies $f*g \in C_\Phi$ for arbitrary polynomial.

It remains to apply an approximation argument in order to assert that $f*g \in C_\Phi$ for arbitrary $g \in C$. □

3. Mikusiński type operational calculus, based on convolution (2.3)

In Dimovski [2] an operational calculus for the right inverse operator L of $\dfrac{d}{dt}$, defined by the solution $y = Lf(t)$ of the elementary BVP

$$y' = f(t),\ \Phi\{y\} = 0$$

is developed.

By our assumption $\Phi\{1\} = 1$, the solution of this BVP is

$$Lf(t) = \int_0^t f(\tau)\,d\tau - \Phi_\tau\left\{\int_0^\tau f(\tau)\,d\tau\right\}. \tag{3.1}$$

The operator L is the convolution operator

$$Lf(t) = \{1\} * f, \tag{3.2}$$

i.e. $L = \{1\} *$.

Denote by D the subset of C of all non-zero non-divisors of 0 of the convolution algebra $(C,*)$. The set D is not empty, since at least $\{1\} \in D$. Following the idea of Volterra and Pèrés [9] (see also Pèrés [10]), used later by Mikusiński [6], we consider the ring $\mathcal{M}$ of the convolution fractions $\dfrac{f}{g}$ with $f \in \mathcal{C}$ and $g \in D$.

The formal definition of $\mathcal{M}$ is the following

Definition 3.1. $\mathcal{M}$ is the quotient ring of $\mathcal{C} \times D$ with respect to the equivalence relation

$$(f, g) \sim (f_1, g_1) \Leftrightarrow f * g_1 = g * f_1,$$

i.e.

$$\mathcal{M} = \mathcal{C} \times D / \sim$$

For our next considerations, the most important element of $\mathcal{M}$ is the algebraic inverse of L, i.e. the convolution fraction

$$S = \frac{1}{\{1\}}.$$

The systems of real (or complex) numbers, considered as a ring with the basic operations of addition and multiplication, may be considered as part of $\mathcal{M}$, due to the embedding

$$\mathbb{R} \hookrightarrow \mathcal{M} : \ \alpha \mapsto \frac{\{\alpha\}}{\{1\}}.$$

The same is true to C, considered as a ring with the operation of addition and the convolution $*$, but due to the embedding

$$(C, *) \hookrightarrow \mathcal{M} : \ f \mapsto \frac{Lf}{\{1\}}.$$

The basic formula of our operational calculus concerns the relationship between the derivative f' and the product Sf.

Theorem 3.2. *Let $f \in C^1$. Then*

$$f' = Sf - \Phi\{f\}, \tag{3.3}$$

where $\Phi\{f\}$ is considered as a number, different from the constant function $\{\Phi\{f\}\}$, except in the case $\Phi\{f\} = 0$.

Proof. From (13) we get

$$Lf'(t) = f(t) - f(0) - \Phi_\tau\{f(t) - f(0)\} = f(t) - \Phi\{f\}$$

This identity in $\mathcal{C}$ may be written in $\mathcal{M}$ as

$$Lf' = f - \Phi\{f\}\, L.$$

Multiplying by $S = \dfrac{1}{L}$, we get (3.3) as the **basic formula of the operational calculus**, we are developing. □

Corollary 3.3. *If $f \in C^{(n)}$, then*

$$f^{(n)} = S^n f - S^{n-1}\Phi\{f\} - S^{n-2}\Phi\{f'\} - \dots - S\Phi\{f^{(n-2)}\} - \Phi\{f^{(n-1)}\} \tag{3.4}$$

Very important for the solution of nonlocal Cauchy problems are the convolution fractions of the form $\dfrac{1}{(S-\mu)^k}$, $k = 1, 2, \dots$.

In order such a fraction to be meaningful, it is necessary $S - \mu$ to be a non-divisor of zero.

