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Square roots of bicomplex number and quadratic equation of bicomplex variable

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Square roots of bicomplex number and quadratic equation of bicomplex variable*†

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Abstract

Square roots of bicomplex number are found. Quadratic equation of bicomplex variable is solved.

1 Square roots of bicomplex number.

The bicomplex numbers are introduced by C. Segre in [3]. Algebraic investigations of these numbers and of the hyperbolic numbers, which form their subalgebra, are made in [2], [5]. Functions of bicomplex variable are study in [1], [4].

Let us recall the definition of the algebra of bicomplex numbers $B\mathbb{C}$. It is true

$$B\mathbb{C} = \{x + iy + ju + ijv : \ i^2 = j^2 = -1, \ ij = ji, \ x, y, u, v \in \mathbb{R}\}.$$

The addition and the multiplication by real scalar are defined componentwise, and the multiplication of elements of the algebra is defined by opening the brackets and using the identities of the units i and j. The algebra $B\mathbb{C}$ is an associative, commutative algebra with divisors of zero. So are for example the numbers $X(ij-1)$, where X is an arbitrary bicomplex number. Actually, the product of this number with $ij+1$ is equal to zero, as $X(ij-1)(ij+1) = X(i^2j^2 - 1) = X((-1)(-1) - 1) = X \cdot 0 = 0$.

We would like to find the square roots of bicomplex number $A = a + ib + jc + ijd$, i.e. to solve the equation $X^2 = A$, where $X = x + iy + ju + ijv$ and a, b, c, d, x, y, u and v are real numbers.

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To do this we write

$$(x + iy + ju + ijv)^2 = a + ib + jc + ijd \tag{1}$$

and we obtain

$$\begin{aligned}(x + iy + ju + ijv)^2 &= x^2 + ixy + jxu + ijxv + iyx - \\ -y^2 + ijyu - jyv + jux + jiuy - u^2 - iuv + ijvx - jvy - ivu + v^2 \\ = x^2 - y^2 - u^2 + v^2 + 2ixy + 2jxu + 2ijxv + 2ijyu - 2jyv - 2iuv = \\ = a + ib + jc + ijd.\end{aligned} \tag{2}$$

So arise the following system of four quadratic equations with four real variables x, y, u, v and four real parameters a, b, c, d

$$x^2 - y^2 - u^2 + v^2 = a, \tag{3}$$

$$2xy - 2uv = b, \tag{4}$$

$$2xu - 2yv = c \tag{5}$$

$$2xv + 2yu = d. \tag{6}$$

To find the real numbers x, y, u and v and the bicomplex number X, we make some transformations with the equations of this system.

The equations (3) and (6) implies the following system of two equations, equivalent to the system of these two equations

$$(x + v)^2 - (y - u)^2 = a + d, \qquad i)$$

$$(x - v)^2 - (y + u)^2 = a - d. \qquad ii)$$

The system of equations (4) and (5) is equivalent to the following system of two equations

$$2(y + u)x - 2(y + u)v = b + c \iff 2(y + u)(x - v) = b + c, \qquad iii)$$

$$2(y - u)x + 2(y - u)v = b - c \iff 2(u - y)(x + v) = b - c. \qquad iv)$$

Now, the system i), ii), iii), iv) can be written as two systems of two equations. The equations i) and iv) gives the system of two equations for the unknown $x + v$ and $y - u$ as follows

$$(x + v)^2 - (y - u)^2 = a + d, \qquad a)$$

$$2(x+v)(y-u) = b-c. \qquad b)$$

The equations ii) and iii) gives the system of two equations for the unknown $x-v$ and $y+u$ as follows

$$(x-v)^2 - (y+u)^2 = a-d, \qquad c)$$

$$2(x-v)(y+u) = b+c. \qquad d)$$

All three systems of four equations of four real variables x, y, u and v are equivalent one to other.

