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## Точни решения на нелокални гранични задачи за двумерното уравнение на топлопроводността

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## Exact solutions of nonlocal BVPs for the two-dimensional heat equation

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# EXACT SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR ONE- AND TWO-DIMENSIONAL HEAT EQUATION

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## Abstract

It is proposed an operational method for obtaining of explicit solutions of a class of space-nonlocal BVPs for the two-dimensional heat equation. It is based on a direct three-dimensional operational calculus built on a three-dimensional convolution, combining the classical Duhamel convolution with two non-classical convolutions for the operators $\partial_{xx}$ and $\partial_{yy}$. The corresponding operational calculus uses multiplier fractions instead of convolution fractions. An extension of the Duhamel principle to the space variables is proposed. Thus is obtained explicit solutions of BVPs considered. The general approach is specialized to the case when some of the boundary value conditions are of integral type.

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**1. Introduction.** In Gutterman [1] an operational calculus approach to Cauchy problems for PDEs with constant coefficients is proposed. This approach did not apply to mixed initial-boundary value problems. According to Gutterman, such problems need new ideas and approaches. Here we use an operational calculus approach, developed in [9] to cope with BVPs for the two-dimensional heat equation

(1) $\quad u_t = u_{xx} + u_{yy} + F(x, y, t), \quad 0 < t, \quad 0 < x < a, \quad 0 < y < b$

with an initial condition

(2) $\quad u(x, y, 0) = f(x, y), \quad 0 \le x \le a, \quad 0 \le y \le b$

and with local and nonlocal BVCs of the form

(3) $\quad u(0, y, t) = 0, \quad \Phi_\xi\{ u(\xi, y, t)\} = p(y, t), \quad 0 \le t, \quad 0 \le y \le b,$

$\quad u(x, 0, t) = 0, \quad \Psi_\eta\{ u(x, \eta, t)\} = q(x, t), \quad 0 \le t, \quad 0 \le x \le a,$

where $\Phi$ and $\Psi$ are non-zero linear functionals on $C^1[0, a]$ and $C^1[0, b]$, correspondingly. Here $F(x, y, t)$, $f(x, y)$, $p(y, t)$ and $q(x, t)$ are given functions. We

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suppose that each of the carriers supp$\Phi$ and supp$\Psi$ of the functionals $\Phi$ and $\Psi$ contains at least one point, different from 0. In the next considerations we suppose also that $\Phi$ and $\Psi$ satisfy the normalizing restrictions:

$$\Phi_\xi\{\xi\} = 1 \qquad \text{and} \qquad \Psi_\eta\{\eta\} = 1.$$

These restrictions are made for the sake of simplification and they can be ousted by some unessential technical involvements.

**2. Weak solutions of BVP (1) – (3)**

It is natural to look for a classical solution of the BVP (1)-(3), but, in general, the sufficient conditions for the existence of such solutions may happen to be too restrictive. That's why we introduce the notion of a *weak solution* of (1)-(3). In order to give an exact meaning of this notion, we introduce some notations: in the domain $D = [0, a] \times [0, b] \times [0, \infty)$, we consider the integral operators

$$(4) \qquad l_t\{u(x,y,t)\} = \int_0^t u(x,y,\tau)d\tau,$$

and the right inverse operators $L_x$ and $L_y$ of $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial^2}{\partial y^2}$ given by

$$(5) \qquad L_x\{u(x,y,t)\} = \int_0^x (x-\xi)u(\xi,y,t)d\xi - x\Phi_\xi\{\int_0^\xi (\xi-\eta)u(\eta,y,t)d\eta\},$$

and

$$(6) \qquad L_y\{u(x,y,t)\} = \int_0^y (y-\eta)u(x,\eta,t)d\eta - y\Psi_\eta\{\int_0^\eta (\eta-\varsigma)u(x,\varsigma,t)d\varsigma\},$$

correspondingly. These operators are considered on $C(D)$. They satisfy the boundary value conditions $\Phi_x\{L_x u\} = 0$ and $\Psi_y\{ L_y u \} = 0$.

**Definition 1.** A function $u(x, y, t) \in C^1(D)$ is said to be a *weak solution* of problem (1)-(3), iff it satisfies the integral relation

$$(7) \qquad L_x L_y u - l_t L_y u - l_t L_x u = L_x L_y f(x, y) + l_t L_x L_y F(x, y, t) - \\ - x\, l_t L_y p(y, t) - y\, l_t L_x q(x, t).$$

Formally, (7) is obtained from equation (1) by application of the product operator $l_t L_x L_y$, followed by using BVCs (2)-(3). It is easy to show that each classical solution of (1)-(3) is a weak solution too. If it is happens $u \in C^2(D)$ then the converse is true. Nevertheless, we can prove that each *weak solution* satisfies the BVCs (2)-(3).

**Lemma 1.** Let $u \in C^1(D)$ satisfy (7). Then $u$ satisfies BVCs (2)-(3).

**Proof.** Taking $t = 0$ in (7), we find $L_x L_y\, u(x, y, 0) = L_x L_y\, f(x, y)$. Hence $u(x, y, 0) = f(x, y)$. For $x = 0$ we find $-l_t L_y u(0, y, t) = 0$ and, hence $u(0, y, t) = 0$. Next, applying $\Phi$ to (7), we get $-l_t L_y \Phi_\xi\{u(\xi, y, t)\} = -l_t L_y p(y, t)$. If apply $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial^2}{\partial y^2}$, we get $\Phi_\xi\{ u(\xi, y, t)\} = p(y, t)$. Analogously, for $y = 0$, from (7) we find $-l_t L_x u(x, 0, t) = 0$, and hence $u(x, 0, t) = 0$. Applying $\Psi$ to (7), we obtain

$-l_t L_x \Psi_\eta\{u(x,\eta,t)\} = -l_t L_x q(x,t)$. Then applying $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ we get

$\Psi_\eta\{u(x,\eta,t)\} = q(x,t)$. □

**Lemma 2.** Assume that $u$ is a solution of (7) with continuous partial derivatives $u_{xx}$, $u_{yy}$, $u_t$. Then $u$ is a classical solution of (1)-(3).

**Proof.** Applying the operator $\frac{\partial}{\partial t}\frac{\partial^4}{\partial x^2 \partial y^2}$ to (7), we get $u_t = u_{xx} + u_{yy} + F(x,y,t)$.