Theorem 3.4. *The element $S - \mu$ is a divisor of 0 in $\mathcal{M}$ iff μ is a zero of the indicatrix $E(\lambda)$, i.e. iff $E(\mu) = 0$.*

Proof. Let $E(\mu) = 0$. By (15) we get

$$(S-\mu)\{e^{\mu t}\} = S\{e^{\mu t}\} - \mu\{e^{\mu t}\} = \{\mu e^{\mu t}\} + \Phi_\tau\{e^{\mu t}\} - \mu\{e^{\mu t}\} = E(\mu) = 0$$

Hence, $S - \mu$ is a divisor of 0 in $\mathcal{M}$.

In order to prove the necessity of the condition $E(\mu) = 0$, assume that $S - \mu$ is a divisor of 0. Then there should exist a non-zero function $u \in \mathcal{C}$, such that

$$(S-\mu)\, u = 0.$$

This equation for u is equivalent to $u - \mu L u = 0$.

From it, it follows that $u \in \mathcal{C}^1$ and $u' - \mu u = 0$, $\Phi\{u\} = 0$.

All the non-zero solutions of $u' - \mu u = 0$ are $u = C e^{\mu t}$, $C \neq 0$.

The boundary condition $\Phi\{u\} = 0$ gives

$$\Phi\{e^{\mu t}\} = E(\mu) = 0.$$

Thus, the necessity is proven. □

Theorem 3.5. *Let $\mu \in \mathcal{C}$ be such that $E(\mu) \neq 0$. Then*

$$\frac{1}{S-\mu} = \left\{\frac{e^{\mu t}}{E(\mu)}\right\} \tag{3.5}$$

and

$$\frac{1}{(S-\mu)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial \mu^{k-1}}\frac{e^{\mu t}}{E(\mu)}\right\} \tag{3.6}$$

Proof. In order to prove (3.5), we use identity (2.4) from Theorem 2.1:

$$R_\mu f(t) = \left\{\frac{e^{\mu t}}{E(\mu)}\right\} * f(t).$$

$y = R_\mu f(t)$ is the solution of the nonlocal BVP

$$y' - \mu y = f, \ \Phi\{y\} = 0.$$

By (3.3) we have

$$Sy - \mu y = f$$

and hence

$$y = \frac{f}{S-\mu}.$$

Thus we proved the identity

$$\frac{1}{S-\mu} f = \left\{ \frac{e^{\mu t}}{E(\mu)} \right\} f \text{ in } \mathcal{M}.$$

Taking f to be a non-divisor of 0, we may cancel it, thus obtaining

$$\frac{1}{S-\mu} = \left\{ \frac{e^{\mu t}}{E(\mu)} \right\}.$$

Identity (3.6) can be proven by induction, using the obvious algebraic identity

$$\frac{1}{(S-\mu)^{k+1}} = \frac{1}{S-\mu} \cdot \frac{1}{(S-\mu)^k}.$$

□

Example 3.1. Let $\Phi\{f\} = \int_0^1 f(\tau)\, d\tau$. Then

$$(f * g)(t) = \int_0^1 \left[\int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma \right] d\tau\,,$$

$$E(\lambda) = \frac{e^\lambda - 1}{\lambda}\,,$$

$$\frac{1}{S-\mu} = \left\{ \frac{\mu\, e^{\mu t}}{e^\mu - 1} \right\}, \text{ provided } \mu \neq 2\pi i n,\ n \in \mathbb{Z} \setminus \{0\}\,,$$

$$\frac{1}{(S-\mu)^2} = \left\{ \frac{e^{\mu t}}{e^\mu - 1} - \frac{\mu\, e^{\mu(t+1)}}{(e^\mu - 1)^2} + \frac{\mu\, t\, e^{\mu t}}{e^\mu - 1} \right\}, \text{ etc.}$$

Example 3.2. Let $\Phi\{f\} = \dfrac{f(0)+f(1)}{2}$. Then

$$(f * g)(t) = \frac{1}{2} \int_0^t f(t-\tau)\, g(\tau)\, d\tau - \frac{1}{2} \int_t^1 f(1+t-\tau)\, g(\tau)\, d\tau$$

$$E(\lambda) = \frac{1+e^\lambda}{2},\ \lambda_n = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

$$\frac{1}{S-\mu} = \left\{ \frac{2\, e^{\mu t}}{1+e^\mu} \right\},\ \mu \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

$$\frac{1}{(S-\mu)^2} = \left\{ \frac{2\, e^{\mu t}}{1+e^\mu} - \frac{2\, e^{\mu(t+1)}}{(1+e^\mu)^2} \right\}, \text{ etc.}$$