1.1 Case $b \neq \pm c$.

As $b \neq c$, from the equations a) and b) follows the equation

$$(x+v)^2 - \frac{b-c}{4(x+v)^2} = a+d \tag{7}$$

and as $b \neq -c$ from the equations c) and d) follows

$$(x-v)^2 - \frac{b+c}{4(x-v)^2} = a-d. \tag{8}$$

From the equation(7) we obtain the biquadratic equation

$$4(x+v)^4 - 4(a+d)(x+v)^2 = (b-c)^2,$$

which is equivalent to the equation

$$(2(x+v)^2 - a - d)^2 = (b-c)^2 + (a+d)^2.$$

As we ask the real solutions of this equation, we obtain the solution, satisfying the inequality

$$2(x+v)^2 = a+d+\sqrt{(a+d)^2+(b-c)^2} \geq 0,$$

namely,

$$x+v = \varepsilon_1\sqrt{\frac{a+d}{2}+\frac{1}{2}\sqrt{(a+d)^2+(b-c)^2}}, \tag{9}$$

where $\varepsilon_1 = \pm 1$.

From the equation(8) we obtain the biquadratic equation

$$4(x-v)^4 - 4(a-d)(x-v)^2 = (b+c)^2,$$

which is equivalent to the equation

$$(2(x-v)^2 - a + d)^2 = (b+c)^2 + (a-d)^2.$$

As we ask the real solutions of this equation, we consider the solution, for which

$$2(x-v)^2 = a - d + \sqrt{(a-d)^2 + (b+c)^2} \geq 0$$

and finally

$$x - v = \varepsilon_2 \sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}, \tag{10}$$

where $\varepsilon_2 = \pm 1$.

From the equations (9) and (10) we obtain the following numbers for x and v in the case $b \neq \pm c$

$$x = \frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} + \frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}$$

and

$$v = \frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} - \frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}.$$

To find the real numbers y and u in the considered case we work as follows. As $b \neq c$ from the equations a) and b) follows the equation

$$\frac{b-c}{4(y-u)^2} - (y-u)^2 = a + d, \tag{11}$$

and as $b \neq -c$ from the equations c) and d) follows

$$\frac{b+c}{4(y+u)^2} - (y+u)^2 = a - d. \tag{12}$$

From the equation(11) we obtain the biquadratic equation

$$4(y-v)^4 + 4(a+d)(y-u)^2 = (b-c)^2, \tag{13}$$

which is equivalent to the equation

$$(2(y-u)^2 + a + d)^2 = (a+d)^2 + (b+c)^2.$$

As we ask the real solutions of this equation, we obtain

$$2(y-u)^2 = -a - d + \sqrt{(a+d)^2 + (b-c)^2} \geq 0$$

and finally

$$y - u = \varepsilon_3 \sqrt{-\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}}, \tag{14}$$

where $\varepsilon_3 = \pm 1$.

From the equation(12) we obtain the biquadratic equation

$$4(y+u)^4 + 4(a-d)(y+u)^2 = (b+c)^2, \tag{15}$$

which is equivalent to the equation

$$(2(y+u)^2 + a - d)^2 = (a-d)^2 + (b+c)^2.$$

As we ask the real solutions of this equation, we obtain

$$2(y+u)^2 = -a + d + \sqrt{(a-d)^2 + (b+c)^2} \geq 0$$

and finally

$$y + u = \varepsilon_4 \sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}, \tag{16}$$

where $\varepsilon_4 = \pm 1$.

From the equations (14) and (16) we obtain the following real numbers for y and u

$$y = \frac{\varepsilon_3}{2}\sqrt{-\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} + \frac{\varepsilon_4}{2}\sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}$$

and

$$u = \frac{\varepsilon_4}{2}\sqrt{-\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} - \frac{\varepsilon_3}{2}\sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}$$