The fulfilment of the boundary value conditions follows from Lemma 1. □

Our final aim is to reduce the solution of BVP (1)-(3) to the following two nonlocal one-dimensional BVPs:

(8) $v_t = v_{xx}, \quad 0 < t\,,\ 0 < x < a,$

$v(x,0) = f(x),\ 0 \le x \le a, \qquad v(0,t) = 0, \qquad \Phi_\xi\{v(\xi,t)\} = 0,\ \ 0 \le t$

and

(9) $w_t = w_{yy}, \quad 0 < t, \quad 0 < y < b,$

$w(y,0) = g(y), \qquad 0 \le y \le b, \qquad w(0,t) = 0, \quad \Psi_\eta\{u(\eta,t)\} = 0, \quad 0 \le t.$

Next, with appropriate functions $f(x)$ and $g(y)$, we will consider the one-dimensional problems (8) and (9) independently of problem (1)-(3).

**Definition 2.** The functions $v = v(x,t) \in C^1([0,a]\times[0,\infty))$ and $w = w(y,t) \in C^1([0,b]\times[0,\infty))$ are said to be *weak solutions* of problems (8) and (9), if they satisfy the integral relations

(10) $L_x v - l_t v = L_x f(x)$

and

(11) $L_y w - l_t w = L_y g(y),$

correspondingly.

**Lemma 3.** If $v(x,t) \in C^1([0,a]\times[0,\infty))$ satisfy (10), then $v(x,t)$ satisfies the initial and boundary value conditions:

$v(x,0) = f(x), \qquad v(0,t) = 0, \qquad \Phi_\xi\{v(\xi,t)\} = 0.$

The proof is similar to that of Lemma 1, but a simpler one. We skip it.

Such is the relation between problem (9) and equation (11).

**Lemma 4.** If $v(x,t)$ with $v_{xx}(x,t)$, $v_t(x,t) \in C([0,a]\times[0,\infty))$ satisfy (10), then $v(x,t)$ is a classical solution of (8).

A similar relation holds for (11) too. The proof is similar to that of Lemma 2.

**Lemma 5.** Let $v(x,t) \in C^1([0,a]\times[0,\infty))$ and $w(y,t) \in C^1([0,b]\times[0,\infty))$ be weak solutions of problems (8) and (9), correspondingly. Then $u(x,y,t) = v(x,t)\,w(y,t) \in C(D)$ is a weak solution of the problem:

(12) $u_t = u_{xx} + u_{yy}\,,$

(13) $u(x,y,0) = f(x)\,g(y)\,,$

(14) $u(0,y,t) = 0, \qquad \Phi_\xi\{u(\xi,y,t)\} = 0\,,$

$u(x,0,t) = 0, \qquad \Psi_\eta\{u(x,\eta,t)\} = 0,$

in the sense of Definition 1.

**Remark.** If $v$ and $w$ are classical solutions, then we may assert that $u = v\,w$ is a classical solution of (12)-(14) too.

**Proof.** By Definition 2, we have:

(15) $$L_x v = l_t v + L_x f(x), \qquad L_y w = l_t w + L_y g(y).$$

According to Definition 1 we are to prove that:

$$L_x L_y \, v w - l_t L_y v w - l_t L_x v w = L_x L_y f(x) g(y).$$

Using (13), we find

$$L_x L_y v w - l_t L_y v w - l_t L_x w v = L_x v \, L_y w - l_t (v \, L_y w) - l_t (w L_x v) =$$

$$= (l_t v + L_x f(x)) (l_t w + L_y g(y)) - l_t (v (l_t w + L_y g(y))) - l_t ( w (l_t v + L_x f(x))) =$$

$$= (l_t v) (l_t w) + (l_t v) (L_y g(y)) + (l_t w)(L_x f(x)) + (L_x f(x)) (L_y g(y)) -$$
$$- l_t (v (l_t w) ) - l_t(v) (L_y g(y)) - l_t (w (l_t v)) - l_t (w) (L_x f(x)) =$$

$$= (l_t v) (l_t w) - l_t (v (l_t w) ) - l_t (w (l_t v)) + (L_x f(x)) (L_y g(y)).$$

In order to prove the assertion of the lemma, it remains to show that $(l_t v) (l_t w) - l_t (v (l_t w) ) - l_t (w (l_t v)) = 0$. Indeed,

$$(l_t v) (l_t w) - l_t (v (l_t w) ) - l_t (w (l_t v)) = \left(\int_0^t v(x,\tau) d\tau\right)\left(\int_0^t w(y,\tau) d\tau\right) -$$

$$- \int_0^t v(x,\tau) \left(\int_0^\tau w(y,\theta) d\theta\right) d\tau - \int_0^t w(y,\tau) \left(\int_0^\tau v(x,\theta) d\theta\right) d\tau.$$

Further, we get:

$$\int_0^t v(x,\tau) \left(\int_0^\tau w(y,\theta) d\theta\right) d\tau = \int_0^t \left(\int_0^\tau w(y,\theta) d\theta\right) d \int_0^\tau v(x,\sigma)\, d\sigma =$$

$$= \left(\int_0^t w(y,\theta) d\theta\right)\left(\int_0^t v(x,\sigma) d\sigma\right) - \int_0^t \left(\int_0^\tau v(x,\sigma) d\sigma\right) d \int_0^\tau w(y,\theta)\, d\theta =$$

$$= \left(\int_0^t w(y,\theta) d\theta\right)\left(\int_0^t v(x,\sigma) d\sigma\right) - \int_0^t w(y,\tau) \left(\int_0^\tau v(x,\sigma) d\sigma\right) d\tau.$$

Hence $(l_t v) (l_t w) - l_t (v (l_t w) ) - l_t (w (l_t v)) = 0$. Thus we proved relation (7) for $u = v\,w$, $p(y, t) = 0$, $q(x, t) = 0$ and $f(x, y) = f(x)g(y)$. Hence, $u = v\,w$ is a weak solution of (12)-(14). □

## 3. Convolutions

Here we will briefly remind the convolutions, introduced in [9].

### 3.1. One-dimensional convolutions

**1)** $f, g \in C\,[0, \infty)$, $\qquad (f * g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau$ (Duhamel convolution).

**2)** $f, g \in C_x = C\,[0, a]$, $\qquad (f \overset{x}{*} g)(x) = -\frac{1}{2} \widetilde{\Phi}_\xi \{h(x,\xi)\}$, (Dimovski [2])

with $\widetilde{\Phi} = \Phi_\xi \circ l_\xi$ and

$$h(x,\eta) = \int_x^\eta f(\eta + x - \varsigma) g(\varsigma) d\varsigma - \int_{-x}^\eta f(|\eta - x - \varsigma|) g(|\varsigma|) \operatorname{sgn}(\varsigma(\eta - x - \varsigma)) d\varsigma .$$

**3)** $f, g \in C_y = C[0, b]$, $\quad (f \overset{y}{*} g)(y) = -\frac{1}{2}\widetilde{\Psi}_\eta\{h(y,\eta)\}$,

with $\widetilde{\Psi} = \Psi_\eta \circ l_\eta$ (Dimovski [2]).

