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Математически анализ без формулата на Тейлор?

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Mathematical analysis without Taylor's formula?

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It is shown that the use of the classical TAYLOR formula for obtaining sufficient conditions for existence of local extrema of real functions of real variables and for expanding the familiar elementary functions of a real variable in power series can be avoided. Suitable examples demonstrate the significance of this formula in other parts of the classical analysis

1. The next assertion is more popular as a rule:

If the real function f of a real variable has derivatives up to the order $k+2, k \in \mathbb{N}_0 = N \cup \{0\}$ in a neighborhood of an inner point x_0 of its range of definition, $f^{(j)}(x_0) = 0, j = 1, 2, 3, \ldots, k+1, f^{(k+2)}$ is continuous and non-vanishing at x_0 and k is even, then f has a local extrema at x_0. More precisely, f has maximum when $f^{(k+2)}(x_0) > 0$ and minimum when $f^{(k+2)}(x_0) > 0$. If k is odd, then f has no local extrema at the point x_0.

This rule is a direct consequence of TAYLOR'S formula for real functions of a real variable. The most essential condition for its validity is the requirement for continuity of the $(k+2)$-th derivative at the point x_0. But it can be obtained without appealing to this formula namely by using only the existence and the non-vanishing of $f^{(k+2)}(x_0)$. Indeed, from the existence of the derivatives $f^{(j)}, j = 1, 2, 3, \ldots, k+1$ in the neighborhood of the point x_0 it follows that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{k+2}} = \frac{f^{(k+2)}(x_0)}{(k+2)!}. \tag{1.1}$$

For $k = 0$ the above equality is a consequence of the CAUCHY mean value theorem. Indeed,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^2} = \lim_{x \to x_0} \frac{f'(\xi(x))}{2(\xi(x) - x_0)} = \lim_{x \to x_0} \frac{f'(\xi(x)) - f'(x_0)}{2(\xi(x) - x_0)} = \frac{f''(x_0)}{2!},$$

since $0 < |\xi(x) - x_0| < |x - x_0|$. The validity of (1.1) for $k \geq 1$ can be proved by induction.

Sufficient conditions for existence of local extrema of a real function of real variables usually are given in the text-book of Mathematical Analysis as consequences of corresponding TAYLOR formula. This also can be avoided by reducing the multi-dimensional problem to the one-dimensional.

Let $n \geq 2$ and let $M_0(x_1^{(0)}, \ldots, x_n^{(0)})$ be an inner point of the range of definition E of the function F. Then, there is a spherical neighborhood

$$U(M_0;r) = \left\{(x_1,\dots,x_n) : \sum_{j=1}^{n}\left(x_j - x_j^{(0)}\right)^2 < r^2\right\}$$

contained in the set E. Suppose that the function F is two-times differentiable in $U(M_0;r)$. If $\lambda_k \in \mathbb{R}, k = 1,2,\dots,n$ are such that

$$\sum_{k=1}^{n} \lambda_k^2 = 1, \tag{1.2}$$

then the function

$$f(\lambda_1,\lambda_2,\dots,\lambda_n;t = F(x_1^{(0)}+\lambda_1 t, x_2^{(0)}+\lambda_2 t,\dots,x_n^{(0)}+\lambda_n t), -r < t < r, \tag{1.3}$$

is differentiable in the interval $(-r,r)$ and, moreover, has second derivative at the point $t = 0$.

Suppose that the function F has extrema at the point M_0. If $\lambda_k, k = 1,2,\dots,n$ satisfy (1.2), then the function (1.3) has extrema at the point $t = 0$ and, hence, for all such $\lambda_j, j = 1,2,\dots,n$,

$$f'(\lambda_1,\lambda_2,\dots,\lambda_n;0) = \sum_{k=1}^{n} \lambda_k F_{x_k}(M_0) = 0.$$

By choosing $\lambda_j = 1$ and $\lambda_k = 0, k = 1,2,\dots,n, k \neq j$, we come to the equalities

$$F_{x_j}(M_0) = 0, \quad j = 1,2,\dots,n,$$

i.e. to the necessary conditions for existence of local extrema of the function F at the point M_0.

