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# Тримерно конволюционно смятане за нелокални еволюционни гранични задачи

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# THREE-DIMENSIONAL OPERATIONAL CALCULI FOR NONLOCAL EVOLUTION BOUNDARY VALUE PROBLEMS

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## Abstract

Direct algebraic operational calculi for functions $u(x, y, t)$, continuous in a domain of the form $D = [0, a] \times [0, b] \times [0, \infty)$, are built. Along with the classical Duhamel convolution, the construction uses also two non-classical convolutions for the operators $\partial_x^2$ and $\partial_y^2$. These three one-dimensional convolutions are combined into one three-dimensional convolution $u * v$ in $C(D)$. Instead of J. Mikusiński's approach, based on convolution fractions, we develop systematically an alternative approach, based on the multiplier fractions of the convolution algebra $(C(D), *)$.

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**1. Introductions.** Till recently, all the operational calculi for functions of one, or several variables, were intended to cope with initial value problems. In Gutterman [8], direct operational calculi for functions of several real variables are proposed. They are applicable for solution of Cauchy problems for linear partial differential equations with constant coefficients. As for mixed problems, i.e. problems, containing both boundary and initial conditions, Gutterman acknowledges that his method is unpractical, and its extension to them would need essentially new ideas and approaches. Not to speak about nonlocal boundary value problems. Here we intend to propose a direct operational calculus approach to local and nonlocal boundary value problems for functions of one, two, and three real variables.

Let $P$, $Q$, $R$ be polynomials. We consider following BVPs:

(1) $\quad P(\partial_t)u + Q(\partial_x^2)u + R(\partial_y^2)u = F(x,y,t), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < t,$

in a domain $D = [0, a] \times [0, b] \times [0, \infty)$, with initial conditions

(2) $\quad \partial_t^k u(x,y,0) = f_k(x,y), \quad 0 \le x \le a, \quad 0 \le y \le b, \quad k = 0, 1, \ldots, \deg P - 1,$

and boundary value conditions

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$$\partial_x^{2l}u(0,y,t)=\varphi_l(y,t),\quad \Phi_\xi\{\partial_x^{2l}u(\xi,y,t)\}=g_l(y,t),\ 0\le y\le b,\ 0\le t,$$

(3) $$l=0,\,1,\,\ldots,\,\deg Q-1,$$

$$\partial_y^{2m}u(x,0,t)=\psi_m(x,t),\quad \Psi_\eta\{\partial_y^{2m}u(x,\eta,t)\}=h_m(x,t),\ 0\le x\le a,\ 0\le t,$$

$$m=0,\,1,\,\ldots\,\deg R-1.$$

Here $\Phi$ and $\Psi$ are supposed to be non-zero linear functionals on $C^1[0,a]$ and $C^1[0,b]$ correspondingly. They have Stieltjes-type representations of the form:

(4) $$\Phi\{f\}=Af(a)+\int_0^a f'(\xi)d\alpha(\xi),\qquad f\in C^1[0,a]\qquad \text{and}$$

(5) $$\Psi\{g\}=Bg(b)+\int_0^b g'(\eta)d\beta(\eta),\qquad g\in C^1[0,b],$$

with functions $\alpha(x)$ and $\beta(y)$ with bounded variation, and constants $A$ and $B$.
For the sake of simplicity, we suppose that

(6) $$\Phi_\xi\{\xi\}=1\qquad \text{and}\qquad \Psi_\eta\{\eta\}=1.$$

These restrictions may be ousted by some non-essential involvements.

The next considerations are based on the following elementary boundary value problems:

(7) $u''(x)+\lambda^2u(x)=f(x)$, $x\in(0,a)$, $u(0)=0$, $\Phi\{u\}=0$

and

(8) $v''(y)+\mu^2v(y)=g(y)$, $y\in(0,b)$, $v(0)=0$, $\Psi\{v\}=0$,

respectively in $C[0,a]$ and $C[0,b]$.

The solutions of (7) and (8) are the resolvent operators

$$R_x(f,\lambda)=\frac{1}{\lambda}\int_0^x\sin\lambda(x-\xi)\ f(\xi)d\xi-\Phi_\xi\left\{\frac{1}{\lambda}\int_0^\xi\sin\lambda(\xi-\eta)\ f(\eta)d\eta\right\}\frac{\sin\lambda x}{\lambda E(\lambda)}$$

and

$$R_y(g,\mu)=\frac{1}{\mu}\int_0^y\sin\mu(y-\eta)\ g(\eta)d\eta-\Psi_\eta\left\{\frac{1}{\mu}\int_0^\eta\sin\mu(\eta-\xi)\ g(\xi)d\xi\right\}\frac{\sin\mu y}{\mu F(\mu)}$$

where $E(\lambda)=\Phi_\xi\left\{\dfrac{\sin\lambda\xi}{\lambda}\right\}$ and $F(\mu)=\Psi_\eta\left\{\dfrac{\sin\mu\eta}{\mu}\right\}$ are the so called *sine-indicatrices* of the functionals $\Phi$ and $\Psi$.

