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**Normality of the twistor space of a
5-manifold with an irreducible
SO(3)-structure**

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**Нормалност на туисторното пространство
на 5-мерно многообразие с неприводима
SO(3)- структура**

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NORMALITY OF THE TWISTOR SPACE OF A 5-MANIFOLD WITH AN IRREDUCIBLE $SO(3)$ -STRUCTURE

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ABSTRACT. A manifold with an irreducible $SO(3)$ -structure is a 5-manifold M whose structure group can be reduced to the group $SO(3)$, non-standardly imbedded in $SO(5)$. The study of such manifolds has been initiated by M. Bobieński and P. Nurowski who, in particular, have shown that one can define four CR -structures on a twistor-like 7-dimensional space associated to M . In the present paper it is observed that these CR -structures are induced by almost contact metric structures. The purpose of the paper is to study the problem of normality of these structures. The main result gives necessary and sufficient condition for normality in geometric terms of the base manifold M . Examples illustrating this result are presented at the end of the paper.

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Key words: irreducible $SO(3)$ -structures, twistor spaces, almost contact metric structures.

1. INTRODUCTION

It is well-known that the group $SO(3)$ has a unique irreducible representation in dimension five which gives a non-standard embedding of $SO(3)$ in the group $SO(5)$. Recently M. Bobieński and P. Nurowski [4] have studied five-dimensional oriented Riemannian manifolds admitting an irreducible $SO(3)$ -structure, meaning that their tangent frame bundle has a $SO(3)$ -subbundle, $SO(3)$ being non-standardly embedded in $SO(5)$. In general, the Levi-Civita connection does not preserve this subbundle, which provides an example of the so-called non-integrable geometric structures. A framework for studying such structures has been outlined by T. Friedrich in [7]. Among the irreducible $SO(3)$ -structures the so-called nearly integrable structures are of special interest. An important feature of these structures is that they admit a (unique) characteristic connection, i.e. a metric $SO(3)$ -connection with totally skew-symmetric torsion [4]. Interesting geometric properties of such structures on concrete homogeneous manifolds have been given by I. Agricola, J. Becker-Bender, T. Friedrich [1], and a class of nearly integrable irreducible $SO(3)$ -structures on five-dimensional Lie groups has been studied by S. Chiossi, A. Fino [6]. We also note that the topological obstructions for existence of an irreducible $SO(3)$ -structure have been found in [5] and [1].

It has been observed in [4] that every manifold M with an irreducible $SO(3)$ -structure admits a twistor-like space. This is a 2-sphere bundle $\mathbb{T}$ over M on which one can define four almost CR -structures in a way that resembles the twistor construction. In fact, these almost CR -structures are induced by almost contact metric structures and the main purpose of this paper is to find the geometric conditions on the base manifold M under which these almost contact structures are normal. Recall that normality is an important property of an almost contact manifold N ,

which means that the product manifold $N \times S^1$ is a complex manifold with the complex structure induced by the almost contact one (cf., for example, [3]).

In Section 2 of the present paper we collect basic facts about manifolds with an irreducible $SO(3)$ -structure. The twistor space $\mathbb{T}$ of such a manifold and the four almost contact metric structures on it are defined in Section 3. The main result of the paper is proved in Section 4. It is not hard to see that three of the almost contact metric structures on $\mathbb{T}$ are not normal. In order to state the normality result for the fourth one, we introduce a specific tensor Q on the base manifold by means of the curvature of its characteristic connection. We show that this almost contact metric structure on the twistor space $\mathbb{T}$ is normal if and only if Q and certain components of the torsion and the curvature of the characteristic connection vanish. Two examples illustrating this result are given at the end of the paper. One of them provides a new example of a non-normal almost contact metric structure which induces an integrable almost CR -structure (as is well-known, any normal structure induces an integrable structure [8]). Some of the computations in Chapter 4 are used in an Addendum for giving a new proof of the integrability result in [4] for the almost CR -structures on $\mathbb{T}$. This proof reveals the role of the geometric conditions on the base manifold obtained there for integrability/non-integrability of these structures.

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2. IRREDUCIBLE $SO(3)$ -STRUCTURES

2.1. The irreducible representation of $SO(3)$ on $\mathbb{R}^5$.

It is well-known that the irreducible finite-dimensional representations of $SO(3)$ are odd-dimensional and there is a unique irreducible representation of $SO(3)$ on the space $\mathbb{R}^{2l+1}$, $l = 0, 1, \dots$. Following [4], we shall describe the unique irreducible (orthogonal) representation of $SO(3)$ on $\mathbb{R}^5$ as well as an $SO(3)$ -invariant symmetric 3-form on $\mathbb{R}^5$ which plays a crucial role for defining 5-manifolds with $SO(3)$ -structure.