From the equality

$$f''(\lambda_1,\lambda_2,\dots,\lambda_n;0) = \sum_{j,k=1}^{n} F_{x_j x_k}(M_0)\lambda_j\lambda_k$$

it follows that the condition

$$f''(_1,\lambda_2,\dots,\lambda_n;0) \neq 0, \quad \sum_{j=1}^{n} \lambda_j^2 = 1,$$

which ensures the existence of local extrema for the function F at the point M_0, holds if and only if the quadratic form

$$\Delta(F,M,\lambda_1,\lambda_2,\dots,\lambda_n) = \sum_{j,k=1}^{n} F_{x_j x_k}(M_0)\lambda_j\lambda_k \tag{1.4}$$

is definite. More precisely, if it is positively definite, then the function F has local minimum at the point M_0 and if it is negatively definite, then it has local maximum at this point.

The well-known criterion of SILVESTER says that a real quadratic form is positively definite if all principal minors of its KRONECKER'S matrix are positive. For the form (1.4) this matrix looks as follows

$$||1/2(F_{x_jx_k}(M_0)+F_{x_kx_j}(M_0)||_{j,k=1}^n \tag{1.5}$$

As an example let us consider the case $n=2$. More concrete, let (x_0, y_0) be an inner point of the range of definition of a real function F of the real variables x, y. Suppose that this function has partial derivatives up to the second order in a neighborhood of the point (x_0, y_0) and, moreover, that F_{xy}, F_{yx} are continuous at this point, which leads to the equality $F_{xy}(x_0, y_0) = F_{yx}(x_0, y_0)$. KRONECKER'S matrix in the case under consideration has the form

$$\begin{Vmatrix} F_{xx}(x_0, y_0) & F_{xy}(x_0, y_0) \\ F_{yx}(x_0, y_0) & F_{yy}(x_0, y_0) \end{Vmatrix}$$

Its principal minors are $F_{xx}(x_0, y_0)$ and

$$D(F; x_0, y_0) = F_{xx}(x_0, y_0)F_{yy}(x_0, y_0) - (F_{xy}(x_0, y_0))^2.$$

If they are positive, then the function F has local minimum at the point (x_0, y_0). Thus we come to the well-known rule:

If the real function F of the real variables x, y has continuous partial derivatives up to the second order in the neighborhood of the point (x_0, y_0), $F_x(x_0, y_0) = F_y(x_0, y_0) = 0, D(F; x_0, y_0) < 0$ and $F_{xx}(x_0, y_0) > 0$, then it has local minimum at this point.

If the function F has local maximum at the point (x_0, y_0), then $-F$ has local minimum there. Since $D(-F; x_0, y_0) = D(F; x_0, y_0)$, we obtain also the rule:

If the real function F of the real variables x, y has continuous partial derivatives up to the second order in the neighborhood of the point (x_0, y_0), $F_x(x_0, y_0) = F_y(x_0, y_0) = 0, D(F; x_0, y_0) < 0$ and $F_{xx}(x_0, y_0) < 0$, then it has local maximum at this point.

2. As it is well-known, if the function f is defined as the sum of a convergent power series with center at the point $x_0 \in \mathbb{R}$, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \quad |x-x_0| < R \leq \infty, \tag{2.1}$$

then it is in the class C^∞ in the interval $(x_0 - R, x_0 + R)$. That means it has all the derivatives in this interval. Moreover,

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

i.e. each convergent power series with center $x_0 \in \mathbb{R}$ is in, fact, the TAYLOR series with the same center of the function defined by it. But the converse of the last assertion is, in general, not true. Indeed, the TAYLOR series of a function f of the class C^∞ in a neighborhood of the point x_0 of its range of definition may be convergent there with sum different from the function f. An example confirming this hypothesis is the function f defined by $f(x) = \exp(-x^{-2})$ for $x \neq 0$ and $f(0) = 0$. It is of the class $C^\infty(\mathbb{R}), f^{(n)}(0) = 0, n = 0, 1, 2, \ldots$, hence its TAYLOR'S series with center at the point 0 is everywhere convergent, but its sum is not the function $\exp(-x^{-2})$. The function $(1 + x^2)^{-1}, x \in \mathbb{R}$ is also in the class $C^\infty(\mathbb{R})$, but its TAYLOR'S series with center at the zero point represents it only in the interval $(-1, 1)$, i.e.