The resolvent operators $R_x(f,\lambda)$ and $R_y(g,\mu)$ are defined for $\lambda=0$ and $\mu=0$, correspondingly, due to restrictions (6). Denoting $L_xf(x)=R_x(f,0)$ and $L_yg(y)=R_y(g,0)$ we have

(9) $$L_xf(x)=\int_0^x(x-\xi)f(\xi,)d\xi-x\Phi_\xi\left\{\int_0^\xi(\xi-\eta)f(\eta)d\eta\right\},$$

(10) $$L_yg(y)=\int_0^y(y-\eta)g(\eta)d\eta-y\Psi_\eta\left\{\int_0^\eta(\eta-\varsigma)g(\varsigma)d\varsigma\right\}.$$

## 2. One-dimensional operational calculi

**2.1. The Duhamel convolution.** This is the operation

(11) $$(\varphi\overset{t}{*}\psi)(t)=\int_0^t\varphi(t-\tau)\psi(\tau)d\tau,\qquad \varphi,\psi\in C[0,\infty).$$

It bears the name of Duhamel, but sometimes it is called either Borel, or Laplace convolution. It is connected with the integration operator

$$l_t\varphi(t) = \int_0^t \varphi(\tau)\,d\tau, \tag{12}$$

to which sometimes it is given the name of Volterra integration operator. The connection is the convolutional representation

$$l_t\,\varphi = \{1\} \overset{t}{*} \varphi. \tag{13}$$

The operational calculus for operator $l_t$ is developed in a purely algebraic way by J. Mikusiński [11] using the Duhamel convolution. Here we suppose that at least the elements of it are known to the reader.

### 2.2. A family of convolutions in $C[0, a] = C_x$

**Theorem 1.** (Dimovski [2], p.119) *If $f, g \in C\,[0, a]$, then the operation*

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\widetilde{\Phi}_\xi\{h(x,\xi)\}, \tag{14}$$

*with* $\widetilde{\Phi}_\xi = \Phi_\xi \circ l_\xi$ and

$$h(x,\xi) = \int_x^\xi f(\xi + x - \varsigma)g(\varsigma)d\varsigma - \int_{-x}^\xi f(|\,\xi - x - \varsigma\,|)g(|\,\varsigma\,|)\,\mathrm{sgn}(\varsigma(\xi - x - \varsigma))d\varsigma$$

*is a bilinear, commutative and associative operation in $C$ [0, $a$], such that the resolvent operator $R_x(f\,;\,\lambda)$ has the representation*

$$R_x(f,\lambda) = \left\{\frac{\sin\lambda x}{\lambda\,E(\lambda)}\right\} \overset{x}{*} f. \tag{15}$$

Due to (4), the functional $\widetilde{\Phi}$ can be written in the explicit form

$$\widetilde{\Phi}\{f\} = A\int_0^a f(\xi)d\xi + \int_0^a f(\xi)d\alpha(\xi).$$

For $\lambda = 0$, we get

$$L_x f(x) = \{x\} \overset{x}{*} f. \tag{16}$$

### 2.3. A family of convolutions in $C[0, b] = C_y$

Completely analogical to Theorem 1 is the following Theorem 2. The difference is only in the notations.

**Theorem 2.** *If $f, g \in C\,[0, b]$, then the operation*

$$(f \overset{y}{*} g)(y) = -\frac{1}{2}\widetilde{\Psi}_\eta\{h(y,\eta)\} \tag{17}$$

*with* $\widetilde{\Psi}_\eta = \Psi_\eta \circ l_\eta$ and

$$h(y,\eta) = \int_y^\eta f(\eta + y - \varsigma)g(\varsigma)d\varsigma - \int_{-y}^\eta f(|\,\eta - y - \varsigma\,|)g(|\,\varsigma\,|)\,\mathrm{sgn}(\varsigma(\eta - y - \varsigma))d\varsigma$$

*is a bilinear, commutative and associative operation in $C$ [0, $b$], such that the resolvent operator $R_y(g\,;\,\mu)$ has the representation*

$$R_y(g,\mu) = \left\{\frac{\sin\mu y}{\mu\,F(\mu)}\right\} \overset{y}{*} g. \tag{18}$$

Due to (5), the functional $\widetilde{\Psi}$ can be written in the explicit form

$$\widetilde{\Psi}\{g\} = B\int_0^b g(\eta)d\eta + \int_0^b g(\eta)d\beta(\eta).$$

For $\mu = 0$, we get

(19) $$L_y g(y) = \{y\} \overset{y}{*} g.$$

**2.4. Rings $\mathcal{M}_x$ and $\mathcal{M}_y$ of the multiplier fractions for the convolution algebras $(C_x, \overset{x}{*})$ and $(C_y, \overset{y}{*})$**

We will describe briefly the construction of the rings $\mathcal{M}_x$ and $\mathcal{M}_y$ of the multiplier fractions of the convolution algebras $(C_x, \overset{x}{*})$ and $(C_y, \overset{y}{*})$. These constructions are completely analogical. We restrict our considerations in more detail on $(C_x, \overset{x}{*})$ only (see Dimovski [2]).

**Definition 1.** (See [9], p. 14) A linear operator $M : C_x \to C_x$ is said to be a multiplier of the convolution algebra $(C_x, \overset{x}{*})$, iff the relation

$$M(f \overset{x}{*} g) = (Mf) \overset{x}{*} g$$

holds for all $f, g \in C_x$.

These multipliers form a commutative ring $\mathfrak{m}_x$. Denote by $\mathfrak{n}_x$ the subset of non-zero nondivisors of zero of $\mathfrak{m}_x$. The standard procedure of construction of the ring $\mathcal{M}_x =$ $= \mathfrak{n}_x^{-1}\mathfrak{m}_x$ of multiplier fractions $\dfrac{M}{N}$ with $M \in \mathfrak{m}_x$, $N \in \mathfrak{n}_x$, is described in S. Lang [10]. This procedure bears the name of "localization". Considered as a ring, the algebra $(C_x, \overset{x}{*})$ embeds into $\mathcal{M}_x$ by the map

$$f \mapsto \frac{f\overset{x}{*}}{I_x},$$

where $f\overset{x}{*}$ denotes the convolution operator $(f\overset{x}{*})g = f\overset{x}{*}g$ and $I_x$ is the identity operator of $C_x$. Further, due to this embedding, we denote the convolution operator $f\overset{x}{*}$ by $f$. Especially, $L_x = \{x\}\overset{x}{*} = \{x\}$.