### 3.2. Two-dimensional convolutions

**1)** $f, g \in C([0, a] \times [0, \infty))$, $\quad f(x,t) \overset{(x,t)}{*} g(x,t) = \int_0^t f(x, t-\tau) \overset{x}{*} g(x,\tau)\, d\tau$, (see [4]).

**2)** $f, g \in C([0, b] \times [0, \infty))$, $\quad f(y,t) \overset{(y,t)}{*} g(y,t) = \int_0^t f(y, t-\tau) \overset{y}{*} g(y,\tau)\, d\tau$, (see [4]).

**Theorem 1.** (See [5]) *If* $f, g \in C([0, a] \times [0, b])$, *then*

$$f(x,y) \overset{(x,y)}{*} g(x,y) = \frac{1}{4}\widetilde{\Phi}_\xi \widetilde{\Psi}_\eta\{h(x,y,\xi,\eta)\}, \tag{16}$$

*with*

$$h(x,y,\xi,\eta) = \int_x^\xi \int_y^\eta f(\xi + x - \sigma, \eta + y - \tau)\, g(\sigma,\tau)\, d\sigma\, d\tau -$$

$$- \int_{-x}^\xi \int_y^\eta f(|\xi - x - \sigma|, \eta + y - \tau)\, g(|\sigma|, \tau) \operatorname{sgn}(\xi - x - \sigma)\sigma\, d\sigma\, d\tau -$$

$$- \int_x^\xi \int_{-y}^\eta f(\xi + x - \sigma, |\eta - y - \tau|)\, g(\sigma, |\tau|) \operatorname{sgn}(\eta - y - \tau)\tau\, d\sigma\, d\tau +$$

$$+ \int_{-x}^\xi \int_{-y}^\eta f(|\xi - x - \sigma|, |\eta - y - \tau|)\, g(|\sigma|, |\tau|) \operatorname{sgn}(\xi - x - \sigma)(\eta - y - \tau)\sigma\tau\, d\sigma\, d\tau.$$

*is a bilinear, commutative and associative operation in* $C([0, a] \times [0, b])$, such that

$$L_x L_y u(x,y,t) = \{x\, y\} \overset{(x,y)}{*} u(x,y,t).$$

For a proof see [5].

### 3.3. A 3-dimensional convolution in *C*(*D*)

**Theorem 2.** *Let* $u, v \in C(D)$. *Then the operation*

$$u(x,y,t) * v(x,y,t) = \int_0^t u(x,y,t-\tau) \overset{(x,y)}{*} v(x,y,\tau)\, d\tau, \tag{17}$$

*where by* $\overset{x,y}{*}$ *is denoted operation* (16), *is a bilinear, commutative and associative operation in* $C(D)$, *such that*

$$l_t L_x L_y\, u(x,y,t) = \{x\, y\} * u(x,y,t). \tag{18}$$

**Proof.** If $u(x, y, t) = f_1(x)\, g_1(y)\, h_1(t)$ and $v(x, y, t) = f_2(x)\, g_2(y)\, h_2(t)$ we have $u * v = (f_1 \overset{x}{*} f_2)(g_1 \overset{y}{*} g_2)(h_1 \overset{t}{*} h_2)$. The assertions of the Theorem 2 follow from the corresponding assertions about one-dimensional convolutions $\overset{t}{*}$, $\overset{x}{*}$ and $\overset{y}{*}$.

Let us prove (18). First we prove this for a product $u(x, y, t) = f(x)\, g(y)\, h(t)$. We have

$$l_t\, L_x L_y u(x,y,t) = l_t\, L_x L_y\, [f(x)g(y)h(t)] = L_x f(x).L_y g(y).l_t h(t) =$$

$$= [\{1\} \overset{t}{*} h(t)][\{x\} \overset{x}{*} f(x)][\{y\} \overset{y}{*} g(y)] = \{x\,y\} * \{f(x)g(y)h(t)\} = \{x\,y\} * u(x,y,t).$$

Then, (18) is true for linear combinations of such products. But every function $u(x, y, t) \in C(D)$ can be approximated by linear combinations of product functions $f(x)\, g(y)\, h(t)$ where $f(x) \in C[0, a]$ , $g(y) \in C[0, b]$ and $h(t) \in C[0, \infty)$, e.g. by polynomials of the variables $x, y, t$. Hence, (18) is true for arbitrary $u \in C(D)$.□

## 4. Ring of the multiplier fractions of $(C(D), *)$

We consider the convolution algebra $(C, *)$, where $C = C(D)$. Our direct operational calculus approach which we will apply to the two-dimensional heat equation is outlined in [10]. Here we will remind only some notations.

The multipliers of the form $\{u(x, y, t)\}*$ will be denoted by $\{u\}$ or $u$ and the result of the application of the operator $u*$ to a function $F \in C(D)$ will be denoted simply by $\{u\}F$ or $u\,F$.

**Definition 3.** Let $f$ be a function of the variable $x$ only in $C[0, a]$ and $g$ be a function of the variable $y$ only in $C[0, a]$, but both considered as functions of $C(D)$. The operators $[f]_x$ and $[g]_y$ defined by $[f]_x\, u = f \overset{x}{*} u$ and $[g]_y\, u = g \overset{y}{*} u$ are said to be partially numerical operators with respect to $y$ and $x$, correspondingly.

In this notations we have $L_x = [x]_x$ and $L_y = [y]_y$ .

The set of all the multipliers of the convolution algebra $(C, *)$ is a commutative ring $\mathcal{M}$. The multiplicative set $\mathcal{N}$ of the non-zero non-divisors of 0 in $\mathcal{M}$ is non-empty, since at least the operators $\{x\} \overset{x}{*} = [x]_x$ and $\{y\} \overset{y}{*} = [y]_y$ are non-divisors of 0.

Next we introduce the ring $\mathcal{M} = \mathcal{N}^{-1}\, \mathcal{M}$ of the multiplier fractions of the form $\dfrac{A}{B}$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. The standard algebraic procedure of constructing of this ring, named "localization", is described, e.g. in Lang [7]. Basic for our construction are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $S_y = \dfrac{1}{L_y}$ of the multipliers $L_x$ and $L_y$ in $\mathcal{M}$ correspondingly. If $u \in C^2(D)$, then, in general $S_x u$ and $S_y u$ are different from $u_{xx}$ and $u_{yy}$, but they are connected with them.