Then we compute the asking solution $x+iy+ju+ijv$. We obtain for the square roots the following expression

$$\sqrt{A}=\sqrt{a+ib+jc+ijd}=x+iy+ju+ijv=X(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=$$

$$=\frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2}+\frac{1}{2}\sqrt{(a+d)^2+(b-c)^2}}+\frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2}+\frac{1}{2}\sqrt{(a-d)^2+(b+c)^2}}+$$

$$+i\frac{\varepsilon_3}{2}\sqrt{-\frac{a+d}{2}+\frac{1}{2}\sqrt{(a+d)^2+(b-c)^2}}+i\frac{\varepsilon_4}{2}\sqrt{-\frac{a-d}{2}+\frac{1}{2}\sqrt{(a-d)^2+(b+c)^2}}+$$

$$+j\frac{\varepsilon_4}{2}\sqrt{-\frac{a+d}{2}+\frac{1}{2}\sqrt{(a+d)^2+(b-c)^2}}-j\frac{\varepsilon_3}{2}\sqrt{-\frac{a-d}{2}+\frac{1}{2}\sqrt{(a-d)^2+(b+c)^2}}+$$

$$+ij\frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2}+\frac{1}{2}\sqrt{(a+d)^2+(b-c)^2}}-ij\frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2}+\frac{1}{2}\sqrt{(a-d)^2+(b+c)^2}}=$$

$$\begin{aligned}&=\varepsilon_1\frac{1+ij}{2\sqrt{2}}\sqrt{a+d+\sqrt{(a+d)^2+(b-c)^2}}+\\&+\varepsilon_2\frac{1-ij}{2\sqrt{2}}\sqrt{a-d+\sqrt{(a-d)^2+(b+c)^2}}+\\&+i\varepsilon_3\frac{1+ij}{2\sqrt{2}}\sqrt{-(a+d)+\sqrt{(a+d)^2+(b-c)^2}}+\\&+i\varepsilon_4\frac{1-ij}{2\sqrt{2}}\sqrt{-(a-d)+\sqrt{(a-d)^2+(b+c)^2}},\end{aligned}\tag{17}$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4=\pm 1$.

Let us note that the coefficients $\frac{1+ij}{2}$ and $\frac{1-ij}{2}$ are idempotent elements of the algebra $B\mathbb{C}$ of bicomplex numbers, i.e. $\left(\frac{1+ij}{2}\right)^2=\frac{1+ij}{2}$ and $\left(\frac{1-ij}{2}\right)^2=\frac{1-ij}{2}$.

1.2 Case $b = c \neq 0$.

In this case the equation b) seems as follows

$$2(x+v)(y-u) = 0.$$

There arise two subcases: 2a) $x + v = 0$ and 2b) $y - u = 0$.

Case 1.2 a) $x = -v$.

In this case the equations a), c) and d) are the following ones

$$-(y-u)^2 = a + d, \qquad a2a)$$

$$4x^2 - (y+u)^2 = a - d, \qquad c2a)$$

$$2x(y+u) = b. \qquad d2a)$$

From a2a) follows the following necessary condition for existing of solution: $a + d \leq 0$. Then

$$y - u = \pm\sqrt{-a-d}. \qquad (18)$$

From d2a) follows that $y + u = \dfrac{b}{2x}$ and $4x^2 - \dfrac{b^2}{4x^2} = a - d$, from where

$$16x^4 - 4(a-d)x^2 = b^2$$

and

$$\left(4x^2 - \frac{a-d}{2}\right)^2 = b^2 + \frac{(a-d)^2}{4} \iff 4x^2 = \frac{a-d}{2} \pm \frac{1}{2}\sqrt{4b^2 + (a-d)^2}.$$

As we ask real solutions of the system, the sign - doesn'd give a solution. For x and v we obtain the values

$$x = -v = \frac{\varepsilon_1}{2}\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}},$$

where $\varepsilon_1 = \pm 1$.

From d2a) follows

$$y + u = \frac{b}{2x} = \frac{b}{\varepsilon_1\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}} =$$

$$= \varepsilon_1 \ \text{sign} \ b\sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}.$$

As $y - u = \pm\sqrt{-a-d}$, we obtain for y and u the values

$$y = \frac{\varepsilon_1}{2} \ \text{sign} \ b\sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}} + \frac{\varepsilon_2}{2}\sqrt{-a-d}$$

and

$$u = \frac{\varepsilon_1}{2} \ \text{sign} \ b\sqrt{-\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}} - \frac{\varepsilon_2}{2}\sqrt{-a-d}.$$