4. Application of the operational method for solving nonlocal Cauchy problems

Let's consider the general nonlocal Cauchy problem (1.4)–(1.5) with $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$:

$$P\left(\frac{d}{dt}\right) y = F(t),\ \Phi\left\{y^{(k)}\right\} = \alpha_k,\ k = 0,1,2,\ldots,\deg P - 1,$$

in an interval Δ with $0 \in \Delta$.

For the linear functional Φ we suppose only that $\Phi\{1\} \neq 0$ and, more precisely, that $\Phi\{1\} = 1$.

In the ring $\mathcal{M}$ the nonlocal Cauchy problem is equivalent to a single algebraic equation of the form

$$P(S)\,y = F + Q(S), \tag{4.1}$$

where $Q(S)$ is the polynomial expression

$$Q(s) = \sum_{j=1}^{n}\left(\sum_{k=j}^{n} a_{n-k}\,\alpha_{k-j}\right) S^{j-1}.$$

Indeed, by the basic formula (3.4) of our operational calculus, we have

$$y^{(k)} = S^k y - S^{k-1}\Phi\{y\} - \ldots - S\Phi\left\{y^{(k-2)}\right\} - \Phi\left\{y^{(k-1)}\right\} =$$

$$= Sy^k - S^{k-1}\alpha_0 - S^{k-2}\alpha_1 - \ldots - S\alpha_{k-2} - \alpha_{k-1}$$

Then $P\left(\dfrac{d}{dt}\right) y = P(S)\,y - Q(S)$.

Equation (4.1) has a unique solution if $P(S)$ is a nondivisor of zero. In this case

$$y = \frac{1}{P(S)}F + \frac{Q(S)}{P(S)}. \tag{4.2}$$

Theorem 4.1. *Let $\mu_1, \mu_2, \ldots, \mu_m$ be the different zeros of a polynomial $P(\mu)$. Then $P(S)$ is a nondivisor of 0 in $\mathcal{M}$ iff $E(\mu_k) \neq 0$, $k = 1,2,\ldots,m$.*

Proof. Since $P(S) = a_0(S-\mu_1)^{l_1}\ldots(S-\mu_m)^{l_m}$, then the assertion follows from Theorem 3.4.

In order to find explicitly $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ as functions of C, we use the well known expansions of these expressions in elementary fractions of the form

$$\frac{1}{(S-\mu_k)^l},\ k = 1,2,\ldots,m,\ l = 1,2,\ldots,l_k.$$

According to Theorem 4.1 each of these fractions can be expressed as a function of C.

Let

$$G(t) = \frac{1}{P(S)} = \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{A_{k\,l}}{(S-\mu_k)^l},$$

$$R(t) = \frac{Q(S)}{P(S)} = \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{B_{k\,l}}{(S-\mu_k)^l}$$

Then (4.2) takes the form

$$y = G * f + R \tag{4.3}$$

□

This procedure is only a modification of the classical Heaviside algorithm.

5. The homogenous nonlocal Cauchy problem

Let's consider problem (1.4)–(1.5) in the case when

$$\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0,$$

i.e. the problem

$$P\left(\frac{d}{dt}\right) y = F(t),\ \Phi\left\{y^{(k)}\right\} = 0,\ k = 0, 1, 2, \ldots, \deg P - 1. \tag{5.1}$$

This problem reduces to the equation

$$P(S)\,y = F \tag{5.2}$$

in $\mathcal{M}$.

Here two cases may arise:

Case 1: $P(S)$ is a nondivisor of 0 in $\mathcal{M}$ (non–resonance case).

Case 2: $P(S)$ is a divisor of 0 in $\mathcal{M}$ (resonance case).