**Lemma 6.** Let $u_{xx}$, $u_{yy}$, $u_t$ be continuous on $D$. Then

$$u_{xx} = S_x u + S_x\{\, (x\Phi_\xi\{1\} - 1)u(0,y,t) - x\Phi_\xi\{u(\xi,y,t)\}\,\},$$

$$u_{yy} = S_y u + S_y\{\, (y\Psi_\eta\{1\} - 1)u(x,0,t)\} - y\Psi_\eta\{u(x,\eta,t)\}\,\},$$

$$u_t = s\,u - [u(x,y,0)]_t\,,$$

(See [4] and [10]).

## 5. Formal (generalized) solution of (1)-(3)

Let us consider problem (1)-(3). The equation (1) $u_t = u_{xx} + u_{yy} + F(x, y, t)$ together with the initial and boundary conditions (2) and (3) can be reduced to a single algebraic equation for $u$ in $\mathcal{M}$. Indeed, by Lemma 6, using (2) and (3), we get:

$$u_{xx} = S_x u - [p(y,t)]_{y,t}, \quad u_{yy} = S_y u - [q(x,t)]_{x,t}, \quad u_t = s\,u - [f(x,y]_{x,y}.$$

Then, (1)-(3) takes the following algebraic form in $\mathcal{M}$:

(19) $$(s - S_x - S_y)u = [f(x,y)]_{x,y} - [p(y,t)]_{y,t} - [q(x,t)]_{x,t} + F(x,y,t).$$

We may solve (19) in $\mathcal{M}$, provided $s - S_x - S_y$ is a non-divisor of zero in $\mathcal{M}$. Next, a sufficient condition of this is given.

**Theorem 3.** *If* $a \in$ supp $\Phi$ *and* $b\in$ supp $\Psi$, *then the element* $s - S_x - S_y$ *is a non-divisor of zero in* $\mathcal{M}$.

**Proof.** Assume the contrary. It is easy to see, that $s - S_x - S_y$ would be a divisor of zero in $\mathcal{M}$ iff there is a function $u$ with $u_{xx}, u_{yy}, u_t \in C$, $u \neq 0$, such that $(s - S_x - S_y)\,u = 0$. This relation is equivalent to

(20) $$(L_x L_y - l_t L_y - l_t L_x)\,u = 0.$$

Let $\lambda_n$ be an arbitrary eigenvalue of the elementary boundary value problems

$$u''(x) + \lambda^2 u(x) = f(x)\,, \quad x\in (0, a)\,, \quad u(0) = 0\,, \quad \Phi\{u\} = 0.$$

Then $\lambda_n$ is a zero of the sine-indicatrix $E(\lambda) = \Phi_\xi\left\{\frac{\sin \lambda\xi}{\lambda}\right\}$ of the functional $\Phi$. Let $\alpha_n$ be the multiplicity of $\lambda_n$ as a zero of $E(\lambda)$. To $\lambda_n$ it corresponds the following finite sequence of the eigenfunction $\sin\lambda_n x$ and the $\alpha_n - 1$ associated eigenfunctions:

$$\varphi_{n,s}(x) = \left(S_x + \lambda_n^2\right)^s \varphi_{n,0}\,, \quad 1 \le s \le \alpha_n - 1,$$

where

$$\varphi_{n,0}(x) = \frac{1}{\pi i}\int_{\Gamma_{\lambda_n}} \frac{\sin \lambda x}{\lambda E(\lambda)}\, d\lambda.$$

Here $\Gamma_{\lambda_n}$ is a contour in the complex plane, containing only the zero $\lambda_n$ of $E(\lambda)$ (see Dimovski and Petrova [6], p.94). Note that $\varphi_{n,\alpha_n-1}(x) = b_n \sin \lambda_n x$ with some $b_n \neq 0$ is the eigenfunction corresponding to $\lambda_n$.

The $\alpha_n$-dimensional eigenspace corresponding to $\lambda_n$ is

$$\mathcal{E}_{\lambda_n}^{(\alpha_n)} = \operatorname{span}\{\varphi_{n,s}(x),\ s = 0, 1, \ldots, \alpha_n - 1\}.$$

The spectral projector $P_{\lambda_n} : C_x \to \mathcal{E}_{\lambda_n}^{(\alpha_n)}$ is given by $P_{\lambda_n}\{f\} = f \overset{x}{*} \varphi_n$. According to a theorem of N. Bozhinov [6], in the case $a \in$ supp $\Phi$, the projectors $P_{\lambda_n}$ form a total system, i.e. a system for which $P_{\lambda_n}\{f\} = 0$, $\forall\ n\in \mathbb{N}$ imply $f \equiv 0$. For a simple proof of Bozhinov's theorem for our case, see [6] p. 97–98.

In a similar way, we consider the elementary eigenvalue problem

$$v''(y) + \mu^2 v(y) = g(y)\,, \quad y\in (0, b)\,, \quad v(0) = 0\,, \quad \Psi\{v\} = 0$$

Let $\mu_m$ be an arbitrary eigenvalue of (8). Let $\beta_m$ be the multiplicity of $\mu_m$ as a zero of the sine-indicatrix $F(\mu) = \Psi_\eta\left\{\frac{\sin \mu\eta}{\mu}\right\}$ of the functional $\Psi$. To $\mu_m$ it corresponds the following finite sequence of the eigenfunction and $\beta_m - 1$ associated eigenfunctions, corresponding to $\mu_m$:

$$\psi_{m,s}(x) = \left(S_y + \mu_m^2\right)^s \psi_{m,0}\,, \qquad 1 \le s \le \beta_m - 1,$$

where

$$\psi_{m,0}(y) = \frac{1}{\pi i}\int_{\Gamma_{\mu_m}} \frac{\sin \mu y}{\mu F(\mu)} d\mu$$

Here $\Gamma_{\mu_m}$ is a contour in the complex plane, containing only the zero $\mu_m$ of $F(\mu)$ (see Dimovski and Petrova [6], p.94). In fact $\psi_{m,\beta_m-1}(y) = c_m \sin \mu_m y$ with some $c_m \neq 0$ is the eigenfunction, corresponding to $\mu_m$. The corresponding $\beta_m$ - dimensional eigenspace is

$$\mathcal{F}_{\mu_m}^{(\beta_m)} = \operatorname{span}\{\psi_{m,s}(y),\ s = 0, 1, \ldots, \beta_m - 1\}.$$

The spectral projector $Q_{\mu_k} : C_y \to \mathcal{F}_{\mu_m}^{(\beta_m)}$ is given by $Q_{\mu_m}\{f\} = f \overset{y}{*} \psi_m$.