In this case, when the condition $a + d \leq 0$ is fulfilled, the asked square roots are the following bicomplex numbers

$$\sqrt{A} = \sqrt{a + (i+j)b + ijd} = x + iy + ju + ijv = X(\varepsilon_1, \varepsilon_2) =$$

$$= (1 - ij)\frac{\varepsilon_1}{2\sqrt{2}}\sqrt{a - d + \sqrt{4b^2 + (a-d)^2}} + i(1 + ij)\frac{\varepsilon_2}{2}\sqrt{-a-d} + \quad (19)$$

$$+i(1 - ij)\frac{\varepsilon_1}{2\sqrt{2}} \ \text{sign} \ b\sqrt{-(a-d) + \sqrt{4b^2 + (a-d)^2}},$$

where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$.

When $a + d > 0$ the square root, satisfying the condition $x = -v$ of the bicomplex number $A = a + (i+j)b + ijd$ does not exist.

Case 1.2 b) $y = u$.

The system, equivalent to the equation (1) in this case seems as follows

$$(x + v)^2 = a + d, \quad a2b)$$

$$(x - v)^2 - 4y^2 = a - d, \quad c2b)$$

$$2(x - v)y = b. \quad d2b)$$

A necessary condition for existing of a real solution of this system is the inequality $a + d \geq 0$. Then

$$x + v = \varepsilon_1\sqrt{a + d}, \quad (20)$$

where $\varepsilon_1 = \pm 1$.

From d2b) follows $y = \dfrac{b}{2(x-v)}$. Setting this value of y in c2b) we obtain the equation $(x-v)^2 - \dfrac{b^2}{(x-v)^2} = a - d$. So the biquadratic equation $(x-v)^4 - (a-d)(x-v)^2 = b^2$ arise, from where

$$((x-v)^2 - \frac{a-d}{2})^2 = b^2 + \frac{(a-d)^2}{4}.$$

Then $(x-v)^2 = \dfrac{a-d}{2} \pm \sqrt{b^2 + \dfrac{(a-d)^2}{4}}$. As we ask real solution of the system above we obtain two solutions for $x - v$ of this equation, namely

$$x - v = \varepsilon_2 \sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}, \tag{21}$$

where $\varepsilon_2 = \pm 1$.

From equations (20) and (21) we obtain the values of x and v in this case. We have

$$x = \frac{1}{2}\varepsilon_1\sqrt{a+d} + \frac{1}{2}\varepsilon_2\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}$$

and

$$v = \frac{1}{2}\varepsilon_1\sqrt{a+d} - \frac{1}{2}\varepsilon_2\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}.$$

To find y and u, we consider the equation d2b), from where

$$y = u = \frac{b}{2(x-v)} = \frac{b}{2\varepsilon_2\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{4b^2 + (a-d)^2}}} =$$

$$= \frac{\text{sign } b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a-d) + \sqrt{4b^2 + (a-d)^2}},$$

where $\varepsilon_2 = \pm 1$.

Finally, in the considered case $b = c \neq 0$, $y = u$ and $a + d \geq 0$, we obtain the following square roots of the bicomplex number A

$$\sqrt{A} = \sqrt{a + (i+j)b + ijd} = x + iy + ju + ijv = X(\varepsilon_1, \varepsilon_2) =$$

$$= \frac{1+ij}{2}\varepsilon_1\sqrt{a+d} + \frac{1-ij}{2\sqrt{2}}\varepsilon_2\sqrt{a-d+\sqrt{4b^2+(a-d)^2}} + \tag{22}$$
$$+\frac{i(1-ij)\ \mathrm{sign}\ b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a-d)+\sqrt{4b^2+(a-d)^2}},$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

1.3 Case $b = -c \neq 0$.

In this case the equation d) seems as follows

$$2(x-v)(y+u) = 0.$$

There arise two subcases, namely, case 3a) $x-v = 0$ and case 3b) $y+u = 0$.

Case 1.3 a) $x = v$.