Further, we denote

$$S_x = \frac{1}{L_x}.$$

$S_x$ is an element of $\mathcal{M}_x$, but not an operator on $C_x$. Nevertheless, if $f \in C^2[0, a]$, then $S_x$ is connected with the operator $\dfrac{d^2}{dx^2}$, due to

**Lemma 1.** If $f \in C^2\,[0, a]$, then

(20) $$f'' = S_x f + S_x\{(x\Phi\{1\} - 1)f(0) - x\Phi\{f\}\}.$$

**Proof.** Let us calculate $L_x f''$:

(21) $$L_x\{f''\} = f(x) + (x\Phi\{1\} - 1)f(0) - x\Phi\{f\}.$$

Multiplying by $S_x$ and using $S_x\{x\} = S_x L_x = 1$, we get (20). □

**Lemma 2.** If $\lambda \in \mathcal{C}$ is a zero of $E(\lambda)$, then $S_x + \lambda^2$ is a divisor of zero in $\mathcal{M}_x$.
Indeed $(S_x + \lambda^2)\ \{\sin \lambda x\} = 0$, since $\Phi_\xi\{\sin \lambda \xi\} = 0$.

**Theorem 3.** *If* $E(\lambda) \neq 0$, *then* $S_x + \lambda^2$ *is a nondivisor of zero in* $\mathcal{M}_x$, *and*

$$\frac{1}{S_x + \lambda^2} = \left\{\frac{\sin \lambda x}{\lambda\ E(\lambda)}\right\}.$$

**Proof.** Assume that $S_x + \lambda^2$ is a divisor of zero in $\mathcal{M}_x$. Then there should exist a non-zero function $u \in C^2[0, a]$, such that $(S_x + \lambda^2)\ u = 0$, or equivalently, $u + \lambda^2 L_x u = 0$. From this equation it follow $u(0) = 0$ and $\Phi\{u\} = 0$. Applying the operator $\left(\frac{d}{dx}\right)^2$, we get $u'' + \lambda^2 u = 0$. Hence $u \neq 0$, $u = C \sin \lambda x$ with $C \neq 0$. Then, $\Phi\{u\} = 0$ implies $C\lambda \dot{E}(\lambda) = 0$ which is a contradiction.
Denoting $u = R_x(f, \lambda)$, we have $u(0) = 0$ and $\Phi\{u\} = 0$. By (7) and (20), we get $u'' = S_x u = f - \lambda^2 u$ and thus we obtain $u = \frac{1}{S_x + \lambda^2} f = \left\{\frac{\sin \lambda x}{\lambda E(\lambda)}\right\} \overset{x}{*} f$. For $f = \{x\}$, we get $u = \frac{1}{S_x(S_x + \lambda^2)} = \left\{\frac{\sin \lambda x}{\lambda E(\lambda)}\right\} \frac{1}{S_x}$ and it remains to multiply the last identity by $S_x$.□

Analogous is the construction of the ring of the multiplier fractions $\mathcal{M}_y$ of $(C_y, \overset{y}{*})$. Here we denote

$$S_y = \frac{1}{L_y}.$$

The relation between $S_y$ and $f''$ is given by

**Lemma 3.** If $f \in C^2\ [0, b]$, then

$$f'' = S_y f + S_y\{\ (y\Psi\{1\} - 1) f(0) - y\ \Psi\{f\}\ \}. \tag{22}$$

**Lemma 4.** If $F(\mu) = 0$, then $S_y + \mu^2$ is a divisor of zero in $\mathcal{M}_y$.
Indeed $(S_y + \mu^2)\ \{\sin \mu y\} = 0$, since $\Psi_\eta\{\sin \mu \eta\} = 0$.

**Theorem 4.** *If* $F(\mu) \neq 0$, *then* $S_y + \mu^2$ *is a nondivisor of zero in* $\mathcal{M}_y$, *and*

$$\frac{1}{S_y + \mu^2} = \left\{\frac{\sin \mu y}{\mu\ E(\mu)}\right\}.$$

The proof is similar to that of Theorema 3.

## 3. Two-dimensional convolutions

### 3.1. Two-dimensional convolution in $C([0, a] \times [0, \infty))$

**Theorem 5.** *Let* $u, v \in C([0, a] \times [0, \infty))$. *Then*

(23) $$u(x,t) \overset{(x,t)}{*} v(x,t) = \int_0^t u(x,t-\tau) \overset{x}{*} v(x,\tau)\, d\tau .$$

*is a bilinear, commutative and associative operation in* $C([0, a] \times [0, \infty))$ *such that* $l_t L_x u = \{x\} \overset{(x,t)}{*} u(x,t)$, *where* $l_t$ *denotes the integration operator*

$$l_t\{u(x,t)\} = \int_0^t u(x,\tau)\, d\tau .$$

**Proof.** First, we prove the assertions of the theorem for product functions $u(x, t) = f(x)\varphi(t)$ and $v(y, t) = g(y)\psi(t)$. It is easy to see that $u \overset{(x,t)}{*} v = (f \overset{x}{*} g)(\varphi \overset{t}{*} \psi)$. Then, the commutativity relation $u \overset{(x,t)}{*} v = v \overset{(x,t)}{*} u$ and the associativity relation $(u \overset{(x,t)}{*} v) \overset{(x,t)}{*} w = u \overset{(x,t)}{*} (v \overset{(x,t)}{*} w)$ follow from the corresponding associativity relations for the one-dimensional convolutions $f \overset{x}{*} g$ and $\varphi \overset{t}{*} \psi$. Further, we may use approximation argument.□