Denote $u_{n,m}(x,y,t) = [u(x,y,t) \overset{x}{*} \varphi_n(x)] \overset{y}{*} \psi_m(y)$. From $(L_x L_y - l_t L_y - l_t L_x)\, u = 0$ it follows

(21) $$(L_x L_y - l_t L_y - l_t L_x)\, u_{n,m} = 0.$$

We will show that this equation has only the trivial solution $u_{n,m} \equiv 0$ in $\mathcal{E}_{\lambda_n}^{(\alpha_n)} \otimes \mathcal{F}_{\mu_m}^{(\beta_m)} \otimes C[0, \infty)$. Assume the contrary, i.e. that there exists a non-zero solution $u_{n,m} \in \mathcal{E}_{\lambda_n}^{(\alpha_n)} \otimes \mathcal{F}_{\mu_m}^{(\beta_m)} \otimes C[0, \infty)$ of (21). It should have the form

(22) $$u_{n,m}(x,y,t) = \sum_{i=p}^{\alpha_n}\sum_{j=q}^{\beta_m} A_{i,j}(t)\, \varphi_{n,i}(x)\, \psi_{m,j}(y)$$

with $A_{p,q}(t) \neq 0$ for some $p$ and $q$, $0 \le p \le \alpha_n - 1,\ 0 \le q \le \beta_m - 1$. We multiply (21) by $\left(S_x + \lambda_n^2\right)^{\alpha_n - p - 1}\left(S_y + \mu_m^2\right)^{\beta_m - q - 1}$ and obtain

$$(L_x L_y - l_t L_y - l_t L_x) A_{p,q}(t)\varphi_{n,\alpha_n-1}(x)\psi_{m,\beta_m-1}(y) = 0,$$

since $\left(S_x + \lambda_n^2\right)^s \varphi_{n,0} = 0$, for $s \ge \alpha_n$ and $\left(S_y + \mu_m^2\right)^j \psi_{m,0} = 0$ for $j \ge \beta_m$.

But $\varphi_{n,\alpha_n-1}(x) = b_n \sin \lambda_n x$ with $b_n \neq 0$ and $\psi_{m,\beta_m-1}(y) = c_m \sin \mu_m y$ with $c_m \neq 0$. Consider $(L_x L_y - l_t L_y - l_t L_x)\, A_{p,q}(t) \sin \lambda_n x \sin \mu_m y = 0$ as an equation for $A_{p,q}(t)$. It is equivalent to the initial value problem

$$A_{p,q}'(t) + (\lambda_n^2 + \mu_m^2) A_{p,q}(t) = 0\,, \qquad A_{p,q}(0) = 0.$$

The only solution is $A_{p,q}(t) \equiv 0$, which is a contradiction. Hence, $u_{n,m}(x, y, t) \equiv 0$ for all $n, m \in \mathbb{N}$. By N. Bozhinov's theorem it follows that $u(x, y, t) \equiv 0$. Thus we proved, that $s - S_x - S_y$ is a non-divisor of 0 in $\mathcal{M}$. □

**Remark.** Theorem 3 is a special case of Theorem 13 from [10].

**Corollary.** If $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$ then boundary value problem (1)-(3) has a unique solution.

Indeed, the homogeneous BVP (1)-(3) reduces to the algebraic equation $(s - S_x - S_y)\, u = 0$ in $\mathcal{M}$ and hence $u \equiv 0$, since $s - S_x - S_y$ is a non-divisor of zero in $\mathcal{M}$.

From now on, we suppose that $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$.
The formal solution of (19) is

$$(23)\qquad u = \frac{1}{s - S_x - S_y}\left( [f(x,y)]_t - [p(y,t)]_x - [q(x,t)]_y + F(x,y,t) \right).$$

Similarly, considering the algebras $(C[0, a]\times[0, \infty), \overset{x,t}{*})$ and $(C[0, b]\times[0, \infty), \overset{y,t}{*})$ and their rings of multiplier fractions $\mathcal{M}_{x,t}$ and $\mathcal{M}_{y,t}$, the problem (8) and (9) have the formal solutions

$$(24)\qquad v = \frac{1}{s - S_x}\left( [f_1(x)]_x - [p_1(t)]_t + F_1(x,t) \right)$$

and

$$(25)\qquad w = \frac{1}{s - S_y}\left( [f_2(y)]_y - [p_2(t)]_t + F_2(y,t) \right)$$

in $\mathcal{M}_{x,t}$ and $\mathcal{M}_{y,t}$ since $s - S_x$ and $s - S_y$ are non-divisors of zero (see [3]).

## 6. Interpretation of the formal (generalized) solution of (1)-(3) as a function

**6.1.** Our next task is to interpret (23) as a function of $C([0, a] \times [0, b] \times [0, \infty))$. To this end, we consider (1)-(3) for $F(x, y, t) \equiv p(y, t) \equiv q(x, t) \equiv 0$ and $f(x, y) = x\,y$. We denote its weak solution, if it exists, by $U = U(x, y, t)$. We have the following algebraic representation of this solution:

$$U = \frac{1}{s - S_x - S_y}[x\,y]_{x,y} = \frac{1}{s - S_x - S_y}(L_x L_y) = \frac{1}{S_x S_y (s - S_x - S_y)}.$$

Analogically, we denote the weak solutions of the problems (8) and (9) for $f(x) = x$ and $g(y) = y$ by $V = V(x, t)$ and $W = W(y, t)$, correspondingly. Then the algebraic representations of these solutions are

$$V = \frac{1}{S_x(s - S_x)} \quad\text{and}\quad W = \frac{1}{S_y(s - S_y)}.$$

**Theorem 4.** *Assume that* $V = \dfrac{1}{S_x(s - S_x)}$ *and* $W = \dfrac{1}{S_y(s - S_y)}$ *are weak solutions of* (8) *and* (9) *for* $f(x) = x$ *and* $g(y) = y$*, correspondingly. Then* $U = \dfrac{1}{S_x S_y (s - S_x - S_y)} = \{V\,W\}$*, where* $W\,V = V(x, t)\, W(y, t)$ *is the ordinary product of* $V$ *and* $W$*, is a weak solution of* (1)-(3) for $p(y, t) = q(x, t) = F(x, y, t) \equiv 0$ *and* $f(x, y) = x\,y$.
The proof follows immediately from Lemma 5.