Let $e_1, \dots, e_5$ be the standard basis of $\mathbb{R}^5$. Denote by μ the isomorphism of $\mathbb{R}^5$ onto the space of symmetric traceless real 3×3 -matrices given by

$$x = \sum_{i=1}^5 x_i e_i \rightarrow \mu(x) = \begin{bmatrix} \frac{x_1}{\sqrt{3}} - x_4 & x_2 & x_3 \\ x_2 & \frac{x_1}{\sqrt{3}} + x_4 & x_5 \\ x_3 & x_5 & -2\frac{x_1}{\sqrt{3}} \end{bmatrix}$$

Then one can define an irreducible faithful representation ρ of $SO(3)$ on $\mathbb{R}^5$ setting

$$\rho(h) \cdot x = \mu^{-1}(h\mu(x)h^{-1}), \quad h \in SO(3).$$

In this way we obtain a non-standard smooth embedding

$$\iota : SO(3) \hookrightarrow SO(5)$$

such that $\iota(SO(3))$ acts on $\mathbb{R}^5$ irreducibly. Further on, we shall often consider $SO(3)$ as a subgroup of $SO(5)$ by means of the embedding ι .

The characteristic polynomial $P_x(\lambda) = \det(\mu(x) - \lambda I)$ of the matrix $\mu(x)$ has the form

$$P_x(\lambda) = -\lambda^3 + g(x, x)\lambda + \frac{2\sqrt{3}}{9}\Upsilon(x, x, x),$$

where

$$g(x, x) = x_1^2 + \dots + x_5^2$$

is the standard metric of $\mathbb{R}^5$ and $\Upsilon(x, x, x) = \frac{3\sqrt{3}}{2}\det \mu(x)$ is given by

$$(1) \quad \begin{aligned} \Upsilon(x, x, x) &= \frac{1}{2}x_1(6x_2^2 + 6x_4^2 - 2x_1^2 - 3x_3^2 - 3x_5^2) \\ &\quad + \frac{3\sqrt{3}}{2}x_4(x_5^2 - x_3^2) + 3\sqrt{3}x_2x_3x_5. \end{aligned}$$

Obviously $P_{\rho(h) \cdot x} = P_x$, hence the polynomial Υ is ρ -invariant. Denote the $SO(3)$ -invariant symmetric 3-form on $\mathbb{R}^5$ corresponding to the $SO(3)$ -invariant polynomial Υ also by Υ .

An oriented orthonormal basis $a_1, \dots, a_5$ of $\mathbb{R}^5$ will be called *adapted* if $\Upsilon(x, x, x)$, $x \in \mathbb{R}^5$, is given by the right-hand side of (1) with $x_1, \dots, x_5$ being the coordinates of x with respect to the basis $a_1, \dots, a_5$. Obviously, the standard basis $e = (e_1, \dots, e_5)$ of $\mathbb{R}^5$ is adapted. In view of the $SO(3)$ -invariance of Υ , it is clear that the action of $SO(3)$ on $\mathbb{R}^5$ preserves the set of the adapted bases. Moreover, this action is transitive since, by [4, Proposition 2.5], the stabilizer of Υ under the standard action of $O(5)$ on the symmetric 3-forms on $\mathbb{R}^5$ coincides with $SO(3)$.

Now consider each $W \in \otimes^2 \mathbb{R}^5$ as an endomorphism of $\mathbb{R}^5$ given by $g(W(x), y) = g(W, x \otimes y)$, $x, y \in \mathbb{R}^5$, and set

$$\widehat{\Upsilon}(W)(x) = 4 \sum_{j=1}^5 \Upsilon_{W(e_j)} \circ \Upsilon_{e_j}(x).$$

Then $\widehat{\Upsilon}$ preserves the decomposition $\otimes^2 \mathbb{R}^5 = \Lambda^2 \mathbb{R}^5 \oplus \odot^2 \mathbb{R}^5$ and according to [4, Proposition 3.3] we have the orthogonal decomposition

$$(2) \quad \otimes^2 \mathbb{R}^5 = \Lambda_3^2 \oplus \Lambda_7^2 \oplus \odot_1^2 \oplus \odot_5^2 \oplus \odot_9^2,$$

where

$$\Lambda_3^2 = \{F \in \otimes^2 \mathbb{R}^5 : \widehat{\Upsilon}(F) = 7F\}, \quad \Lambda_7^2 = \{F \in \otimes^2 \mathbb{R}^5 : \widehat{\Upsilon}(F) = -8F\},$$

$$\odot_1^2 = \{S \in \otimes^2 \mathbb{R}^5 : \widehat{\Upsilon}(S) = 14S\} = \{S = \lambda g : \lambda \in \mathbb{R}\},$$

$$\odot_5^2 = \{S \in \otimes^2 \mathbb{R}^5 : \widehat{\Upsilon}(S) = -3S\}, \quad \odot_9^2 = \{S \in \otimes^2 \mathbb{R}^5 : \widehat{\Upsilon}(S) = 4S\}.$$

The representations Λ_j^2 and $\odot_k^2$ of $SO(3)$ are irreducible; the indices j and k indicate their dimensions.

The space Λ_3^2 will be used in the next section to define a twistor space.