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1.$$

It arises the problem how to "guess" where does the TAYLOR series of a real function $f \in C^\infty$ represent it. The answer may be given by the reminder of its TAYLOR'S formula, namely

$$R_n(f; x_0, x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2.3}$$

If we define

$$\rho(f; x_0) = \sup\{r \geq 0 : \lim_{n\to\infty} \max_{|x - x_0| \leq r} R_n(f; x_0, x) = 0\}, \tag{2.4}$$

then the TAYLOR series of the function f with center x_0 does not represent it in any nonempty subset of the open set $(-\infty, x_0 - \rho(f; x_0)) \cup (x_0 + \rho(f; x_0), \infty)$ even when it converges there. But the application of this criterion may find considerable technical difficulties as it is e.g. in the case of the function $\exp(-x^{-2})$ when $x_0 \neq 0$. This can be avoided by having in view the analytical structure of the real functions of a real variable admitting expansions in convergent power series.

If a real function f has representation in the neighborhood $U(x_0; R) = \{x \in \mathbb{R} : |x - x_0| < R\}, 0 < R \leq \infty$ by its TAYLOR'S series with center x_0, then it has holomorphic extension in the plane if complex numbers. That means there exist a region $G \subset \mathbb{C}$ containing $U(x_0; R)$ and a complex function F holomorphic in G and such that $F(x) = f(x)$ for $x \in U(x_0; R)$. Indeed, it is sufficient to choose $G = \{z \in \mathbb{C} : |z - x_0| < R\}$ and to define F as the sum of the power series obtained from (2.1) by substituting x by z. The converse is also true and this is a consequence of TAYLOR'S theorem for holomorphic function of one complex variable. It is clear now that this "philosophy" gives the answer of the question why does the TAYLOR series of the function $\exp(-x^{-2})$ with center at the point 0 do not represent it in the neighborhood of this point. Indeed, this function does not admit holomorphic extension in any its circular neighborhood since this point is an essentially singular point for the function $\exp(-z^{-2}), z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The function $(1+x^2)^{-1}$ has holomorphic extension in the region $\mathbb{C} \setminus \{i, -i\}$ as a meromorphic function with simple poles at the points $i, -i$. This makes clear why does the radius of convergence of the power series (2.3) have to be equal to 1.

3. The general conclusion we could make now is that in order to represent a real function of one real variable by a convergent power series with center at inner point of its region of definition, we have to convince ourselves that its restriction to a neighborhood of this point has holomorphic extension and then to apply TAYLOR'S theorem for holomorphic functions of one complex variable. But in particular cases this can be realized without using any tools of the classical complex analysis and even without the help of TAYLOR'S formula as it was "promised" at the beginning. This can be done by engaging some ground facts concerning the power series of one variable including the rules for their termwise differentiation and integration. This approach is applied further to the to most usable functions in the Mathematical Analysis and its applications usually named elementary transcendental functions.

The exponential function. This is the function with value e^x for $x \in \mathbb{R}$, where

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n, \tag{3.1}$$

is the NAPIER number. By WALTER RUDIN: "This is the most important function in the mathematics" (Real and complex analysis, New York, 1976, Prologue).

As it is well-known, $(e^x)' = e^x, x \in \mathbb{R}$, and it is clear that $e^0 = 1$. Moreover, this function is the unique differentiable function with these two properties. Indeed, if $f'(x) = f(x), x \in \mathbb{R}$ and $f(0) = 1$, then from $(f(x)e^{-x})' = 0, x \in \mathbb{R}$ it follows that $f(x)e^{-x} = 1, x \in \mathbb{R}$ and, hence, $f(x) = e^x, x \in \mathbb{R}$.

Since $(e^x)^{(n)} = e^x, x \in \mathbb{R}, n = 1, 2, 3, \ldots$, the MACLORAIN series of the exponential function, i.e. its TAYLOR'S series with center at the point 0, is

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\dots. \tag{3.2}$$

D'ALEMBERT'S criterion immediately yields that this series is absolutely convergent for each $x \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and, hence, it is uniformly convergent on each compact subset of $\mathbb{R}$. The same holds for the series obtained after its termwise differentiation. If $E(x)$ is the sum of the series (3.2), then $E'(x) = E(x), x \in \mathbb{R}$ and, moreover, $E(0) = 1$. Therefore, $E(x) = e^x, x \in \mathbb{R}$, i.e.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad x \in \mathbb{R}. \tag{3.3}$$

Remark. Since more than fifty years instead of e^x it is used the denotation $\exp x$.