The generalized solution of problem (1)-(3) for arbitrary $f(x, y)$, $p(y, t)$, $q(x, t)$ and $F(x, y, t)$ can by represented in the form:

$$u = S_x S_y \left( \frac{1}{S_x S_y (s - S_x - S_y)}[f(x,y)]_t - \frac{1}{S_x S_y (s - S_x - S_y)}[p(y,t)]_x - \right.$$
$$\left. - \frac{1}{S_x S_y (s - S_x - S_y)}[q(x,t)]_y + \frac{1}{S_x S_y (s - S_x - S_y)} F(x,y,t) \right).$$

As a function it has the form

(26) $$u=\frac{\partial^4}{\partial x^2\partial y^2}\left[U\overset{x,y}{*}f(x,y)-U\overset{y,t}{*}p(y,t)-U\overset{x,t}{*}q(x,t)+U*F(x,y,t)\right]$$

provided the denoted derivatives exist.

Let us consider the problem (1)-(3) for $p(y,t)=q(x,t)=F(x,y,t)\equiv 0$. Then

$$u=\frac{\partial^4}{\partial x^2\partial y^2}\left(U(x,y,t)\overset{x,y}{*}f(x,y)\right)=\frac{\partial^4}{\partial x^2\partial y^2}\left((V(x,t)\,W(y,t))\overset{x,y}{*}f(x,y)\right)=$$

$$=\frac{\partial^4}{\partial x^2\partial y^2}\left(V(x,t)\overset{x}{*}\left(W(y,t)\overset{y}{*}f(x,y)\right)\right)=\frac{\partial^2}{\partial x^2}\left(V(x,t)\overset{x}{*}\frac{\partial^2}{\partial y^2}\left(W(y,t)\overset{y}{*}f(x,y)\right)\right)=$$

$$=V(x,t)\overset{x}{\tilde{*}}\left(W(y,t)\overset{y}{\tilde{*}}f(x,y)\right).$$

where the operations $\overset{x}{\tilde{*}}$ in $C[0,a]$ and $\overset{y}{\tilde{*}}$ in $C[0,b]$ are defined as

$$f(x)\overset{x}{\tilde{*}}g(x)=\frac{\partial^2}{\partial x^2}(f(x)\overset{x}{*}g(x))\quad\text{and}\quad f(y)\overset{y}{\tilde{*}}g(y)=\frac{\partial^2}{\partial y^2}(f(y)\overset{y}{*}g(y)).$$

If $f(x,y)=f_1(x)f_2(y)$, then

$$u=(V(x,t)\overset{x}{\tilde{*}}f_1(x))(W(y,t)\overset{y}{\tilde{*}}f_2(y)),$$

This is the desired explicit solution of (1)-(3) for $f(x,y)=f_1(x)f_2(y)$.

**6.2.** Let us consider BVP (1)-(3) with $F(x,y,t)\equiv p(y,t)\equiv q(x,t)\equiv 0$ and

$$f(x,y)=L_x\{x\}L_y\{y\}=\frac{1}{S_x^2S_y^2}=\left(\frac{x^3}{6}-\frac{x}{6}\Phi_\xi\{\xi^3\}\right)\left(\frac{y^3}{6}-\frac{y}{6}\Psi_\eta\{\eta^3\}\right).$$

We denote the solution of this problem by $\Omega=\Omega(x,y,t)$. Then we have the following algebraic representation of (23):

$$\Omega=\frac{1}{s-S_x-S_y}(L_x\{x\}\,L_y\{y\})=\frac{1}{S_x^2S_y^2(s-S_x-S_y)}.$$

Analogically we denote the weak solutions of problems (8) and (9) for $f(x)=L_x\{x\}=\frac{1}{S_x^2}=\frac{x^3}{6}-\frac{x}{6}\Phi_\xi\{\xi^3\}$ and $g(y)=L_y\{y\}=\frac{1}{S_y^2}=\frac{y^3}{6}-\frac{y}{6}\Psi_\eta\{\eta^3\}$ by $H=H(x,t)$ and $K=K(y,t)$, correspondingly. Then the algebraic representations of these solutions are

$$H=\frac{1}{S_x^2(s-S_x)}\quad\text{and}\quad K=\frac{1}{S_y^2(s-S_y)}.$$

**Theorem 5.** *Assume that* $H=\frac{1}{S_x^2(s-S_x)}$ *and* $K=\frac{1}{S_y^2(s-S_y)}$ *are weak solutions of (8) and (9) for* $f(x)=\frac{x^3}{6}-\frac{x}{6}\Phi_\xi\{\xi^3\}$ *and* $g(y)=\frac{y^3}{6}-\frac{y}{6}\Psi_\eta\{\eta^3\}$, *correspondingly.*

*Then* $\Omega=\frac{1}{S_x^2S_y^2(s-S_x-S_y)}=\{H\,K\}$, *where* $H\,K=H(x,t)\,K(y,t)$ *is the ordinary product of* $H$ *and* $H$, *is a weak solution of* (1)-(3) for $p(y,t)=q(x,t)=F(x,y,t)\equiv 0$ *and* $f(x,y)=\left(\frac{x^3}{6}-\frac{x}{6}\Phi_\xi\{\xi^3\}\right)\left(\frac{y^3}{6}-\frac{y}{6}\Psi_\eta\{\eta^3\}\right)$.

The proof follows immediately from Lemma 5.

The solution of problem (1)-(3) for arbitrary $f(x, y)$, $p(y, t)$, $q(x, t)$ and $F(x, y, t)$ can by represented in the form:

$$u = S_x^2 S_y^2 \left( \frac{1}{S_x^2 S_y^2 (s - S_x - S_y)} [f(x,y)]_t - \frac{1}{S_x^2 S_y^2 (s - S_x - S_y)} [p(y,t)]_x - \right.$$
$$\left. - \frac{1}{S_x^2 S_y^2 (s - S_x - S_y)} [q(x,t)]_y + \frac{1}{S_x^2 S_y^2 (s - S_x - S_y)} F(x,y,t) \right)$$

which can be interpreted as

$$(27) \qquad u = \frac{\partial^8}{\partial x^4 \partial y^4} \left[ \Omega \overset{x,y}{*} f(x,y) - \Omega \overset{y,t}{*} p(y,t) - \Omega \overset{x,t}{*} q(y,t) + \Omega \overset{x,y,t}{*} F(x,y,t) \right].$$

Assuming some smoothness conditions for the given functions we may assert that (27) is either weak, or classical solution of (1)-(3).
In order to reveal further the structure of the solution, we may introduce the auxiliary operations

$$f(x) \overset{x}{\circ} g(x) = \frac{\partial^2}{\partial x^2} (f(x) \overset{x}{\tilde{*}} g(x)) \quad \text{and} \quad f(y) \overset{y}{\circ} g(y) = \frac{\partial^2}{\partial y^2} (f(y) \overset{y}{\tilde{*}} g(y)).$$