If $a = \{a_1, \dots, a_5\}$ is an adapted basis of $\mathbb{R}^5$, set

$$(3) \quad \begin{aligned} \kappa_1 &= \kappa_1(a) = \sqrt{3}a_1 \wedge a_5 + a_2 \wedge a_3 + a_4 \wedge a_5, \\ \kappa_2 &= \kappa_2(a) = \sqrt{3}a_1 \wedge a_3 + a_2 \wedge a_5 + a_3 \wedge a_4, \\ \kappa_3 &= \kappa_3(a) = 2a_2 \wedge a_4 + a_3 \wedge a_5. \end{aligned}$$

Then $\{\kappa_1, \kappa_2, \kappa_3\}$ is an orthogonal basis of Λ_3^2 with $|\kappa_1|^2 = |\kappa_2|^2 = |\kappa_3|^2 = 5$, the metric on $\Lambda^2 \mathbb{R}^5$ being defined by $g(x \wedge y, u \wedge v) = g(x, u)g(y, v) - g(x, v)g(y, u)$.

2.2. Manifolds with an irreducible $SO(3)$ -structure.

Let M be a 5-dimensional manifold. Suppose that the structure group of the tangent bundle $\pi : TM \rightarrow M$ can be reduced to the group $\imath(SO(3))$. Using an atlas of trivializations of TM whose transition functions take values in $\imath(SO(3))$, we can define a Riemannian metric g and an orientation on M , and can transfer the tensor Υ on $\mathbb{R}^5$ to a rank 3 tensor on M (since Υ is invariant under $\imath(SO(3))$). We denote this tensor on M again by Υ . Taking an adapted basis of $\mathbb{R}^5$ we can get a local oriented orthonormal frame $E_1, \dots, E_5$ of TM such that

$$\begin{aligned} \Upsilon(X, X, X) &= \frac{1}{2}x_1(6x_2^2 + 6x_4^2 - 2x_1^2 - 3x_3^2 - 3x_5^2) \\ &\quad + \frac{3\sqrt{3}}{2}x_4(x_5^2 - x_3^2) + 3\sqrt{3}x_2x_3x_5 \end{aligned}$$

for $X = \sum_{i=1}^4 x_i E_i$. A frame with this property will be called *adapted*. The set of adapted bases of tangent spaces of M constitute the total space of a principal $\imath(SO(3))$ -bundle $\mathcal{AB}(M)$ that is a reduction of the bundle of oriented orthonormal frames of (M, g) .

For every $v \in TM$, the linear map $\Upsilon_v : T_{\pi(v)}M \rightarrow T_{\pi(v)}M$ defined by

$$g(\Upsilon_v(x), y) = \Upsilon(v, x, y), \quad x, y \in T_{\pi(v)}M,$$

has the following properties:

- (i) Υ_v is totally symmetric, i.e.
 $g(u, \Upsilon_v(w)) = g(w, \Upsilon_v(u)) = g(u, \Upsilon_v(w))$, $u, v, w \in T_{\pi(v)}M$;
- (ii) Υ_v is trace-free, $\text{Trace } \Upsilon_v = 0$;
- (iii) $\Upsilon_v^2(v) = g(v, v)v$.

One of the important observations in [4] is that any rank 3 tensor on an oriented Riemannian 5-manifold with the properties (i) - (iii) determines a reduction of the structure group of TM to the group $\imath(SO(3))$. Two reductions determined by tensors Υ and $\tilde{\Upsilon}$ are equivalent iff there is an isometry φ of M such that $\varphi^*(\tilde{\Upsilon}) = \Upsilon$.

We shall say a 5-manifold M has an *irreducible $SO(3)$ -structure* if the structure group of TM can be reduced to the group $\imath(SO(3))$. The topological obstructions for existence of such a structure are discussed in [1, 5]. Examples of manifolds with irreducible $SO(3)$ -structures can be found in [1, 4, 5, 6].

Let (M, g, Υ) be an oriented Riemannian 5-manifold with an irreducible $SO(3)$ -structure determined by a rank 3 tensor Υ having the properties (i) - (iii).

The orthogonal decomposition (2) of $\otimes^2 \mathbb{R}^5$ gives rise to an orthogonal decomposition

$$(4) \quad \otimes^2 TM = \Lambda_3^2 TM \oplus \Lambda_7^2 TM \oplus \odot_1^2 TM \oplus \odot_5^2 TM \oplus \odot_9^2 TM,$$

where $\Lambda_3^2 TM$ is the associated bundle $\mathcal{AB}(M) \times_{\imath(SO(3))} \Lambda_3^2$, etc.

Every $SO(3)$ -connection ∇ on M is metric ($\nabla g = 0$) and preserves Υ ($\nabla \Upsilon = 0$) as well as the decomposition (4).