### 3.2. Two-dimensional convolution in $C([0, b] \times [0, \infty))$

**Theorem 6.** *Let* $u, v \in C([0, b] \times [0, \infty))$. *Then*

(24) $$u(y,t) \overset{(y,t)}{*} v(y,t) = \int_0^t u(y,t-\tau) \overset{y}{*} v(y,\tau)\, d\tau$$

*is a bilinear, commutative and associative operation in* $C([0, b] \times [0, \infty))$ *such that* $l_t L_y u = \{y\} \overset{(y,t)}{*} u(y,t)$, *where* $l_t$ *denotes the integration operator*

$$l_t\{u(y,t)\} = \int_0^t u(y,\tau)\, d\tau .$$

The proof is identical with that of Theorem 5.

### 3.3. Two-dimensional convolution in $C([0, a] \times [0, b])$

**Theorem 7.** (Dimovski [3]) *Let* $u, v \in C([0, a] \times [0, b])$. *Then*

(25) $$u(x,y) \overset{(x,y)}{*} v(x,y) = \frac{1}{4} \tilde{\Phi}_\xi \tilde{\Psi}_\eta \{h(x,y,\xi,\eta)\},$$

*with*

$$h(x,y,\xi,\eta)=\int_x^{\xi}\int_y^{\eta}u(\xi+x-\sigma,\eta+y-\tau)\,v(\sigma,\tau)\,d\sigma\,d\tau-$$

$$-\int_{-x}^{\xi}\int_y^{\eta}u(|\,\xi-x-\sigma\,|,\eta+y-\tau)\,v(|\,\sigma\,|,\tau)\,\mathrm{sgn}(\xi-x-\sigma)\sigma\,d\sigma\,d\tau-$$

$$-\int_x^{\xi}\int_{-y}^{\eta}u(\xi+x-\sigma,|\,\eta-y-\tau\,|)\,v(\sigma,|\,\tau\,|)\,\mathrm{sgn}(\eta-y-\tau)\tau\,d\sigma\,d\tau+$$

$$+\int_{-x}^{\xi}\int_{-y}^{\eta}u(|\,\xi-x-\sigma\,|,|\,\eta-y-\tau\,|)\,v(|\,\sigma\,|,|\,\tau\,|)\,\mathrm{sgn}(\xi-x-\sigma)(\eta-y-\tau)\sigma\tau\,d\sigma\,d\tau.$$

*is a bilinear, commutative and associative operation in* $C([0, a] \times [0, b])$, *such that*

$$L_x L_y u(x,y,t)=\{x\,y\}\overset{(x,y)}{*}u(x,y,t)$$

*and the operator* $L_x$ *and* $L_y$ *are multipliers of the convolution algebra* $(C([0, a] \times [0, b]), \overset{(x,y)}{*})$.

The proof of the last assertion follows from the considerations in the next section.

## 4. Three-dimensional convolution in $C(D)$

**Theorem 8.** *Let* $u, v \in C = C([0, a] \times [0, b] \times [0, \infty)) = C(D)$. *Then*

$$u(x,y,t)*v(x,y,t)=\int_0^t u(x,y,t-\tau)\overset{(x,y)}{*}v(x,y,\tau)\,d\tau \tag{26}$$

*is a bilinear, commutative and associative operation in* $C(D)$, *such that*

$$l_t\,L_x L_y\,u(x,y,t)=\{x\,y\}*u(x,y,t). \tag{27}$$

*and the operators* $l_t$, $L_x$ *and* $L_y$ *are multipliers of the convolution algebra* $(C, *)$.

**Proof.** For $u(x, y, t) = f_1(x)\, g_1(y)\, h_1(t)$ and $v(x, y, t) = f_2(x)\, g_2(y)\, h_2(t)$ we have $u*v=(f_1\overset{x}{*}f_2)(g_1\overset{y}{*}g_2)(h_1\overset{t}{*}h_2)$. The assertions of Theorem 8 follow from the corresponding assertions about one-dimensional convolutions $\overset{t}{*}$, $\overset{x}{*}$ and $\overset{y}{*}$. For example, let us prove (27). First we consider a product function $u(x, y, t) = f(x)\, g(y)\, h(t)$. We have

$$l_t\,L_x L_y u(x,y,t)=l_t\,L_x L_y\,[f(x)g(y)h(t)]=L_x f(x).L_y g(y).l_t h(t)=$$

$$=[\{1\}\overset{t}{*}h(t)][\{x\}\overset{x}{*}f(x)][\{y\}\overset{y}{*}g(y)]=\{x\,y\}*\{f(x)g(y)h(t)\}=\{x\,y\}*u(x,y,t).$$

Then, (27) is true for linear combinations of such products. But every function $u(x, y, t) \in C(D)$ can be approximated by linear combinations of product functions $f(x)\, g(y)\, h(t)$ where $f(x) \in C[0, a]$, $g(y) \in C[0, b]$ and $h(t) \in C[0, \infty)$, e.g. by polynomials of the variables $x, y, t$, using Weierstrass' approximation theorem. Hence, (27) is true for arbitrary $u \in C(D)$.□

## 5. Multipliers of $(C, *)$

Further, we consider the algebra $\mathcal{M}$ of the multipliers of $(C, *)$. Let us remind the definition of a multiplier of $(C, *)$. It is analogical to those of $(C_x, \overset{x}{*})$ and $(C_y, \overset{y}{*})$.