The logarithmic function. The exponential function is positive and increasing in the interval $(-\infty,\infty)$. Its inverse function is the logarithmic function whose value for $x \in \mathbb{R}^+$ is usually denoted by $\log x$. Typical property of this function is that $\log(xy) = \log x + \log y, x, y \in \mathbb{R}$, which is a consequence of the equality $\exp(u+v) = \exp u \exp v, u, v \in \mathbb{R}$. The last one means that the mapping $\exp : \mathbb{R} \to \mathbb{R}^+$ defined by the exponential function is, in fact, an (algebraic) isomorphism of the additive group $\{\mathbb{R},+\}$ of real numbers and the multiplicative group $\{\mathbb{R}^+,.\}$ of positive real numbers. Hence, the mapping defined by the logarithmic function is the inverse of this isomorphism.

In the Mathematical Analysis they use more often the function defined for $x > -1$ by $\log(1+x)$. The MACLORAIN series of this function, namely

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$

has radius of convergence equal to 1. If L denotes its sum in the interval $(-1,1)$, then

$$L'(x) = \sum_{n=0}^{\infty}(-1)^n x^n = (1+x)^{-1}, \quad -1 < x < 1.$$

Hence,

$$\log(1+x) = \int_0^t (1+t)^{-1}\,dt = \int_0^x L'(t)\,dt = L(x), \quad -1 < x < 1,$$

i.e.

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x < 1. \tag{3.5}$$

The function$(1+x)^m$. If m is a nonnegative integer, then the region of definition of this function is $\mathbb{R}$. If m is a negative integer, this region is $\mathbb{R}\setminus\{-1\}$. If $m \notin \mathbb{Z}$, this function is defined by the equality

$$(1+x)^m = \exp(m\log(1+x)), \quad -1 < x < \infty. \tag{3.6}$$

Its consequence is that

$$((1+x)^m)^{(n)} = m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n}.$$

Hence, the (formal) MACLORAIN series of the function $(1+x)^m$ is

$$\sum_{n=0}^{\infty}\binom{m}{n}x^n \tag{3.7}$$

mostly named binomial series.

Except the case when m is a nonnegative integer, its radius of convergence is equal to 1. Let us denote its sum by $B_m(x)$. Then, $(B_m(x))' = mB_{m-1}(x)$ and, hence, $(B_m(x)(1+x)^{-m})' = 0, -1 < x < 1$. Since $B_m(0) = 1$, it follows that $B_m(x)(1+x)^{-m} = 1, -1 < x < 1$, i.e. $B_m(x) = (1+x)^m, -1 < x < 1$. Thus we come to the MACLORAIN expansion of the function $(1+x)^m$, namely

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n, \quad m \in \mathbb{R}\setminus\mathbb{N}_0, -1 < x < 1. \tag{3.8}$$

The functions $\sin x$ **and** $\cos x$. This is the only pair of differentiable functions $S(x), C(x), x \in \mathbb{R}$ such that $S'(x) = C(x), C'(x) = -S(x), S(0) = 0, C(0) = 1$. Indeed, for the derivative of the function $F(x) = (S(x) - \sin x)^2 + (C(x) - \cos x)^2$ one obtains that $F'(x) = 0, x \in \mathbb{R}$. Hence, $F(x) = F(0) = 0, x \in \mathbb{R}$, i.e. $S(x) = \sin x, C(x) = \cos x, x \in \mathbb{R}$.

The functions $\sin x, \cos x$ are in the class $C^\infty(\mathbb{R})$. Their MACLORAIN'S series are

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \tag{3.9}$$

and

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \tag{3.10}$$

respectively, and they converge for each $x \in \mathbb{R}$. Moreover, if $s(x)$ is the sum of (3.9) and $c(x)$ is that of (3.10), then $s'(x) = c(x), c'(x) = -s(x), x \in \mathbb{R}$, $s(0) = 0, c(0) = 1$, hence, $s(x) = \sin x, c(x) = \cos x$, i.e.