The curvature operator of ∇ at any point $p \in M$ takes its values in $\Lambda_3^2 T_p M$ ($\cong so(3)$). Note that for the curvature tensor R and the curvature operator $\mathcal{R} : \Lambda^2 TM \rightarrow \Lambda^2 TM$ we adopt the following definitions:

$$\begin{aligned} R(X, Y, Z) &= -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z, \\ g(\mathcal{R}(X \wedge Y), Z \wedge U) &= g(R(X, Y)Z, U), \end{aligned}$$

for $X, Y, Z, U \in TM$.

3. THE TWISTOR SPACE OF A MANIFOLD WITH AN IRREDUCIBLE $SO(3)$ -STRUCTURE AND ALMOST CONTACT METRIC STRUCTURES ON IT

As in [4], define the twistor space of M as the sphere bundle

$$\mathbb{T} = \{\sigma \in \Lambda_3^2 TM : |\sigma|^2 = 5\}.$$

The radius of the fibre is chosen so that $\kappa_1, \kappa_2, \kappa_3$ defined via (3) by means of an adapted frame be sections of $\mathbb{T}$. We can define four CR -structures on the manifold $\mathbb{T}$ in the following way ([4]).

Henceforward the restrictions to $\Lambda_3^2 TM$ and $\mathbb{T}$ of the bundle projection $\pi : \Lambda^2 TM \rightarrow M$ will be denoted again by π . The tangent bundle of $\mathbb{T}$ will be considered as a subbundle of the tangent bundle of the manifold $\Lambda_3^2 TM$. For every $\sigma \in \mathbb{T}$, the vertical space $\mathcal{V}_\sigma$ at σ of the bundle $\pi : \mathbb{T} \rightarrow M$ (i.e. the tangent space at σ of the fibre through σ) will be denoted by $\mathcal{V}_\sigma$; clearly $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\Lambda_3^2 T_{\pi(\sigma)} M$.

Let ∇ be a $SO(3)$ -connection on M . Then, as we have mentioned, ∇ preserves $\Lambda_3^2 TM$, so it induces a connection on this vector bundle which we shall denote also by ∇ . For every $\sigma \in \mathbb{T}$, the horizontal subspace $\mathcal{H}_\sigma$ of $T_\sigma(\Lambda_3^2 TM)$ with respect to ∇ is tangent to $\mathbb{T}$ since ∇ is metric. Thus, the tangent bundle of the twistor space is the direct sum of its vertical bundle $\mathcal{V}$ and the horizontal bundle $\mathcal{H}$ of ∇ : $T\mathbb{T} = \mathcal{V} \oplus \mathcal{H}$.

For $\sigma \in \mathbb{T}$, set

$$\xi_\sigma = \frac{1}{4} * (\sigma \wedge \sigma),$$

where $*$: $\Lambda^4 TM \rightarrow TM$ is the Hodge star operator. Denote by H^σ the orthogonal complement of ξ_σ in $T_{\pi(\sigma)} M$ and give H^σ the orientation that yields the orientation of $T_{\pi(\sigma)} M$ via the decomposition $T_{\pi(\sigma)} M = \mathbb{R}\xi_\sigma \oplus H^\sigma$. Since (H^σ, g) is an oriented Euclidean four-dimensional vector space, $\Lambda^2 H^\sigma$ decomposes into self-dual and anti-self-dual parts, $\Lambda^2 H^\sigma = \Lambda_+^2 H^\sigma \oplus \Lambda_-^2 H^\sigma$. Denote by $\sigma_\pm$ the components of σ with respect to this decomposition. The 2-vectors $\sigma_\pm$ determine two complex structures $J_\pm^\sigma$ on the vector space H^σ compatible with its metric and $\pm$ -orientation. These are given by

$$(5) \quad g(J_\pm^\sigma X, Y) = \frac{2}{2 \pm 1} g(\sigma_\pm, X \wedge Y), \quad X, Y \in H^\sigma.$$

For $\sigma \in \mathbb{T}$, denote by $(H^\sigma)_\sigma^h$ the horizontal lift at σ of the vector space H^σ with respect to ∇ . Let $\mathcal{D} \rightarrow \mathbb{T}$ be the subbundle of $T\mathbb{T}$ whose fibre at $\sigma \in \mathbb{T}$ is

$$\mathcal{D}_\sigma = \mathcal{V}_\sigma \oplus (H^\sigma)_\sigma^h.$$

The fibre of the twistor bundle $\pi : \mathbb{T} \rightarrow M$ is the unit sphere and we denote its standard complex structure by $\mathcal{J}$. Then, as is usual in the twistor theory, we define almost complex structure $\mathcal{J}_\pm^{(n)}$, $n = 1, 2$, on the bundle $\mathcal{D}$ setting

$$\mathcal{J}_\pm^{(n)} V = (-1)^{n+1} \mathcal{J} \text{ for } V \in \mathcal{V}_\sigma, \quad \mathcal{J}_\pm^{(n)} X_\sigma^h = (J_\pm^\sigma X)_\sigma^h \text{ for } X \in H^\sigma.$$

In this way we obtain four almost CR -structures $(\mathcal{D}, \mathcal{J}_\pm^{(n)})$ on the manifold $\mathbb{T}$.