**Definition 2.** A linear operator $M : C \to C$ is said to be a multiplier of the convolution algebra $(C, *)$, iff the relation

$$M(u * v) = (Mu) * v$$

holds for all $u, v \in C$.

If $f \in C_x$, then the convolution operator $f \overset{x}{*}$ in $C_x = C[0, a]$ can be lifted to an operator in the space $C(D)$ by the natural lifting $(f \overset{x}{*})\{u(x,y,t)\} = \{f(x)\} \overset{x}{*} \{u(x,y,t)\}$, where the variables $y$ and $t$ are considered as parameters. The same is true for convolution operators: $g \overset{y}{*}$, where $g = g(y) \in C_y = C[0, b]$; $\varphi \overset{t}{*}$, $\varphi = \varphi(t) \in C[0, \infty)$; $v \overset{x,t}{*}$ where $v = v(x, t) \in C([0, a] \times [0, \infty))$; $w \overset{y,t}{*}$ where $w = w(y, t) \in C([0, b] \times [0, \infty))$; and $G \overset{x,y}{*}$ where $G = G(x, y) \in C([0, a] \times [0, b])$, correspondingly. Of course, the operator $\{u\} *$, where $u = u(x, y, t) \in C(D) = C([0, a] \times [0, b] \times [0, \infty))$, obviously, is also a multiplier of $(C, *)$.
Further, we use the notations

$$(28) \qquad [\varphi]_t = \{\varphi(t)\} \overset{t}{*}, \qquad [f]_x = \{f(x)\} \overset{x}{*}, \qquad [g]_y = \{g(y)\} \overset{y}{*},$$

$$[v]_{x,t} = \{v(x,t)\} \overset{x,t}{*}, \qquad [w]_{y,t} = \{w(y,t)\} \overset{y,t}{*}, \qquad [G]_{x,y} = \{G(x,y)\} \overset{x,y}{*}$$

and call them "numerical operators" with respect to the absent variables.
**Remark.** Here we deviate slightly from the notations, accepted in [6].

**Theorem 9.** *The convolution operators* (28), *lifted to* $C(D)$, *are multipliers of the convolution algebra* $(C, *)$.
**Proof.** Let $\varphi = \varphi(t) \in C[0, \infty)$. We are to prove that $[\varphi]_t (v * w) = \{[\varphi]_t v\} * w$ or, which is the same, $\varphi \overset{t}{*} (v * w) = (\varphi \overset{t}{*} v) * w$, where $v, w \in C(D)$. First we prove this for products: $v(x, y, t) = v_1(x)\, v_2(y)\, v_3(t)$, $w(x, y, t) = w_1(x)\, w_2(y)\, w_3(t)$.
We have

$$\varphi \overset{t}{*} (v * w) = \varphi \overset{t}{*} (v_1 v_2 v_3 * w_1 w_2 w_3) = \varphi \overset{t}{*} ((v_1 v_2 \overset{x,y}{*} w_1 w_2)(v_3 \overset{t}{*} w_3)) =$$

$$= (v_1 v_2 \overset{x,y}{*} w_1 w_2)(\varphi \overset{t}{*} (v_3 \overset{t}{*} w_3)) = (v_1 v_2 \overset{x,y}{*} w_1 w_2)((\varphi \overset{t}{*} v_3) \overset{t}{*} w_3) = (\varphi \overset{t}{*} v) * w.$$

But every function $v(x, y, t) \in C(D)$ can be approximated by linear combinations of product functions $v_1(x)\, v_2(y)\, v_3(t)$ with $v_1(x) \in C[0, a]$, $v_2(y) \in C[0, b]$ and $v_3(t) \in C[0, \infty)$, e.g. by polynomials of the variables $x, y, t$.

Next, let us take $f \in C([0, a] \times [0, \infty])$. We are to show that $[f]_{x,t} (v * w) = \{[f]_{x,t}\, v\} * w$ or $f \overset{x,t}{*} (v * w) = (f \overset{x,t}{*} v) * w$ where $v, w \in C(D)$.
Again, first we prove this for product functions

$v(x, y, t) = v_1(x)\, v_2(y)\, v_3(t)$ and $w(x, y, t) = w_1(x)\, w_2(y)\, w_3(t)$.

We have

$$f \overset{x,t}{*} (v * w) = f \overset{x,t}{*} (v_1 v_2 v_3 * w_1 w_2 w_3) = f \overset{x,t}{*} ((v_1 v_3 \overset{x,t}{*} w_1 w_3)\ (v_2 \overset{y}{*} w_2)) =$$

$$= (f \overset{x,t}{*} (v_1 v_3 \overset{x,t}{*} w_1 w_3))\ (v_2 \overset{y}{*} w_2) = ((f \overset{x,t}{*} v_1 v_3) \overset{x,t}{*} w_1 w_3)\ (v_2 \overset{y}{*} w_2) = (f \overset{x,t}{*} v) * w.$$

Any function $v(x, y, t) \in C(D)$ can be approximated by linear combination of functions of the form $v_1(x)\, v_2(y)\, v_3(t)$ where $v_1(x) \in C[0, a]$, $v_2(y) \in C[0, b]$, $v_3(t) \in C[0, \infty)$, e.g. by polynomials of $x, y, t$. In a similar way, we may prove that the operators

$$[f]_x = \{f(x)\} \overset{x}{*}, \quad [g]_y = \{g(y)\} \overset{y}{*}, \quad [w]_{y,t} = \{w(y,t)\} \overset{y,t}{*}, \quad [G]_{x,y} = \{G(x,y)\} \overset{x,y}{*}$$

are multipliers of $(C, *)$. □

## 5.1. Ring of the multiplier fractions of (C, ∗)

In $\mathcal{M}$ there are elements which are non-divisors of 0. Indeed, such elements are the multipliers $\{x\} \overset{x}{*} = [x]_x$ and $\{y\} \overset{y}{*} = [y]_y$, i.e. the operators $L_x$ and $L_y$.