In the non-resonance Case 1 (5.3) has the unique solution

$$y = \frac{1}{P(S)}\,F \tag{5.3}$$

In the resonance Case 2 equation (5.2) well may have no solution. Necessary condition on order to exist a solution of (5.2) is F to be divisor of 0 of a special kind. Obviously, in Case 2, there may exist infinitely many solutions. In the frames of our approach this case can be settled completely too, but here we will restrict our consideration only to the non-resonance case. We leave the modifications needed for the resonance case for a forthcoming publication.

In the non-resonance case it is possible the following extension of the classical Duhamel principle.

Theorem 5.1. *Let $P(S)$ be a non-divisor of* 0. *If $G(t)$ is the solution of* (5.1) *for the special choice $F(t) \equiv 1$, then*

$$y = \frac{d}{dt}[G(t) * F(t)] \tag{5.4}$$

is the solution of the general problem (5.1) .

Proof. From (5.3) we have $y = \frac{1}{SP(S)}F = \frac{1}{S}\left(\frac{1}{P(S)}F\right) = \frac{d}{dt}(G * F)$, since $\Phi(G) = 0$.

If $F \in C^1$, then the differentiation of (5.4) gives

$$y = \Phi\{F\}\, G(t) + G(t) * F'(t).$$

Now we specialize our consideration to the case of the mean-periodic solutions of an arbitrary LODE with constant coefficients

$$P\left(\frac{d}{dt}\right) y = F(t).$$

According to Theorem 2.1, this problem is equivalent to the homogenous nonlocal Cauchy problem (5.1) for $\mathcal{C} = \mathcal{C}(\mathbb{R})$. Denoting the space of the mean-periodic functions with respect to the linear functional Φ on $\mathcal{C}$ by $\mathcal{C}_\Phi$, a necessary condition for existence of such solution is $F \in \mathcal{C}_\Phi$. In the non-resonance case, this condition is also sufficient. □

Theorem 5.2. *Let $F \in \mathcal{C}_\Phi$ and $P(\lambda)$ and $E(\lambda)$ have no common zero. Then the equation $P\left(\frac{d}{dt}\right) y = F(t)$ has a unique mean-periodic solution in $\mathcal{C}_\Phi$ with the explicit representation (5.4), where $G(t)$ is the solution of the non-local Cauchy problem*

$$P\left(\frac{d}{dt}\right) G = 1,\ \Phi\left\{G^{(k)}\right\} = 0,\ k = 0, 1, 2, \ldots, \deg P - 1\,.$$

Proof. (5.4) is the solution of the non-local Cauchy problem (5.1) in $\mathcal{C} = \mathcal{C}(\mathbb{R})$. According to Theorem 1.1 if $F \in \mathcal{C}_\Phi$, then $G * F \in \mathcal{C}_\Phi$. Since $G * F \in C^1$, then $y = \frac{d}{dt}(G * F) \in \mathcal{C}_\Phi$. □

The solution (5.4) can be represented in the form

$$y = G'(t) * F(t) \tag{5.5}$$

Remark 5.3. Note that $G(t)$ is by no means a mean–periodic function, but according to Theorem 1.1, the convolution product $G * F$ is a mean–periodic function, provided F is a such function. Its derivative is also a mean–periodic function. In other words, if $F \in \mathcal{C}_\Phi$, then $y = \frac{d}{dt}[G * F]$ is automatically Φ-mean-periodic.

Example 5.1. Let us consider the case when $P(\mu)$ has only simple zeros $\mu_1, \mu_2, \ldots, \mu_n$. Then

$$G'(t) = \frac{1}{P(S)} = \sum_{k=1}^{n} \frac{1}{P'(\mu_k)} \frac{1}{S-\mu} = \sum_{k=1}^{n} \frac{e^{\mu_k t}}{P'(\mu_k)\, E(\mu_k)}$$

and the solution (24) takes the form

$$y = \sum_{k=1}^{n} \frac{1}{P'(\mu_k)} R_{\mu_k} F(t),$$

where $R_{\mu_k} F(t)$ are given by (2.2).