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}, \tag{3.11}$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}. \tag{3.12}$$

"Algebraic intermezo". Let us denote by $\mathcal{C}_{x_0}^{\infty}, x_0 \in \mathbb{R}$, the set of real functions of one real variable such that each of them is of the class $\mathcal{C}^{\infty}$ in a neighborhood of the point x_0 and let $\mathcal{T}_{x_0}$ be the set of such functions which are representable around the point x_0 by their TAYLOR'S series centered at this point. In view of (2.4),

$$\mathcal{T}_{x_0} = \{f \in \mathcal{C}_{x_0} : \rho(f; x_0) > 0\},$$

but $\mathcal{T}_{x_0}$ is a proper subset of $\mathcal{C}_{x_0}^{\infty}$. Each of these sets is a $\mathbb{R}$-vector space. Moreover, the following assertion holds:

The space $\mathcal{T}_{x_0}$ is an integral domain. Its group of units consists of those $f \in \mathcal{T}_{x_0}$ such that $f(x_0) \neq 0$.

The fact that $\mathcal{T}_{x_0}$ is a ring is a consequence of the CAUCHY rule for multiplication of power series in the systems of monomials $(x - x_0)^n, n \in \mathbb{N}_0$ as well as of LEIBNITZ'S rule for computation of the derivatives of a product of two differentiable functions.

Suppose that $f, g \in \mathcal{T}_{x_0}, f \neq 0$ and $fg = 0$. Since $f(x) = (x-x_0)^k \varphi(x), k \in \mathbb{N}_0, \varphi \in \mathcal{G}_{x_0}$, from $f(x)g(x) = (x - x_0)^k \varphi(x)g(x) = 0$ in a neighborhood of the point x_0, it follows that $g(x) = 0$ in its neighborhood, i.e. $g = 0$.

Each function $f \in \mathcal{T}_{x_0}$ has holomorphic extension in a disk with center x_0 and if $f(x_0) \neq 0$, the same holds for the function $1/f$. Then, by TAYLOR'S theorem for holomorphic functions of a complex variable it follow that $1/f \in \mathcal{T}_{x_0}$. Let us point out that the last fact could be proved without using this theorem as it is done e.g. in G. M. FICHTENGOLZ, *A course in differential and integral calculus*, Moskow 1966, II, **448**.

Bernoulli's numbers. These are the numbers $B_n, n \in \mathbb{N}_0$ introduced by J. BERNOULLI in 1713. They make possible finite sums of powers of positive integers to be represented in a "closed" form, namely

(3.13)
$$\sum_{k=0}^{m-1} k^\nu = \frac{1}{\nu+1}\sum_{n=0}^{\nu}\binom{\nu+1}{n}B_n m^{\nu+1-n}, \quad \nu = 0,1,2,\ldots; m = 1,2,3,\ldots.$$

Here are some of the first BERNOULLI'S number:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0,$$

$$B_8 = -\frac{1}{30}, B_9 = 0, B_{10} = \frac{5}{66}, B_{11} = 0, \ldots.$$

It holds the equality

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0, \quad n \geq 1,$$

which yields that in fact all the BERNOULLI numbers are rational numbers.

The function G, defined by $G(x) = x^{-1}(\exp x - 1), G(0) = 1$ is in $\mathcal{G}_0$. Hence, the function F, defined by $F(x) = 1/G(x), x \in \mathbb{R}$ is also in $\mathcal{G}_0$. If $b_n = F^{(n)}(0), n \in \mathbb{N}_0$, then the equality

$$F(x)(\exp x - 1) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}\sum_{n=1}^{\infty}\frac{x^n}{n!} = x$$

holds in a neighborhood of the point 0. Hence,

$$\sum_{k=0}^{n}\binom{n+1}{k}b_k = 0, \quad n = 1,2,3,\ldots.$$

Taking in view (3.13) as well as that $b_0 = B_0 = 1$, we find that $b_n = B_n, n \in \mathbb{N}$, i.e. the MACLORAIN series of the function F is

(3.14)
$$F(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

i.e. F is a generating function for the Bernoulli numbers

The function F admits holomorphic extension as a meromorphic function with simple poles at the points $2k\pi i, k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Hence, the radius of convergence of the series in (3.15) is equal to 2π. Further, since the function