The system, which determine the square roots of the bicomplex number A seems in this case as follows

$$4x^2 - (y-u)^2 = a + d, \tag{a3a)}$$
$$2x(y-u) = b, \tag{b3a)}$$
$$-(y+u)^2 = a - d. \tag{c3a)}$$

Necessary condition for existing of real solutions $y+u$ of this system in the inequality $a - d \leq 0$. Then

$$y + u = \varepsilon_1\sqrt{-(a-d)}, \tag{23}$$

where $\varepsilon_1 = \pm 1$.

Moreover from the equation b3a) follows $y - u = \dfrac{b}{2x}$ and putting this value in the equation a3a) we obtain $4x^2 - \dfrac{b^2}{4x^2} = a + d$. This equation leads to the biquadratic equation

$$(4x^2)^2 - 4(a+d)x^2 = b^2,$$

or

$$\left(4x^2 - \frac{a+d}{2}\right)^2 = b^2 + \frac{(a+d)^2}{4},$$

from where we obtain the following real solutions for x and v

$$x = v = \frac{\varepsilon_2}{2}\sqrt{\frac{a+d}{2} + \sqrt{b^2 + \frac{(a+d)^2}{4}}},$$

where $\varepsilon_2 = \pm 1$.

Now we shall find $y - u$. From b3a) follows

$$y - u = \frac{b}{2x} = \frac{b}{2\varepsilon_2\sqrt{\frac{a+d}{2} + \sqrt{b^2 + \frac{(a+d)^2}{4}}}} = \tag{24}$$

$$= \frac{\text{sign } b}{2}\varepsilon_2\sqrt{-\frac{a+d}{2} + \sqrt{b^2 + \frac{(a+d)^2}{4}}}.$$

Then from (23) and (24) follows

$$y = \frac{1}{2}\varepsilon_1\sqrt{-(a-d)} + \frac{\text{sign } b}{2}\varepsilon_2\sqrt{-\frac{a+d}{2} + \sqrt{b^2 + \frac{(a+d)^2}{4}}}$$

and

$$u = \frac{1}{2}\varepsilon_1\sqrt{-(a-d)} - \frac{\text{sign } b}{2}\varepsilon_2\sqrt{-\frac{a+d}{2} + \sqrt{b^2 + \frac{(a+d)^2}{4}}}.$$

The asked square roots of the bicomplex number A in the case $b = -c \neq 0$, $x = v$, $a - d \leq 0$ will be the bicomplex numbers

$$\sqrt{A} = \sqrt{a + (i+j)b + ijd} = x + iy + ju + ijv = X(\varepsilon_1, \varepsilon_2) =$$

$$= \frac{(1+ij)\varepsilon_2}{\sqrt{2}}\sqrt{a + d + \sqrt{4b^2 + (a+d)^2}} + \frac{i(1-ij)}{2}\varepsilon_1\sqrt{-(a-d)} + \tag{25}$$

$$+ \frac{i(1+ij)\text{ sign } b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a+d) + \sqrt{4b^2 + (a+d)^2}},$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

Case 1.3 b) $y + u = 0$.

The system, which determine the square roots of the bicomplex number A is the following one

$$(x+v)^2 - 4y^2 = a + d, \qquad a3b)$$

$$2(x+v)y = b. \qquad b3b)$$

$$(x-v)^2 = a - d. \qquad c3b)$$

From the equation c3b) follows that $a - d \geq 0$. This is a necessary condition for the existing of real solution $x - v$ in this case.

From c3b) follows that

$$x - v = \varepsilon_1 \sqrt{a-d}, \qquad (26)$$

where $\varepsilon_1 = \pm 1$.

From b3b) follows $y = \dfrac{b}{2(x+v)}$ and using a3b) we obtain the following equation for $x + v$

$$(x+v)^2 - \left(\frac{b}{x+v}\right)^2 = a + d.$$

Then

$$\left((x+v)^2 - \frac{(a+d)}{2}\right)^2 = (a+d)^2 + b^2$$

and the real solutions x, v of this equation satisfies the equation

$$x + v = \varepsilon_2 \sqrt{\frac{a+d}{2} + \sqrt{\frac{(a+d)^2}{4} + b^2}}, \qquad (27)$$

where $\varepsilon_2 = \pm 1$.