We extend $J_\pm^\sigma$ to partially complex structures $\varphi_\pm^\sigma$ on the space $T_{\pi(\sigma)} M$, setting $\varphi_\pm^\sigma(\xi_\sigma) = 0$ and $\varphi_\pm^\sigma = J_\pm^\sigma$ on H^σ . Then $\varphi_\pm^\sigma$ is g -skew-symmetric, $(\varphi_\pm^\sigma)^2 X = -X + g(X, \xi_\sigma)\xi_\sigma$ for every $X \in T_{\pi(\sigma)} M$, and $\text{Im } \varphi_\pm^\sigma = H^\sigma$, $\text{Ker } \varphi_\pm^\sigma = \mathbb{R}\xi_\sigma$. Now

we define partially complex structures $\Phi_{\pm}^{(n)}$, $n = 1, 2$, of rank 6 on the manifold $\mathbb{T}$ setting $\Phi_{\pm}^{(n)} = \mathcal{J}_{\pm}^{(n)}$ on $\mathcal{V}$ and $\Phi_{\pm}^{(n)} X_{\sigma}^h = (\varphi_{\pm}^{\sigma} X)_{\sigma}^h$ for $X \in T_{\pi(\sigma)}M$.

Remark. By "partially complex structure" of rank $2k$ on a manifold N , $0 < 2k \leq \dim N$, we mean an endomorphism $\mathcal{F}$ of TN of rank $2k$ such that $\mathcal{F}^3 + \mathcal{F} = 0$. Clearly, given such a structure, we have $TN = \text{Im}\mathcal{F} \oplus \text{Ker}\mathcal{F}$ and $\mathcal{F}$ is an almost complex structure on the vector bundle $\text{Im}\mathcal{F}$, so the name. Partially complex structures are also called f -structures.

For any fixed $t > 0$, define a metric h_t on $\mathbb{T}$ by

$$h(X_{\sigma}^h + V, Y_{\sigma}^h + W) = g(X, Y) + tg(V, W)$$

where $V, W \in \mathcal{V}_{\sigma}$, $X, Y \in T_{\pi(\sigma)}M$ (this is the so-called canonical variation of the Riemannian metric g [2]). Set $\chi_{\sigma} = (\xi_{\sigma})_{\sigma}^h$. Then $(\Phi_{\pm}^{(n)}, \chi, h_t)$ are almost contact metric structures inducing the CR -structures $(\mathcal{D}, \mathcal{J}_{\pm}^{(n)})$. We refer to [3] for general facts about (almost) contact metric structures.

4. NORMALITY OF THE ALMOST CONTACT METRIC STRUCTURES ON THE TWISTOR SPACE

Recall that any almost contact structure (φ, ξ, η) on a manifold N induces an almost complex structure on the manifold $N \times S^1$ in a natural way ([3]). An almost contact metric structure on N is said to be normal if the induced almost complex structure on $N \times S^1$ is integrable. It is well-known that (φ, ξ, η) is a normal structure if and only the tensor $N^{(1)}(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + d\eta(X, Y)\xi$ vanishes (see, for example, [3]; the definition of $d\eta$ used here is twice the one in [3]). This tensor for the almost contact structure $(\Phi_{\pm}^{(n)}, \chi, h_t)$ will be denote by $N_{\pm}^{(n)}$.

4.1. Certain technicalities.

For any adapted basis $a = \{a_1, \dots, a_5\}$ of $\mathbb{R}^5$, define an orthogonal basis of Λ_3^2 $\{\kappa_1(a), \kappa_2(a), \kappa_3(a)\}$ by means of (3).

The proof of the next statement contains formulas that will also be used latter on.

Lemma 1. All bases $\{\kappa_1(a), \kappa_2(a), \kappa_3(a)\}$ determine the same orientation on the vector space Λ_3^2 .

Proof. Let $a = \{a_1, \dots, a_5\}$ and $a' = \{a'_1, \dots, a'_5\}$ be two adapted bases of $\mathbb{R}^5$. As we have remarked, the group $SO(3)$ acts transitively on the set of adapted bases, so there is a matrix h in $SO(3)$ such that $\iota(h)$ sends the basis a to the basis a' . Any such a matrix h can be represented as the product $h_{\psi} \cdot h_{\theta} \cdot h_{\varphi}$ of the $SO(3)$ -matrices

$$h_{\psi} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, h_{\theta} = \begin{bmatrix} 0 & 0 & 1 \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{bmatrix}, h_{\varphi} = \begin{bmatrix} -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \\ \cos \varphi & \sin \varphi & 0 \end{bmatrix}.$$