6. Periodic and antiperiodic solutions of LODE with constant coefficients

The operational method for obtaining of mean-periodic solutions of LODEs with constant coefficients can be specialized for periodic and antiperiodic solutions of such equations.

6.1. Periodic solutions with a given period T (T-periodic solutions)

We are looking for solutions $y(t)$ of $P\left(\frac{d}{dt}\right) y = F(t)$, such that $y(t+T) = y(t)$ for $-\infty < t < \infty$.

Each T-periodic solution is also a mean-periodic solution with respect to the functional$\tilde{\tilde{\Phi}}\{t\} = f(T) - f(0)$. Unfortunately, the approach, developed in Section 5 is not directly applicable, since $\Phi\{1\} = 0$. We will show, that the problem for determining of the T-periodic solutions for LODEs with constant coefficients can be reduced to determining of mean-periodic solutions with respect to the functional

$$\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)\, d\tau.$$

Theorem 6.1. *If $y = y\{t\}$ is a T-periodic solution of*

$$P\left(\frac{d}{dt}\right) y = F(t), \tag{6.1}$$

then $\tilde{y}(t) = y(t) - \frac{1}{T}\int_0^T y(\tau)\, d\tau$ is a mean-periodic solution of the equation

$$P\left(\frac{d}{dt}\right) \tilde{y} = \tilde{F}(t), \tag{6.2}$$

where $\tilde{F}(t) = F(t) - \frac{1}{T}\int_0^T y(\tau)\, d\tau$ with respect to the functional $\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)\, d\tau$.

Proof. Let us, for the simplicity sake, denote:

$$y_0 = \frac{1}{T}\int_0^T y(\tau)\,d\tau \text{ and } F_0 = \frac{1}{T}\int_0^T F(\tau)\,d\tau.$$

Integrating (6.1) from 0 to T, we get

$$a_n\,y_0 = F_0, \text{ i.e. } a_n\,\frac{1}{T}\int_0^T y(\tau)\,d\,\tau = \frac{1}{T}\int_0^T F(\tau)\,d\,\tau \tag{6.3}$$

It remains to subtract this identity from (6.1) to obtain(6.2). $\tilde{y}$ is mean-periodic with respect to the functional $\Phi\{f\} = \dfrac{1}{T}\displaystyle\int_0^T f(\tau)\,d\,\tau$, since it satisfies the BVCs

$$\Phi\{\tilde{y}^{(k)}\} = \frac{\tilde{y}^{(k-1)}(T) - \tilde{y}^{(k-1)}(0)}{T} = \frac{y^{(k-1)}(T) - y^{(k-1)}(0)}{T} = 0,$$

for $k = 1, 2, \ldots, n-1$ and also $\Phi\{\tilde{y}\} = 0$.

There one should distinguish two cases: 1) $a_n = P(0) \neq 0$ and 2) $a_n = P(0) = 0$.

1) The case $a_n \neq 0$. Then the unknown constant $y_0 = \dfrac{1}{T}\displaystyle\int_0^T y(\tau)\,d\,\tau$ can be determined from (29):

$$y_0 = \frac{1}{a_n T}\int_0^T F(\tau)\,d\,\tau$$

2) In the case $a_n = 0$, the constant y_0 remains arbitrary. The periodic solution $y = y_0 + \tilde{y}$ depends on an arbitrary constant y_0. Then, in order to exist a T-periodic solution, the necessary condition $\displaystyle\int_0^T F(\tau)\,d\,\tau = 0$ should be satisfied. □

Thus, the practical determining of T-periodic solution reduces to the solution of the nonlocal Cauchy problem

$$P\left(\frac{d}{dt}\right) G = 1, \; \frac{1}{T}\int_0^T G^{(k)}(\tau)\,d\tau = 0, \; k = 0, 1, 2, \ldots, n-1$$

or, more explicitly,

$$G^{(k)}(T) - G^{(k)}(0) = 0, \; k = 0, 1, 2, \ldots, n-2$$

and

$$G_0 = \frac{1}{T}\int_0^T G(\tau)\,d\tau = 0.$$

It is equivalent to the following equation in $\mathcal{M}$: $P(S)\,y = \dfrac{1}{S}$.