From (26) and (27) follows

$$x = \frac{\varepsilon_1}{2}\sqrt{a-d} + \frac{\varepsilon_2}{2}\sqrt{\frac{a+d}{2} + \sqrt{\frac{(a+d)^2}{4} + b^2}}$$

and

$$v = -\frac{\varepsilon_1}{2}\sqrt{a-d} + \frac{\varepsilon_2}{2}\sqrt{\frac{a+d}{2} + \sqrt{\frac{(a+d)^2}{4} + b^2}}.$$

Then

$$y = -u = \frac{b}{2(x+v)} = \frac{b}{2\varepsilon_2\sqrt{\frac{a+d}{2} + \sqrt{\frac{(a+d)^2}{4} + b^2}}} =$$

$$= \frac{\varepsilon_2 \operatorname{sign} b}{2}\sqrt{-\frac{a+d}{2} + \sqrt{\frac{(a+d)^2}{4} + b^2}}.$$

Then we obtain the asked square roots

$$\sqrt{A} = \sqrt{a + (i-j)b + ijd} = x + iy + ju + ijv = X(\varepsilon_1, \varepsilon_2) =$$

$$= \varepsilon_1 \frac{(1-ij)}{2}\sqrt{a-d} + \frac{1+ij}{2\sqrt{2}}\varepsilon_2\sqrt{a+d+\sqrt{(a+d)^2+4b^2}} + \tag{28}$$

$$+ \frac{i(1+ij) \operatorname{sign} b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a+d)+\sqrt{(a+d)^2+4b^2}}.$$

1.4 Case $b = c = 0$.

The system determined the square root of the bicomplex number $A = a + dij$ is the following one

$$x^2 - y^2 - u^2 + v^2 = a, \quad 2xy - 2uv = 0, \tag{29}$$

$$2xu - 2yv = 0, \quad 2xv + 2yu = d.$$

From the first and the last equations follows

$$(x+v)^2 - (y+u)^2 = a+d \text{ and } (x-v)^2 - (y-u)^2 = a-d.$$

From the second and the third equation of the system (29) follows

$$(x-v)(y+u)=0 \text{ and } (x+v)(y-u)=0.$$

Case a) $x-v=0 \wedge x+v=0$, i.e. $x=v=0$.

In this case it is fulfilled $-(y+u)^2=a+d>0 \wedge -(y-u)^2=a-d>0 \Rightarrow y+u=\varepsilon_1\sqrt{-(a+d)} \wedge y-u=\varepsilon_2\sqrt{-(a-d)}$ from where follows

$$y=\frac{\varepsilon_1}{2}\sqrt{-(a+d)}+\frac{\varepsilon_2}{2}\sqrt{-(a-d)}, \quad u=\frac{\varepsilon_1}{2}\sqrt{-(a+d)}-\frac{\varepsilon_2}{2}\sqrt{-(a-d)}$$

and

$$\begin{aligned}\sqrt{A}=\sqrt{a+ijd}=iy+ju=X(\varepsilon_1,\varepsilon_2)= \\ =\frac{i(1-ij)\varepsilon_1}{2}\sqrt{-(a+d)}+\frac{i(1+ij)\varepsilon_2}{2}\sqrt{-(a-d)},\end{aligned} \tag{30}$$

where $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$.

Case b) $y+u=0 \wedge y-u=0$, i.e. $y=u=0$.

Then $x^2+v^2=a$, $2xv=d$, from where $(x+v)^2=a+d$ and $(x-v)^2=a-d$. From here follows necessary condition for existing of solution is $a\geq|d|$. In this case we obtain $x+v=\varepsilon_1\sqrt{a+d}$ and $x-v=\varepsilon_2\sqrt{a-d}$, where $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$.

Then $x=\frac{\varepsilon_1}{2}\sqrt{a+d}+\frac{\varepsilon_2}{2}\sqrt{a-d}$ and $v=\frac{\varepsilon_1}{2}\sqrt{a+d}-\frac{\varepsilon_2}{2}\sqrt{a-d}$.