$F(x)+x/2, -2\pi < x < 2\pi$ is even, it follows that $B_{2n+1}=0, n\in\mathbb{N}$, i.e. all the Bernoulli numbers with odd index greater or equal to 3 are equal to zero. Hence, the series representation (3.15) has, in fact, the form

$$F(x)=1-\frac{x}{2}+\sum_{n=1}^{\infty}B_{2n}x^{2n},\quad -2\pi<x<2\pi. \tag{3.15}$$

Let us point out that L. EULER has proved in 1740 that

$$s_{2n}=\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\quad n=1,2,3,\dots.$$

The functions $x\cot x$ and $\tan x$. The function defined by $\dfrac{\sin x}{x}, x\in\mathbb{R}^*$ and $\left.\dfrac{\sin x}{x}\right|_{x=0}=1$ is in $\mathcal{G}_0$. Hence, the function $\dfrac{x}{\sin x}$ is also in $\mathcal{G}_0$ and so does $\dfrac{x}{\sin x}\cos x = x\cot x$.

Let us "extend" the ring $\mathcal{T}_{x_0}, x_0\in\mathbb{R}$ by adding to it the complex-valued functions of a real variable which are representable by their Taylor's series with center x_0. In particular, $\exp ix$ is in $\mathcal{T}_{x_0}$ and even in $\mathcal{G}_{x_0}$ for each $x_0\in\mathbb{R}$. Then, EULER's formulas

$$\cos x=\frac{1}{2}(\exp ix+\exp(-ix)),\quad \sin x=\frac{1}{2i}(\exp ix-\exp(-ix))$$

yield that

$$x\cot x = ix\frac{\exp(2ix)+1}{\exp(2ix)-1}=ix+\frac{2ix}{\exp(2ix)-1}=ix+F(2ix),\quad x\in(-\pi,\pi).$$

Then, from (3.16) it follows that

$$x\cot x=1+\sum_{n=1}^{\infty}(-1)^n 2^{2n}B_{2n}\frac{x^{2n}}{(2n)!},\quad -\pi<x<\pi.$$

Further, the equality $\tan x=\cot x-2\cot 2x, x\in(-\pi/2,\pi/2)$ leads to the representation

$$\tan x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}x^{2n-1},\quad -\pi/2<x<\pi/2.$$

The functions $\arcsin x$ and $\arctan x$. From (3.7) it follows that

$$(1-t^2)^{-1/2}=\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}t^{2n}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}t^{2n},\quad -1<t<1.$$

Then,

$$\int_0^x (1-t^2)^{-1/2}\,dt = x + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}, \quad -1 < x < 1,$$

i.e.

$$\arcsin x = x + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}, \quad -1 < x < 1.$$

In the same way from the representation

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad -1 < x < 1,$$

we obtain that

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 < x < 1.$$

4. Let us suppose that $x_0 \in \mathbb{R}$ is an inner point of the range of definition of the complex-valued function f of real variable and, moreover, that this function is in the class $C^{n+1}, n \in \mathbb{N}_0$ in a neighborhood of x_0. Then, after replacing a by x_0, b by x, $u(t)$ by $f(t)$ and $v(t)$ by $(x-t)^n$ in the generalized formula

$$\int_a^b u(t)dv^{(n)} = \sum_{k=0}^{n}(-1)^k u^{(k)}(t)v^{(n-k)}(t)\Big|_a^b + (-1)^{n+1}\int_a^b v(t)du^{(n)}(t) \tag{4.1}$$

for integration by parts, we obtain the equality

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt, \tag{4.2}$$

which is valid in a neighborhood of the point x_0. This is the TAYLOR formula with reminder in an integral form.

Let us suppose that the function f is real-valued. Since $(x-t)^n$ as a function of t is monotonic, there is ξ such that $|\xi - x_0| \le |x - x_0|$ and, moreover,

$$\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

i.e. from (4.2) we obtain TAYLOR'S formula with reminder in a form of LAGRANGE.