In order this equation to have a solution in $\mathcal{M}$, it is necessary and sufficient $P(S)$ to be a non-divisor of 0 (the non-resonance case). Then

$$G = \frac{1}{S\,P(S)}.$$

Theorem 6.2. (Extended Duhamel principle). *Let $P(S)$ be a non-divisor of zero in $\mathcal{M}$, i.e. $P\left(\frac{2n\pi i}{T}\right) \neq 0$ for $n \in \mathbb{Z}\setminus\{0\}$ and $F(t)$ be a T-periodic function. Then the equation $P\left(\frac{d}{dt}\right) y = F(t)$ has a T-periodic solution of the form*

$$y(t) = y_0 - \frac{1}{T}\int_0^T F(t-\tau)\,G(\tau)\,dt, \tag{6.4}$$

where $G = \dfrac{1}{S\,P(S)}$ and $y_0 = \dfrac{1}{P(0)\,T}\int_0^T F(\tau)\,d\tau$ for $P(0) \neq 0$. For $P(0) = 0$, y_0 is an arbitrary constant, provided $\displaystyle\int_0^T F(\tau)\,d\tau = 0$.

Proof. According to Theorem 5.1, the equation $P\left(\frac{d}{dt}\right)\tilde{y} = \tilde{F}$ has a unique mean-periodic solution with respect to the functional $\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)\,d\tau$. It has the Duhamel representation (formula (5.4))

$$\tilde{y}(t) = \frac{d}{dt}[G * (F - F_0)], \tag{6.5}$$

where by $*$ it is denoted the convolution:

$$(f * g)(t) = \frac{1}{T}\int_0^T \left[\int_\tau^t f(t+\tau-\sigma)\,g(\sigma)\,d\sigma\right] d\tau.$$

By Theorem 2.2 we have

$$\frac{d}{dt}(f*g)(t) = f_0\,g(t) + f(t)\,g_0 - \frac{1}{T}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau + \int_t^T f(T+t-\tau)\,g(\tau)\,d\tau\right]$$

and hence

$$\tilde{y} = -\frac{1}{T}\left[\int_0^t (F(t-\tau) - F_0)\,G(\tau)\,dt + \int_t^T (F(T+t-\tau) - F_0)\,G(\tau)\,dt\right]$$

Using the T-periodicity of F and the BVC $\displaystyle\int_0^T G(\tau)\,d\tau = 0$, we obtain the explicit representation (6.4). The T-periodicity of the function in the right-hand side of (6.4) is obvious. □

Corollary 6.3. *Representation (6.4) is equivalent to*

$$y = y_0 - \frac{1}{T}\left[\int_0^t G(t-\tau)\,F(\tau)\,d\tau + \int_0^T G(T+t-\tau)\,F(\tau)\,d\tau\right] \tag{6.6}$$

Indeed, (6.6) follows from (6.5) by the commutativity of convolution $f * g$.

Example 6.1. Find the π-periodic solution of the equation $y^{(6)}+2y^{(4)}+y'' = F(t)$ for π-periodic $F(t)$ with $\int_0^{\pi} F(\tau)\,d\tau = 0$.

Solution. According to Theorem 6.2 all the π-periodic solutions have the form

$$y(t) = y_0 - \frac{1}{\pi}\int_0^{\pi} F(t-\tau)\,G(\tau)\,d\tau, \tag{6.7}$$

where y_0 is an arbitrary constant, and $G(t)$ is the solution of the nonlocal Cauchy problem

$$G^{(6)} + 2\,G^{(4)} + G'' = 1, \quad \int_0^{\pi} G^{(k)}(\tau)\,d\tau = 0, \ \ k = 0,1,2,\ldots,5.$$

Using the corresponding operational calculus, we find easily

$$G = \frac{1}{S(S^2+1)} = \frac{1}{S^3} - \frac{2}{S} + \frac{2\,S}{S^2+1} + \frac{S}{(S^2+1)^2}$$