The matrix $\iota(h_\varphi)$ transforms the bases $a = \{a_1, \dots, a_5\}$ to the bases $b = \{b_1, \dots, b_5\}$ where

$$(6) \quad \begin{aligned} b_1 &= -\frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2 \sin 2\varphi + \frac{\sqrt{3}}{2}a_4 \cos 2\varphi, \\ b_2 &= -a_3 \sin \varphi + a_5 \cos \varphi, \quad b_3 = a_2 \cos 2\varphi + a_4 \sin 2\varphi, \\ b_4 &= -\frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_2 \sin 2\varphi - \frac{1}{2}a_4 \cos 2\varphi, \quad b_5 = a_3 \cos \varphi + a_5 \sin \varphi. \end{aligned}$$

For this basis, we have

$$(7) \quad \begin{aligned} \kappa_1(b) &= -\sin \varphi \kappa_1(a) - \cos \varphi \kappa_2(a), \quad \kappa_2(b) = \kappa_3(a), \\ \kappa_3(b) &= \cos \varphi \kappa_1(a) - \sin \varphi \kappa_2(a). \end{aligned}$$

Hence $\{\kappa_1(a), \kappa_2(a), \kappa_3(a)\}$ and $\{\kappa_1(b), \kappa_2(b), \kappa_3(b)\}$ yield the same orientation.

The matrix $\iota(h_\theta)$ transforms $b = \{b_1, \dots, b_5\}$ to $c = \{c_1, \dots, c_5\}$ with

$$(8) \quad \begin{aligned} c_1 &= -\frac{1}{2}b_1 + \frac{\sqrt{3}}{2}b_2 \sin 2\theta - \frac{\sqrt{3}}{2}b_4 \cos 2\theta, \quad c_2 = b_3 \cos \theta + b_5 \sin \theta, \\ c_3 &= -b_3 \sin \theta + b_5 \cos \theta, \quad c_4 = \frac{\sqrt{3}}{2}b_1 + \frac{1}{2}b_2 \sin 2\theta - \frac{1}{2}b_4 \cos 2\theta, \\ c_5 &= b_2 \cos 2\theta + b_4 \sin 2\theta. \end{aligned}$$

It follows that

$$(9) \quad \begin{aligned} \kappa_1(c) &= \kappa_3(b), \quad \kappa_2(c) = -\cos \theta \kappa_1(b) + \sin \theta \kappa_2(a), \\ \kappa_3(c) &= -\sin \theta \kappa_1(b) - \cos \theta \kappa_2(b). \end{aligned}$$

Finally, the matrix $\iota(h_\psi)$ sends any adapted basis $c = \{c_1, \dots, c_5\}$ to the adapted bases $a' = \{a'_1, \dots, a'_5\}$ for which

$$(10) \quad \begin{aligned} a'_1 &= c_1, \quad a'_2 = c_2 \cos 2\psi + c_4 \sin 2\psi, \\ a'_3 &= c_3 \cos \psi + c_5 \sin \psi \\ a'_4 &= -c_2 \sin 2\psi + c_4 \cos 2\psi, \quad a'_5 = -c_3 \sin \psi + c_5 \cos \psi. \end{aligned}$$

This implies that

$$(11) \quad \begin{aligned} \kappa_1(a') &= \cos \psi \kappa_1(c) - \sin \psi \kappa_2(c), \quad \kappa_2(a') = \sin \psi \kappa_1(c) + \cos \psi \kappa_2(a), \\ \kappa_3(a') &= \kappa_3(c). \end{aligned}$$

Now, the lemma follows from (7), (9), (11). $\square$

For any $\kappa \in \Lambda^2 \mathbb{R}^5$, denote by S_κ the skew-symmetric endomorphism of $\Lambda^2 \mathbb{R}^5$ corresponding to κ : $g(S_\kappa(x), y) = g(\kappa, x \wedge y)$.

Let $(\kappa_1, \kappa_2, \kappa_3)$ be the basis of Λ_3^2 defined via (3) by means of the standard basis of $\mathbb{R}^5$. Then

$$[S_{\kappa_1}, S_{\kappa_2}] = -S_{\kappa_3}, \quad [S_{\kappa_2}, S_{\kappa_3}] = -S_{\kappa_1}, \quad [S_{\kappa_3}, S_{\kappa_1}] = -S_{\kappa_2}.$$

Thus the map $\Lambda_3^2 \ni \kappa \rightarrow -S_\kappa$ can be used to defined on Λ_3^2 the structure of a 3-dimensional Lie algebra isomorphic to $so(3)$. Considering Λ_3^2 with this structure, the negative Killing form coincides with the metric of Λ_3^2 induced by the metric of $\Lambda^2 \mathbb{R}^5$.

Fix the orientation on Λ_3^2 defined by Lemma 1. Then we have the following.