Denote by $\mathcal{N}$ the set of the non-zero non-divisors of zero on $\mathcal{M}$. The set $\mathcal{N}$ is a multiplicative subset on $\mathcal{M}$, i.e. such that $p, q \in \mathcal{N}$ implies $p\, q \in \mathcal{N}$.

Further, we consider the multiplier fractions of the form $\dfrac{M}{N}$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$. They are introduced in a standard manner, using the well-known method of "localisation" from the general algebra [10].

Let $\mathcal{M} = \mathcal{N}^{-1}\mathcal{M}$ be the set of multiplier fractions of $(C, *)$. It is a commutative ring, containing the basic field ( ℝ or ℂ), the algebras $(C[0, a], \overset{x}{*})$, $(C[0, b], \overset{y}{*})$, $(C[0, \infty), \overset{t}{*})$, $(C[0, a] \times [0, \infty), \overset{x,t}{*})$, $(C[0, b] \times [0, \infty), \overset{y,t}{*})$, $C([0, a] \times [0, b], \overset{x,y}{*})$, $(C, *)$ and $\mathcal{M}$, due to the embeddings

i) $\mathbb{R} \hookrightarrow \mathcal{M}$ or $\mathbb{C} \hookrightarrow \mathcal{M}$ by $\alpha \mapsto \dfrac{\alpha L_x}{L_x}$,

ii) $(C[0, a], \overset{x}{*}) \hookrightarrow \mathcal{M}$ and $(C[0, b], \overset{y}{*}) \hookrightarrow \mathcal{M}$ by $f(x) \mapsto \dfrac{(L_x f) \overset{x}{*}}{L_x}$ and $g \mapsto \dfrac{(L_y g) \overset{y}{*}}{L_y}$,

iii) $(C[0, \infty), \overset{t}{*}) \hookrightarrow \mathcal{M}$ by $\varphi \mapsto \dfrac{[l_t\, \varphi]_t}{l_t} = \dfrac{(l_t\, \varphi) \overset{t}{*}}{l_t}$,

iv) $(C[0, a] \times [0, \infty), \overset{x,t}{*}) \hookrightarrow \mathcal{M}$ and $(C[0, b] \times [0, \infty), \overset{y,t}{*}) \hookrightarrow \mathcal{M}$ by

$$v \mapsto \frac{(l_t\, L_x v) \overset{x,t}{*}}{l_t\, L_x} \quad \text{and} \quad w \mapsto \frac{(l_t\, L_y w) \overset{y,t}{*}}{l_t\, L_y},$$

v) $(C([0, a] \times [0, b]), \overset{x,y}{*}) \hookrightarrow \mathcal{M}$ by $G \mapsto \dfrac{[L_x G]_{x,y}}{L_x} = \dfrac{[L_y G]_{x,y}}{L_y} = \dfrac{(L_x L_y G) \overset{x,y}{*}}{L_x L_y}$,

vi) $(C([0, a] \times [0, b] \times [0, \infty)), *) \hookrightarrow \mathcal{M}$ by $u \mapsto \dfrac{(l_t\, L_x L_y u) *}{l_t\, L_x L_y}$.

Further, we consider all numbers, functions, multiplier and multiplier fractions as elements of a *single algebraic system*: the ring $\mathcal{M}$ of the multiplier fractions.

**5.2. Elements of $\mathcal{M}$.**

In the ring $\mathcal{M}$ we introduce the algebraic inverses $s = \frac{1}{l_t}$, $S_x = \frac{1}{L_x}$ and $S_y = \frac{1}{L_y}$ of the multipliers $l_t$, $L_x$ and $L_y$, correspondingly.

**Lemma 5.** Each element $\frac{P}{Q}$ of $\mathcal{M}$ can be represented as $\frac{P}{Q} = \frac{p}{q}$ with $p, q \in C$.

**Proof.** Relation (27) can be written as $l_t L_x L_y u = \{x y\} u$. Denote $p = P\{x y\}$ and $q = Q\{x y\}$. We have $P u = s S_x S_y (p u)$ and $Q u = s S_x S_y (q u)$. Taking $u$ to be a nondivisor of zero in $\mathcal{M}$, we have $\frac{P}{Q} = \frac{Pu}{Qu} = \frac{pu}{qu} = \frac{p}{q}$.□

The elements $s$, $S_x$ and $S_y$ of $\mathcal{M}$ are connected with $\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ by relations, similar to those of Lemmas 1 and 3.

**Theorem 10.** *If* $u_{xx}$, $u_{yy}$ *and* $u_t$ *are continuous in D, then*

$$u_{xx} = S_x u + S_x\{(x\Phi_\xi\{1\} - 1)u(0, y, t)\} - [\Phi_\xi\{u(\xi, y, t)\}]_x,$$

(29)
$$u_{yy} = S_y u + S_y\{(y\Psi_\eta\{1\} - 1)u(x, 0, t)\} - [\Psi_\eta\{u(x, \eta, t)\}]_y,$$

$$u_t = s u - [u(x, y, 0)]_t.$$

**Theorem 11.** *If u is a classical solution of boundary value problem* (1)-(3), *then it reduces in* $\mathcal{M}$ *to a single algebraic equation*

(30)
$$[P(s) + Q(S_x) + R(S_y)]\, u = \widetilde{F},$$

*where* $\widetilde{F}$ *is a known element of* $\mathcal{M}$.