The functional interpretation of the elementary fractions gives:

$$\frac{1}{S^3} = L^2\{1\} = \frac{1}{2}\,t^2 - \frac{\pi}{2}\,t + \frac{\pi^2}{12}, \quad \frac{2}{S} = 2$$

$$\frac{2\,S}{S^2+1} = \pi\,\sin\,t, \quad \frac{1}{S^2+1} = -\frac{\pi}{2}\,\cos\,t$$

$$\frac{S}{(S^2+1)^2} = -\frac{\pi^2}{4}\,(\sin\,t\,*\,\cos\,t) = -\frac{\pi}{4}\,t\cos t + \frac{\pi^2}{8}\,\cos\,t + \frac{\pi}{4}\,\sin\,t$$

For $G(t)$ we obtain

$$G(t) = \frac{1}{2}\,t^2 - \frac{\pi}{2}\,t + \frac{\pi^2}{12} - 2 - \frac{\pi}{4}\,t\cos t + \frac{\pi^2}{8}\,\cos\,t + \frac{3\pi}{4}\,\sin\,t$$

and (6.7) gives all π-periodic solutions.

7. Antiperiodic solutions with a given antiperiod $T > 0$ (T–antiperiodic solutions)

We are looking for solutions $y(t)$ of $P\left(\dfrac{d}{dt}\right)\,y = F(t)$, such that $y(t+T) = -y(t)$ for $-\infty < y < \infty$.

Each T–antiperiodic solution is also a mean-periodic solution with respect to the functional $\Phi\{t\} = \dfrac{f(T)+f(0)}{T+1}$ (Example 1.2 of Section 1).

So, the general theory of Section 5 is directly applicable to this case. The convolution (2.3) has the form

$$(f*g)(t) = \frac{1}{T+1}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau - \int_t^T f(T+t-\tau)\,g(\tau)\,d\tau\right]$$

Here the indicatrix is $E(\lambda) = \dfrac{e^{\lambda T}+1}{T+1}$ with zeros $\lambda_n = \dfrac{(2n+1)\pi i}{2T}$, $n \in \mathbb{Z}$.

If $\mu \neq \lambda_n$, then (Theorem 3.5):

$$\frac{1}{S-\mu} = \frac{(T+1)\, e^{\mu T}}{e^{\mu T}+1}.$$

Theorem 7.1. *Let $P(S)$ be a nondivisor of 0 on $\mathcal{M}$, i.e. $P\left(\dfrac{(2n+1)\,\pi\, i}{2T}\right) \neq 0$, $n \in \mathbb{Z}$. Then $\dfrac{1}{P(S)} = U(t)$ is the solution of the nonlocal Cauchy problem*

$$P\left(\frac{d}{dt}\right) U = 0, \;\; U^{(k)}(T) + U^{(k)}(0) = 0, \;\; k = 0, 1, 2, \ldots, n-2$$

and

$$U^{(n-1)}(T) + U^{(n-1)}(0) = \frac{1}{a_0}.$$

The proof follows from Theorem 4.1.

Theorem 7.2. (Extended Duhamel principle for antiperiodic solutions). *Let $P(S)$ is a nondivisor of 0 in $\mathcal{M}$, i.e. $P\left(\dfrac{(2n+1)\,\pi\, i}{2T}\right) \neq 0$, $n \in \mathbb{Z}$, and $F(t)$ is a T-antiperiodic function of $C(\mathbb{R})$. Then the equation $P\left(\dfrac{d}{dt}\right) y = F(t)$ has a unique T-antiperiodic solution of the form*

$$y(t) = \frac{1}{T+1} \int_0^T F(t-\tau)\, U(\tau)\, d\tau \tag{7.1}$$

Proof. From Theorem 5.1 we have

$$y = \frac{1}{P(S)} F = U * F = F * U =$$

$$= \frac{1}{T+1} \left[\int_0^t F(t-\tau)\, U(\tau)\, d\tau + \int_T^t F(t+T-\tau)\, U(\tau)\, d\tau \right] =$$

$$= \frac{1}{T+1} \left[\int_0^t F(t-\tau)\, U(\tau)\, d\tau - \int_T^t F(t-\tau)\, U(\tau)\, d\tau \right] =$$

$$= \frac{1}{T+1} \int_0^T F(t-\tau)\, U(\tau)\, d\tau$$

□

Corollary 7.3. (7.1) *is equivalent to*

$$y(t) = \frac{1}{T+1} \left[\int_0^t U(t-\tau)\, d\tau - \int_t^T U(T+t-\tau)\, F(\tau)\, d\tau \right]$$