Let us consider problem (1)-(3) for $p(y, t) = q(x, t) = F(x, y, t) \equiv 0$. We get

$$u = \frac{\partial^2}{\partial x^2} \left( V(x,t) \overset{x}{\tilde{*}} \frac{\partial^2}{\partial y^2} \left( W(y,t) \overset{y}{\tilde{*}} f(x,y) \right) \right) = V(x,t) \overset{x}{\circ} (W(y,t) \overset{y}{\circ} f(x,y)).$$

If $f(x, y) = f_1(x) f_2(y)$, then

$$u = (V(x,t) \overset{x}{\circ} f_1(x))(W(y,t) \overset{y}{\circ} f_2(y)).$$

## 7. Examples

**Problem 1:** $u_t = u_{xx} + u_{yy} + x\,y$,
$u(x, y, 0) = 0$, $\quad u(0, y, t) = 0$, $\quad \Phi_\xi\{u(\xi, y, t)\} = y\,t$,
$u(x, 0, t) = 0$, $\quad \Psi_\eta\{u(x, \eta, t)\} = x\,t$.

**Solution:**

$F(x, y, t) = x\,y = l_t L_x L_y$,

$u_{xx} = S_x u - [yt]_{y,t} = S_x u - L_y l_t^2$,

$u_{yy} = S_y u - [xt]_{x,t} = S_y u - L_x l_t^2$, $\qquad u_t = s\,u$.

The problem reduces to the single algebraic equation:

$(s - S_x - S_y) u = -L_y l_t^2 - L_x l_t^2 + l_t L_x L_y$, with the solution

$$u = \frac{1}{s - S_x - S_y}(-L_y l_t^2 - L_x l_t^2 + l_t L_x L_y) = \frac{1}{s - S_x - S_y} \left( -\frac{1}{S_y s^2} - \frac{1}{S_x s^2} + \frac{1}{s S_x S_y} \right) =$$
$$= \frac{1}{s - S_x - S_y} \frac{s - S_y - S_x}{S_x S_y s^2} = \frac{1}{s^2 S_x S_y} = l_t^2 L_x L_y = t\,x\,y.$$

Thus we obtain the classical solution $u = t\,x\,y$.

**Problem 2:** $u_t = u_{xx} + u_{yy}$, $\qquad u(x, y, 0) = x\,y$,
$u(0, y, t) = 0$, $\qquad \Phi_\xi\{u(\xi, y, t)\} = y$,

$$u(x, 0, t) = 0, \qquad \Psi_\eta\{u(x, \eta, t)\} = x,$$

**Solution:**

$$[f(x, y)]_{x,y} = [x\, y]_{x,y} = L_x L_y ,$$
$$u_{xx} = S_x u - [y]_y = S_x u - L_y l_t ,$$
$$u_{yy} = S_y u - [xt]_{t,x} = S_y u - L_x l_t , \qquad u_t = su .$$

The problem reduces to the single algebraic equation:

$$(s - S_x - S_y)u = L_x L_y - L_y l_t - L_x l_t ,$$

with the solution

$$u = \frac{1}{s - S_x - S_y}(L_x L_y - L_y l_t - L_x l_t) = \frac{1}{s - S_x - S_y}\left(\frac{1}{S_x S_y} - \frac{1}{S_y s} - \frac{1}{S_x s}\right) =$$
$$= \frac{1}{s - S_x - S_y}\,\frac{s - S_x - S_y}{S_x S_y s} = \frac{1}{s S_x S_y} = l_t\, L_x L_y = x\, y.$$

**Remark.** Note that the solutions of problems 1) and 2) do not depend on the specific choice of the functionals Φ and Ψ.

In the next Problem 3, the functionals Φ and Ψ are of Samarskii-Ionkin type (see [8]). Here we look for a classical solution of the BVP considered.

**Problem 3.** Solve the boundary value problem:

$$
\begin{aligned}
&u_t = u_{xx} + u_{yy} , \quad 0 < x < a, \quad 0 < y < b, \; t > 0\\
&u(x, y, 0) = f(x, y),\\
(28) \qquad &u(0, y, t) = 0 , \qquad u(x, 0, t) = 0,\\
&\int_0^a u(\xi, y, t)\,d\xi = 0 , \qquad \int_0^b u(x, \eta, t)\,d\eta = 0 .
\end{aligned}
$$

**Solution:**

We consider the following two one-dimensional BVPs:

(29) $\quad v_t = v_{xx} , \quad 0 < x < a, \quad t > 0,$

$$v(x, 0) = f(x), \quad v(0, t) = 0 , \qquad \int_0^a v(\xi, t)\,d\xi = 0 ,$$

(here $\Phi\{f\} = \dfrac{2}{a^2}\displaystyle\int_0^a f(\xi)\,d\xi$ )

and

(30) $\quad w_t = w_{yy} , \quad 0 < y < b, \quad t > 0,$

$$w(y, 0) = g(y), \quad w(0, t) = 0 , \qquad \int_0^b w(\eta, t)\,d\eta = 0 ,$$

(here $\Psi\{g\} = \dfrac{2}{b^2}\displaystyle\int_0^b g(\eta)\,d\eta$ ).

We find two representation of the solution of (28).

First we use representation (26) from 6.1. of solution (23).
Applying the method, used in [8] for $f(x) = x$, we find the weak solution of problem (29):

$$V(x,t)=2\sum_{n=1}^{\infty}e^{-\lambda_n^2 t}(2\lambda_n\, t\sin\lambda_n x - x\cos\lambda_n x), \qquad \text{with } \lambda_n=\frac{2n\pi}{a}, \quad n=1,2,\ldots.$$

Similarly, if $g(y)=y$, then for problem (30) we find

$$W(y,t)=2\sum_{m=1}^{\infty}e^{-\mu_m^2 t}(2\mu_m\, t\sin\mu_m y - y\cos\mu_m y), \qquad \text{with } \mu_m=\frac{2m\pi}{b}, m=1,2,\ldots.$$