So in this case

$$\begin{aligned}\sqrt{A}=\sqrt{a+ib+jc+ijd}=x+ijv=X(\varepsilon_1,\varepsilon_2)= \\ =\frac{(1+ij)\varepsilon_1}{2}\sqrt{a+d}+\frac{(1-ij)\varepsilon_2}{2}\sqrt{a-d},\end{aligned} \tag{31}$$

where $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$.

Case c) $x-v=0 \wedge y-u=0$.

In this case it is fulfilled $x = v$, $y = u$, $(x+v)^2 - (y+u)^2 = a+d \wedge a-d = 0 \Rightarrow 2x^2 - 2y^2 = a$ from where follows

$$x = v = \cosh \varepsilon_1 \frac{a}{\sqrt{2}}, \quad y = u = \sinh \varepsilon_1 \frac{a}{\sqrt{2}}$$

and

$$\sqrt{A} = \sqrt{a(1+ij)} = x + iy + ju + ijv = X(\varepsilon_1) = \\ = (1+ij)\cosh \varepsilon_1 \frac{a}{\sqrt{2}} + i(1-ij)\sinh \varepsilon_1 \frac{a}{\sqrt{2}}, \tag{32}$$

where $\varepsilon_1 = \pm 1$.

Case d) $y + u = 0 \wedge x + v = 0$.

In this case it is fulfilled $x = -v$, $y = -u$, $2x^2 - 2y^2 = a \wedge a + d = 0$ from where follows

$$x = -v = \cosh \varepsilon_1 \frac{a}{\sqrt{2}}, \quad y = -u = \sinh \varepsilon_1 \frac{a}{\sqrt{2}}$$

and

$$\sqrt{A} = \sqrt{a(1-ij)} = x + iy + ju + ijv = X(\varepsilon_1) = \\ = (1-ij)\cosh \varepsilon_1 \frac{a}{\sqrt{2}} + i(1+ij)\sinh \varepsilon_1 \frac{a}{\sqrt{2}}, \tag{33}$$

where $\varepsilon_1 = \pm 1$.

So we proved the following theorem

Theorem 1. *The bicomplex number $a + ib + jc + ijd$, where a, b, c, d are real numbers and i, j the imaginary units, has square roots as follows*
- in the case $b \neq \pm c$ there exist 16 square roots given by the formula (17);
- in the case $b = c \neq 0$ if $a + d \leq 0$ there exist 4 square roots, given by the formula (19) and if $a + d \geq 0$ there exists 4 square roots, given by the formula (22);
- in the case $b = -c \neq 0$ if $a - d \leq 0$ there exist 4 square roots, given by the formula (25) and if $a - d \geq 0$ there exists 4 square roots, given by the formula (28);

- *in the case* $b = c = 0$ *if* $-a \geq |d|$ *there exist 2 square roots, given by the formula (30);*

- *in the case* $b = c = 0$ *if* $a \geq |d|$ *there exist 2 square roots, given by the formula (31);*

- *in the case* $b = c = 0$ *if* $|a| < |d|$ *there doesn't exist square roots of the bicomplex number;*

- *in the case* $b = c = 0$ *if* $a = \pm d$ *there exist 2 square roots, given by the formula (32) and (33), respectively.*

2 Quadratic equation.

We shall write the solutions of the quadratic equation in the algebra $B\mathbb{C}$ of the bicomplex numbers, using the find above square roots of bicomplex number.

Theorem 2. *The quadratic equation*

$$x^2 + px + q = 0$$

with bicomplex coefficients $p,\ q \in B\mathbb{C}$, $p = p_0 + ip_1 + jp_2 + ijp_3,\quad q = q_0 + iq_1 + jq_2 + ijq_3,\ \ p_k, q_k \in \mathbb{R}$ *for* $k = 0, 1, 2, 3$ *has the following solutions*

$$x_+ = -\frac{p}{2} + X \quad \text{and} \quad x_- = -\frac{p}{2} - X,$$

where

$$X = \sqrt{\frac{p^2}{4} - q}$$

is one of the given in section 1 bicomplex square roots of the bicomplex number $\frac{p^2}{4} - q$.

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