Lemma 2. For every oriented orthogonal basis q_1, q_2, q_3 of Λ_3^2 with $|q_1|^2 = |q_2|^2 = |q_3|^2 = 5$, there is an adapted basis $a = (a_1, \dots, a_5)$ of $\mathbb{R}^5$ such that

$$q_1 = \kappa_1(a), \quad q_2 = \kappa_2(a), \quad q_3 = \kappa_3(a).$$

Proof. Consider Λ_3^2 with the Lie algebra structure isomorphic to $so(3)$ we have defined above. Then it follows from (7), (9), (11) that the standard action of $\iota(SO(3))$ on $\Lambda_3^2 \subset \Lambda^2 \mathbb{R}^5$ coincides with the adjoint action of $SO(3)$: $Ad_h(S_\kappa) = \iota(h)^{-1} \circ S_\kappa \circ \iota(h)$ for $h \in SO(3)$, $\kappa \in \Lambda_3^2$. Let $(\kappa_1, \kappa_2, \kappa_3)$ be the basis of Λ_3^2 defined by means of the standard basis $(e_1, \dots, e_5)$ of $\mathbb{R}^5$. Then (q_1, q_2, q_3) can be sent to $(\kappa_1, \kappa_2, \kappa_3)$ by an orthogonal transformation of Λ_3^2 . Any such a transformation has the form Ad_h for some $h \in SO(3)$, so there is $h \in SO(3)$ for which $S_{q_j} = \iota(h)^{-1} \circ S_{\kappa_j} \circ \iota(h)$ for $j = 1, 2, 3$. Then the basis $a = \{a_l = \iota(h)^{-1}(e_l) : l = 1, \dots, 5\}$ is adapted and computing the values of S_{q_j} at $a_1, \dots, a_5$ we see that $S_{q_j} = S_{\kappa_j(a)}$, $j = 1, 2, 3$. Therefore $q_j = \kappa_j(a)$. $\square$

Now take a point $\sigma \in \mathbb{T}$. By Lemma 2, there is an adapted basis $(a_1, \dots, a_5)$ of $T_{\pi(\sigma)}M$ such that $\sigma = 2a_2 \wedge a_4 + a_3 \wedge a_5$. In this basis, $\xi_\sigma = \frac{1}{4} * (\sigma \wedge \sigma) = -a_1$ and (a_2, a_4, a_3, a_5) is an oriented orthonormal basis of H^σ . Note that $\iota_{\xi_\sigma} \sigma = 0$. Considering $\Lambda^2 H^\sigma$ as a subspace of $\Lambda^2 T_{\pi(\sigma)}M$, we have $\Lambda^2 H^\sigma = \{\tau \in \Lambda^2 T_{\pi(\sigma)}M : \iota_{\xi_\sigma} \tau = 0\}$; here and farther on $\iota : \Lambda^k TM \rightarrow \Lambda^{k-1} TM$, $1 \leq k \leq 5$, stands for the interior product, $g(\iota_X \tau, X_1 \wedge \dots \wedge X_{k-1}) = g(\tau, X \wedge X_1 \wedge \dots \wedge X_{k-1})$ for $\tau \in \Lambda^k TM$ and $X, X_1, \dots, X_{k-1} \in TM$. If $\sigma_\pm$ are the $\Lambda_\pm^2 H^\sigma$ -components of σ , we have $\sigma_+ = \frac{3}{2}(a_2 \wedge a_4 + a_3 \wedge a_5)$, $\sigma_- = \frac{1}{2}(a_2 \wedge a_4 - a_3 \wedge a_5)$ in the adapted basis $(a_1, \dots, a_5)$ we have chosen above. This shows that

$$(12) \quad \sigma_\pm = \frac{1}{2}(\sigma \pm \iota_{\xi_\sigma}(*\sigma)).$$

Note also that, in view of (5),

$$g(\varphi_\pm^\sigma X, Y) = \frac{2}{2 \pm 1} g(\sigma_\pm, X \wedge Y), \quad X, Y \in T_{\pi(\sigma)}M,$$

since $\iota_{\xi_\sigma} \sigma_\pm = 0$.

According to Lemma 1, the vector bundle $\Lambda_3^2 TM$ admits an orientation for which every $(\kappa_1, \kappa_2, \kappa_3)$ defined via (3) is an oriented frame. Denote by $\times$ the usual vector product on the oriented Euclidean 3-dimensional vector space $(\Lambda_3^2 T_p M, g_p)$, $p \in M$. Then the complex structure $\mathcal{J}$ of the fibre of $\mathbb{T}$ through σ is given by

$$\mathcal{J}V = -\sigma \times V, \quad V \in \mathcal{V}_\sigma.$$

Thus

$$\begin{aligned} \Phi_\pm^{(n)} V &= (-1)^n \sigma \times V \quad \text{for } V \in \mathcal{V}_\sigma, \\ \Phi_\pm^{(n)} X_\sigma^h &= (\varphi_\pm^\sigma X)_\sigma^h \quad \text{for } X \in T_{\pi(\sigma)}M. \end{aligned}$$

4.2. Computation of the tensors $N_\pm^{(n)}$.

Further on, for any $\sigma \in \Lambda^2 TM$, the endomorphism S_σ of $T_{\pi(\sigma)}M$ determined by σ and the metric g will also be denoted by σ .