Let $f,\ g\in C_x = C[0,\ a]$. In the case of $\Phi_\xi\{f(\xi)\}=\frac{2}{a^2}\int_0^a f(\xi)\,d\xi$ and $\Psi\{g\}=\frac{2}{b^2}\int_0^b g(\eta)d\eta$ the convolutions $\overset{x}{*}$ and $\overset{y}{*}$ are two times differentiable. Indeed, we have

$$(f\overset{x}{*}g)(x)=-\frac{1}{a^2}\int_0^a\int_0^\xi h(x,\eta)\,d\eta\,d\xi, \quad \text{and} \quad (f\overset{y}{*}g)(x)=-\frac{1}{b^2}\int_0^b\int_0^\xi h(y,\eta)\,d\eta\,d\xi$$

and, after the differentiations, we get

$$(f\overset{\tilde{x}}{*}g)(x)=\frac{\partial^2}{\partial x^2}\left((f\overset{x}{*}g)(x)\right)=-\frac{1}{a^2}\left(h(x,a)-h(x,0)-2f(x)\int_0^a g(\eta)\,d\eta-2g(x)\int_0^a f(\eta)\,d\eta\right)$$

where

$$h(x,a)=\int_x^a f(a+x-\varsigma)g(\varsigma)d\varsigma-\int_{-x}^a f(|a-x-\varsigma|)g(|\varsigma|)\operatorname{sgn}(\varsigma(a-x-\varsigma))d\varsigma$$

and

$$h(x,0)=-2\int_0^x f(x-\varsigma)g(\varsigma)d\varsigma.$$

Let $f,g\in C[0,a]$ and $\int_0^a f(\eta)\,d\eta=\int_0^a g(\eta)\,d\eta=0$. We have

$$(f\overset{\tilde{x}}{*}g)(x)=\frac{\partial^2}{\partial x^2}\left((f\overset{x}{*}g)(x)\right)=$$

$$=-\frac{1}{a^2}\left(\int_x^a f(a+x-\varsigma)g(\varsigma)d\varsigma-\int_{-x}^a f(|a-x-\varsigma|)g(|\varsigma|)\operatorname{sgn}(\varsigma(a-x-\varsigma))d\varsigma-2\int_0^x f(x-\varsigma)g(\varsigma)d\varsigma\right).$$

Analogically we have

$$(f\overset{\tilde{y}}{*}g)(y)=\frac{\partial^2}{\partial y^2}\left((f\overset{y}{*}g)(y)\right)=$$

$$=-\frac{1}{b^2}\left(\int_y^b f(b+y-\varsigma)g(\varsigma)d\varsigma-\int_{-y}^b f(|b-y-\varsigma|)g(|\varsigma|)\operatorname{sgn}(\varsigma(b-y-\varsigma))d\varsigma-2\int_0^y f(y-\varsigma)g(\varsigma)d\varsigma\right).$$

Then the solution of (28) (Problem 3) is

$$(31) \qquad u=\frac{\partial^4}{\partial x^2\partial y^2}\left([V(x,t)W(y,t)]\overset{(x,y)}{*}f(x,y)\right)=V(x,t)\overset{\tilde{x}}{*}\left(W(y,t)\overset{\tilde{y}}{*}f(x,y)\right).$$

It is a weak solution of Problem 3 in the sense of Definition 1.
Looking for a classical solution of Problem 3 we are to impose some smoothness restriction on the function $f(x, y)$.
If $f(x,y)=f_1(x)\,f_2(y)$, then the solution of (28) is:

(32) $$u = (V(x,t) \overset{x}{\tilde{*}} f_1(x))\,(W(y,t) \overset{y}{\tilde{*}} f_2(y)).$$

This time, we are to use representation (27) from 6.2. of the solution of (23).

The solution of (29) for $f(x) = \frac{x^3}{6} - \frac{a^2 x}{12} = L_x\{x\}$ is

$$H(x,t) = -2\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\left(2(\frac{1}{\lambda_n^3} + \frac{1}{\lambda_n^2}\lambda_n t)\sin\lambda_n x - \frac{1}{\lambda_n^2} x\cos\lambda_n x\right), \quad \text{where} \quad \lambda_n = \frac{2n\pi}{a}.$$

Analogically, the solution of (30) for $g(y) = \frac{y^3}{6} - \frac{b^2 y}{12} = L_y\{y\}$ is

$$K(y,t) = -2\sum_{m=1}^{\infty} e^{-\mu_m^2 t}\left(2(\frac{1}{\mu_m^3} + \frac{1}{\mu_m^2}\lambda_n t)\sin\mu_m y - \frac{1}{\mu_m^2} x\cos\mu_m y\right), \quad \text{where} \quad \mu_m = \frac{2m\pi}{b}.$$

$H$ and $K$ are obtained following Ionkin's (see [8]) approach.

**Lemma 7.** Let $f, g \in C^1[0, a]$ and $f(0) = g(0) = \int_0^a f(\xi)d\xi = \int_0^a g(\xi)d\xi = 0$, then

$$f \overset{x}{\circ} g = \frac{\partial^4}{\partial x^4}\left((f \overset{x}{*} g)(x)\right) = \frac{\partial^2}{\partial x^2}\left((f \overset{x}{\tilde{*}} g)(x)\right) =$$

$$= -\frac{1}{a^2}\left(\int_x^a f'(a+x-\varsigma)g'(\varsigma)d\varsigma - \int_{-x}^a f'(|a-x-\varsigma|)\,g'(|\varsigma|)\,d\varsigma + 2\int_0^x f'(x-\varsigma)g'(\varsigma)d\varsigma\right).$$

**Proof.** By direct check. □

**Theorem 5.** Let $f \in C(D)$ be such that $f_x(x, y), f_y(x, y) \in C([0, a] \times [0, b])$ and $\int_0^a f(\xi, y)\,d\eta = \int_0^b f(x,\eta)\,d\eta = 0$, Then

(33) $$u = H(x,t) \overset{x}{\circ} (K(y,t) \overset{y}{\circ} f(x,y))$$

is a weak solution of (28).

If suppose additionally $f(x, y) \in C^2(D)$, then (33) would be a classical solution of (28).

If $f(x, y) = f_1(x)\,f_2(y)$, then the solution of (28) is:

$$u = (H(x,t) \overset{x}{\circ} f_1(x))\,(K(y,t) \overset{y}{\circ} f_2(y)) =$$

$$= \frac{1}{a^4}\left(\int_0^a f_1'(\eta)(H_x(a+x-\eta,t) - H_x(a-x-\eta,t))d\eta - \right.$$

$$\left. - \int_0^x f_1'(\eta)\big(H_x(\eta+a-x,t) + H_x(a+x-\eta,t) - 2H_x(x-\eta,t)\big)d\eta\right) \times$$

$$\times\left(\int_0^b f_2'(\eta)\big(K_y(b+y-\eta,t) - K_y(b-y-\eta,t)\big)d\eta - \right.$$

$$\left. - \int_0^x f_2'(\eta)\big(K_y(\eta+b-y,t) + K_y(b+y-\eta,t) - 2K_y(y-\eta,t)\big)d\eta\right).$$

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