Asymptotic formula for Macdonald's function. The differential equation $z^2w''+zw'+(z^2-\nu^2)w=0$ bears the name of BESSEL. Its solution are called cylinder functions. Such is the function J_ν defined by the equality

$$J_\nu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)},\quad z\in\mathbb{C}\setminus(-\infty,0].$$

It is called BESSEL function of first kind with parameter ν. The function I_ν, defined by the equality $I_\nu(z)=\exp(-i\nu\pi/2)J_\nu(iz)$, is called modified BESSEL function of first kind. It is a solution of the modified BESSEL differential equation $z^2w''+zw'+(z^2+\nu^2)w=0$, which is obtained from BESSEL'S equation after replacing z by iz. If ν is not an itegral multiple of π, then the function

$$K_\nu(z)=\frac{\pi}{2\sin(\nu\pi)}\{I_{-\nu}(z)-I_\nu(z)\}$$

is also a solution of the modified BESSEL equation. It is called modified Bessel function of third kind, or function of MACDONALD. The last one has the the following integral representation

$$\Gamma(\nu+1/2)K_\nu(z)=\sqrt{\frac{\pi}{2z}}\int_0^\infty\left(1+\frac{x}{2z}\right)^{\nu-1/2}x^{\nu-1/2}\exp(-x)\,dx,$$

$$z\in\mathbb{C}\setminus(-\infty,0],\Re\nu>-1/2.$$

From (4.2) with $f(x)=\left(1+\frac{x}{2z}\right)^{\nu-1/2}$ and $x_0=0$ it follows that

$$K_\nu(z)=\sqrt{\frac{\pi}{2z}}\exp(-z)\left\{\sum_{k=0}^{n}\frac{\Gamma(\nu+n+1/2)}{k!(\nu-n+1/2)}(2z)^{-k}+R_{n,\nu}(z)\right\},$$

where

$$n!\Gamma(\nu-n+1/2)(2z)^{-n-1}R_{n,\nu}(z)$$

$$=\int_0^\infty t^{\nu+n+1/2}\exp(-t)\,dt\int_0^1\left(1+\frac{tu}{2z}\right)^{\nu+n+1/2}(1-u)^n\,du.$$

If n and ν with $\Re\nu>-1/2$ are fixed, then from the last representation it follows that $R_{n,\nu}(z)=O(|z|^{-n-1})$ when $|z|\to\infty$ in the region $\mathbb{C}\setminus(-\infty,]$, i.e. that the function $z^{n+1}R_n(z)$ is bounded in each of the regions $\{\mathbb{C}\setminus(-\infty,0]\}\cap D(0;r), 0<r<\infty$.

If we use HANKEL'S symbol

$$(\nu,k)=\frac{2^{-2k}}{k!}(4\nu^2-1^2)(4\nu^2-3^2)\dots(4\nu^2-(2k-1)^2)=\frac{\Gamma(\nu+k+1/2)}{k!\Gamma(\nu-k+1/2)},$$

then the asymptotic formula for the function $K_\nu(z)$ can be written in the following form

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \exp(-z) \left\{ \sum_{k=0}^{n} (\nu, k(2z)^{-k}) + O(|z|)^{-n-1} \right\}.$$

Obrechkoff's formula. In the paper Neue Quadratur formeln, *Abh. preuss. Akad. Wiss. Math.-naturwiss. Kl.*, 1940 No 4, 1-20, of N. OBRECHKOFF is given a formula named by him formula for the aritmetical means of the TAYLOR series. For the function f it is supposed that it is of the class C^{n+k+1} in a neighborhood of the point x_0. The following denotations are introduced, namely

$$S_q^{(0)}(f; x_0, x) = \sum_{j=1}^{q} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j, \quad q = 0, 1, 2, \ldots, n,$$

$$S_n^{(k)}(f; x_0, x) = \sum_{q=0}^{n} S_q^{(k-1)}(f; x_0, x), \quad k = 1, 2, 3, \ldots.$$

OBRECHKOFF'S formula has the form

$$S_n(f; x_0, x) = \sum_{j=0}^{n} \binom{n+k-j}{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \tag{4.3}$$

$$+ \frac{(-1)^k}{k!n!} \int_{x_0}^{x} f^{(n+k+1)}(t)(x-t)^n(t-x_0)^k \, dt.$$

The proof is by induction with respect to k provided n is fixed. If $k = 0$, then the formula (4.3) turns into (4.2). This gives rise it to be considered as a generalization of the classical TAYLOR formula. By its means OBRECHKOFF obtains in the sited paper quadrature formulas having as particular cases those of MACLORAIN and NEWTON.