The problem of uniqueness of the solution of (1)-(3) reduces to the algebraic problem, weather $[P(s) + Q(S_x) + R(S_y)]$ is a divisor of zero or not.

**Lemma 6.** [7] Let $\{\lambda_n\}$ and $\{\mu_m\}$ be the eigenvalues of (7) and (8) for $n, m \in \mathbb{N}$, correspondingly. If there exists a dispersion relation of the form $Q(-\lambda_n^2) + R(-\mu_m^2) = 0$ for some $n, m \in \mathbb{N}$, then $Q(S_x) + R(S_y)$ is a divisor of zero in $\mathcal{M}$.

**Proof.** Since $S_x\{\sin \lambda_n x\} = -\lambda_n^2 x$, then $Q(S_x)\{\sin \lambda_n x\} = Q(-\lambda_n^2)\sin \lambda_n x$. Similarly $R(S_y)\{\sin \mu_m y\} = R(-\mu_m^2)\sin \mu_m y$. Hence

$$(Q(S_x)\sin\lambda_n x)\sin \mu_m y + \sin\lambda_n x\,(R(S_y)\sin \mu_m y) =$$
$$= (Q(-\lambda_n^2) + R(-\mu_m^2))\,\{\sin\lambda_n x \sin \mu_m y\} = 0.\ \square$$

**Corollary.** [7] Let $\lambda_n$ and $\mu_m$ be the eigenvalues of (7) and (8) for $n, m \in \mathbb{N}$, correspondingly. If there exists a dispersion relation of the form $\lambda_n^2 + \mu_m^2 = 0$ for some $n, m \in \mathbb{N}$, then $S_x + S_y$ is a divisor of zero in $\mathcal{M}$.

**Lemma 7.** [4] Let $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$. Then the elements $s - S_x$ and $s - S_y$ are non-divisors of zero in $\mathcal{M}$.

For a proof see [4].

**Theorem 12.** [7] *Let* $a \in \operatorname{supp} \Phi$, $b \in \operatorname{supp} \Psi$. *Then* $S_x + S_y$ *is a non-divisor of zero in* $\mathcal{M}$, *if and only if* $\lambda_n^2 + \mu_m^2 \neq 0$ *for all* $n, m \in \mathbb{N}$.
For a proof see [7].

**Theorem 13.** *Let both* $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$. *If* $\deg P \geq 1$, *then the element* $P(s) + Q(S_x) + R(S_y)$ *is a non-divisor of zero in* $\mathcal{M}$.
**Proof.** Assume the contrary. It is easy to see that $P(s) + Q(S_x) + R(S_y)$ would be a divisor of zero in $\mathcal{M}$ iff there exist a function $u \in C(D)$ with $\partial_x^{2l} u$, $\partial_y^{2m} u$ $\partial_t^k u \in C$, $l = 0, 1, \ldots, \deg Q - 1$, $m = 0, 1, \ldots \deg R - 1$, $k = 0, 1, \ldots, \deg P - 1$, $u \neq 0$, such that

(31) $$[P(s) + Q(S_x) + R(S_y)]\, u = 0.$$

Let $\lambda_n$ be an arbitrary eigenvalue of (7). Then $\lambda_n$ is a zero of the sine-indicatrix $E(\lambda) = \Phi_\xi \left\{ \frac{\sin \lambda \xi}{\lambda} \right\}$ of the functional $\Phi$. Let $\alpha_n$ be the multiplicity of $\lambda_n$ as a zero of $E(\lambda)$, i.e. $E(\lambda_n) = E'(\lambda_n) = \ldots = E^{(\alpha_n - 1)}(\lambda_n) = 0$, but $E^{(\alpha_n)}(\lambda_n) \neq 0$. To $\lambda_n$ it corresponds the following finite sequence of the eigenfunction $\sin \lambda_n x$ and the $\alpha_n - 1$ associated eigenfunctions:

$$\varphi_{n,k}(x) = \left(S_x + \lambda_n^2\right)^k \varphi_{n,0}(x), \quad 1 \leq k \leq \alpha_n - 1,$$

where

$$\varphi_{n,0}(x) = \frac{1}{\pi i} \int_{\Gamma_{\lambda_n}} \frac{\sin \lambda x}{\lambda E(\lambda)} d\lambda .$$

Here $\Gamma_{\lambda_n}$ is a contour in the complex plane, containing only the zero $\lambda_n$ of $E(\lambda)$ (see Dimovski and Petrova [5], p.94).
For the next considerations it is essential to note that $\varphi_{n,\alpha_n - 1}(x) = b_n \sin \lambda_n x$ with some $b_n \neq 0$.
The corresponding $\alpha_n$-dimensional eigenspace is

$$\mathcal{E}_{\lambda_n}^{(\alpha_n)} = \operatorname{span}\{\varphi_{n,s}(x),\ s = 0, 1, \ldots, \alpha_n - 1\} .$$