Example 7.1. Find the π-antiperiodic solution of the equation $y^{(4)} - 4y = F(t)$ if $F(t+\pi) = -F(t)$.

Solution. Here $\Phi\{f\} = \dfrac{f(0)+f(\pi)}{1+\pi}$, $E(\lambda) = \dfrac{1+e^{\lambda\pi}}{1+\pi}$,

$\lambda_n = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$, and

$$(f*g)(t) = \frac{1}{T+1}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau - \int_t^T f(T+t-\tau)\,g(\tau)\,d\tau\right]$$

For obtaining of antiperiodic solutions in explicit form, it is convenient to use the extended Duhamel principle (3.5), i.e.

$$y = F*U,\ U = \frac{1}{P(S)}.$$

Using the π-antiperiodicity of $F(t)$, we obtain

$$y(t) = \frac{1}{\pi+1}\int_0^\pi F(t-\tau)\,U(\tau)\,d\tau,$$

It remains to find $U = \dfrac{1}{P(S)} = \dfrac{1}{S^4-4}$:

$$\frac{1}{S^4-4} = \frac{1}{(S^2-2)(S^2+2)} = \frac{1}{4}\left(\frac{1}{S^2-2} - \frac{1}{S^2+2}\right).$$

$$\frac{1}{S^2-2} = \frac{1}{2\sqrt{2}}\left\{\frac{1}{S-\sqrt{2}} - \frac{1}{S+\sqrt{2}}\right\} = \frac{1+\pi}{2\sqrt{(2)}}\ \text{sech}\,\frac{\pi}{\sqrt{2}}\ \sinh\sqrt{2}(t-\frac{\pi}{2})$$

$$\frac{1}{S^2+2} = \frac{1+\pi}{2\sqrt{2}}\ \sec\frac{\pi}{\sqrt{2}}\ \sin\sqrt{2}\left(t-\frac{\pi}{2}\right)$$

Thus we get

$$U(t) = \frac{1+\pi}{8\sqrt{2}}\left(\text{sech}\,\frac{\pi}{\sqrt{2}}\ \sinh\sqrt{2}\left(t-\frac{\pi}{2}\right) - \sec\frac{\pi}{\sqrt{2}}\ \sin\sqrt{2}\left(t-\frac{\pi}{2}\right)\right)$$

The π-antiperiodic solution $y(t)$ has the explicit representation

$$y(t) = \frac{1}{8\sqrt{(2)}}\int_0^\pi F(t-\tau)\left(\text{sech}\,\frac{\pi}{\sqrt{2}}\ \sinh\sqrt{2}\left(\tau-\frac{\pi}{2}\right) - \sec\frac{\pi}{\sqrt{2}}\ \sin\sqrt{2}\left(\tau-\frac{\pi}{2}\right)\right)d\tau$$

Remark 7.4. It is convenient to use the explicit representations of the solutions for practical computation of mean-periodic solutions of LODEs with constant coefficients and, particulary, for computation of periodic and antiperiodic solutions of such equations, in the environment of a computer algebra system.

In [8] it is considered the obtaining of periodic solutions of LODE with constant coefficients (in the non-resonance and in the resonance cases) with use of an extension of the Heaviside algorithm, proposed by Dimovski [1]. The algorithm is implemented by means of the computer algebra system *Mathematica*. For the purposes of this implementation, a table of interpretation formulae was compiled. A part of the developed programs can be used for computation of mean-periodic solutions. Some interpretation formulae can be used as well.

Thus we ensure the chance to compute periodic solutions of any kind in the program environment of a computer algebra system.

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