The spectral projectors $P_{\lambda_n} : C_x \to \mathcal{E}_{\lambda_n}^{(\alpha_n)}$ are given by $P_{\lambda_n}\{f\} = f \overset{x}{*} \varphi_n$. According to a theorem of N. Bozhinov [1], in the case $a \in \operatorname{supp} \Phi$, the projectors $P_{\lambda_n}$, $n \in \mathbb{N}$ form a total system of projectors, i.e. a system for which $P_{\lambda_n}\{f\} = 0$, $\forall\, n \in \mathbb{N}$ imply $f \equiv 0$. For a simple proof of Bozhinov's theorem for our case, see [5], p. 97–98.
In a similar way, we consider the eigenvalue problem (8). Let $\mu_m$ be an arbitrary eigenvalue of (8). Let $\beta_m$ be the multiplicity of $\mu_m$ as a zero of the sine-indicatrix $F(\mu) = \Psi_\eta \left\{ \frac{\sin \mu \eta}{\mu} \right\}$ of the functional $\Psi$. To $\mu_m$ it corresponds the following finite sequence of the eigenfunction $\sin \mu_m y$ and $\beta_m - 1$ associated eigenfunctions

$$\psi_{m,s}(y) = \left(S_y + \mu_m^2\right)^s \psi_{m,0}(y), \quad 1 \leq s \leq \beta_m - 1,$$

where

$$\psi_{m,0}(y) = \frac{1}{\pi i} \int_{\Gamma_{\mu_m}} \frac{\sin \mu y}{\mu F(\mu)} d\mu .$$

Here $\Gamma_{\mu_m}$ is a contour in the complex plane, containing only the zero $\mu_m$ of $F(\mu)$

(see Dimovski and Petrova [5], p.94).

Let us note that $\psi_{m,\beta_m-1}(y) = c_m \sin \mu_m y$ with some $c_m \neq 0$.

The corresponding $\beta_m$-dimensional eigenspace is

$$\mathcal{F}_{\mu_m}^{(\beta_m)} = \operatorname{span}\{\psi_{m,s}(y),\ s = 0, 1, \ldots, \beta_m - 1\}.$$

The spectral projectors $Q_{\mu_k} : C_y \to \mathcal{F}_{\mu_m}^{(\beta_m)}$ are given by $Q_{\mu_m}\{f\} = f \overset{y}{*} \psi_m$.

Denote $u_{n,m}(x,y,t) = u(x,y,t) \overset{x}{*} \varphi_n(x) \overset{y}{*} \psi_m(y)$. From (31) it follows

(32) $$[P(s) + Q(S_x) + R(S_y)]\, u_{n,m} = 0.$$

We will show that this equation has only the trivial solution $u_{n,m} \equiv 0$ in $\mathcal{E}_{\lambda_n}^{(\alpha_n)} \otimes \mathcal{F}_{\mu_m}^{(\beta_m)} \otimes C[0, \infty)$. Assume the contrary, i.e. that there exists a nonzero solution $u_{n,m} \in \mathcal{E}_{\lambda_n}^{(\alpha_n)} \otimes \mathcal{F}_{\mu_m}^{(\beta_m)} \otimes C[0, \infty)$ of (32). It should have the form

$$u_{n,m}(x,y,t) = \sum_{i=p}^{\alpha_n} \sum_{j=q}^{\beta_m} A_{i,j}(t)\, \varphi_{n,i}(x)\, \psi_{m,j}(y)$$

with $A_{p,q}(t) \neq 0$ for some $p$ and $q$, $0 \leq p \leq \alpha_n - 1$, $0 \leq q \leq \beta_m - 1$. We multiply (32) by the element $\left(S_x + \lambda_n^2\right)^{\alpha_n - p - 1} \left(S_y + \mu_m^2\right)^{\beta_m - q - 1}$ and obtain

$$[P(s) + Q(S_x) + R(S_y)]\, A_{p,q}(t)\, \varphi_{n,\alpha_n-1}(x)\, \psi_{m,\beta_m-1}(y) = 0,$$

since $\left(S_x + \lambda_n^2\right)^{\sigma} \varphi_{n,0} = 0$, for $\sigma \geq \alpha_n$ and $\left(S_y + \mu_m^2\right)^{j} \psi_{m,0} = 0$ for $j \geq \beta_m$.

But $\varphi_{n,\alpha_n-1}(x) = b_n \sin \lambda_n x$ with $b_n \neq 0$ and $\psi_{m,\beta_m-1}(y) = c_m \sin \mu_m y$ with $c_m \neq 0$.

Consider $[P(s) + Q(S_x) + R(S_y)]\, A_{p,q}(t) \sin \lambda_n x \sin \mu_m y = 0$ as an equation for $A_{p,q}(t)$. It is equivalent to the initial value problem

$$P\left(\frac{d}{dt}\right) A_{p,q}(t) + \left(Q(-\lambda_n^2) + R(-\mu_m^2)\right) A_{p,q}(t) = 0,$$

$$\frac{d^k}{dt^k} A_{p,q}(0) = 0, \qquad k = 0, \ldots, \deg P - 1.$$

The only solution is $A_{p,q}(t) \equiv 0$, which is a contradiction. Hence, $u_{n,m}(x, y, t) \equiv 0$ for all $n, m \in \mathbb{N}$. By N. Bozhinov's theorem it follows that $u(x, y, t) \equiv 0$. Thus we proved, that $P(s) + Q(S_x) + R(S_y)$ is a non-divisor of 0 in $\mathcal{M}$. □

**Corollary.** If $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$, then boundary value problem (1)-(3) has a unique solution.

**Proof.** Indeed, the homogeneous BVP (1)-(3) reduces to the algebraic equation $[P(s) + Q(S_x) + R(S_y)]\, u = 0$ in $\mathcal{M}$. Since $P(s) + Q(S_x) + R(S_y)$ is a nondivisor of zero in $\mathcal{M}$, then $u \equiv 0$. □

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