Let $(x_1, \dots, x_5)$ be a local coordinate system of M and $(E_1, \dots, E_5)$ an adapted frame of TM on U . Using this frame, define sections $(\kappa_1, \kappa_2, \kappa_3)$ of $\Lambda_3^2 TM$ by (3). Set

$$(13) \quad \tilde{x}_\alpha(\tau) = x_\alpha \circ \pi(\tau), \quad 1 \leq \alpha \leq 5, \quad y_i(\tau) = \frac{1}{5} g(\tau, \kappa_i \circ \pi(\tau)), \quad 1 \leq i \leq 3,$$

for $\tau \in \Lambda_3^2 TM$. Then $(\tilde{x}_\alpha, y_i)$ is a local coordinate system of the manifold $\Lambda_3^2 TM$. For each vector field

$$X = \sum_{\alpha=1}^5 X^\alpha \frac{\partial}{\partial x_\alpha}$$

on U , its horizontal lift X^h is given by

$$(14) \quad X^h = \sum_{\alpha=1}^5 (X^\alpha \circ \pi) \frac{\partial}{\partial \tilde{x}_\alpha} - \sum_{i,j=1}^3 y_i (g(\nabla_X \kappa_i, \kappa_j) \circ \pi) \frac{\partial}{\partial y_j}.$$

For $\tau \in \Lambda_3^2 TM$, we have

$$(15) \quad [X^h, Y^h]_\tau = [X, Y]_\tau^h + R(X, Y)\tau,$$

where $R(X, Y)\tau$ is the curvature of the connection ∇ on the vector bundle $\Lambda_3^2 TM$.

Set

$$f_{\alpha\beta}^\pm(\sigma) = g(\varphi_\pm^\sigma E_\alpha, E_\beta)$$

for $\sigma \in (\pi|T)^{-1}(U)$. Then

$$(16) \quad \begin{aligned} \Phi_\pm^{(n)} X^h &= \sum_{\alpha,\beta=1}^5 (g(X, E_\alpha) \circ \pi) f_{\alpha\beta}^\pm E_\beta^h, \\ (\Phi_\pm^{(n)})^2 X^h &= \sum_{\alpha,\beta,\gamma=1}^5 (g(X, E_\alpha) \circ \pi) f_{\alpha\beta}^\pm f_{\beta\gamma}^\pm E_\gamma^h. \end{aligned}$$

It is easy to see that, in the local coordinates $\tilde{x}_\alpha, y_i$ on T ,

$$(17) \quad \xi_\sigma = \left(\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - y_3^2\right)E_1 - \sqrt{3}y_1y_2E_2 + \sqrt{3}y_1y_3E_3 - \frac{\sqrt{3}}{2}(y_1^2 - y_2^2)E_4 - \sqrt{3}y_2y_3E_5$$

for $\sigma = y_1\kappa_1 + y_2\kappa_2 + y_3\kappa_3$. Then, setting

$$h_\alpha(y) = g(\xi_\sigma, E_\alpha),$$

we have

$$(18) \quad f_{\alpha\beta}^\pm = \frac{1}{2 \pm 1} \sum_{i=1}^3 y_i [g(\kappa_i, E_\alpha \wedge E_\beta) \circ \pi \pm \sum_{\epsilon=1}^5 h_\epsilon(y) g(*\kappa_i, E_\alpha \wedge E_\beta \wedge E_\epsilon) \circ \pi].$$

Notation. Fix a point $\sigma \in T$ and take an adapted frame $(E_1, \dots, E_5)$ in a coordinate neighbourhood U of the point $p = \pi(\sigma)$ such that

$$\nabla E_\alpha|_p = 0, \alpha = 1, \dots, 5, \quad \text{and} \quad \sigma = (\kappa_3)_p,$$

where κ_3 is defined via (3) by means of the frame $(E_1, \dots, E_5)$. We also use this frame to define κ_1 and κ_2 by (3). Then $\nabla \kappa_i|_p = 0$, $i = 1, 2, 3$.

Choose local coordinates $x_1, \dots, x_5$ of M on U and define local coordinates of T by (13).

For $w \in \Lambda_3^2 TM$ and $\tau \in T$ with $\pi(w) = \pi(\tau)$, we set

$$\tilde{w}_\tau = w - \frac{1}{5}g(w, \tau)\tau,$$

the $\mathcal{V}_\tau$ -component of w .

Any section S of $\Lambda_3^2 TM$ near the point p yields a (local) vertical vector field $\tilde{S}$ on T defined by

$$\tilde{S}_\tau = S_{\pi(\tau)} - \frac{1}{5}g(S_{\pi(\tau)}, \tau)\tau.$$

We shall use this notation throughout this section and the following ones without further referring to it.

Lemma 3. Let X be a vector field on M and S a section of $\Lambda_3^2 TM$ defined on a neighbourhood of the point $p = \pi(\sigma)$. Then:

$$\begin{aligned}
[X^h, \tilde{S}]_\sigma &= \widetilde{(\nabla_X S)_\sigma}, \\
[X^h, \Phi_\pm^{(n)} \tilde{S}]_\sigma &= \Phi_\pm^{(n)} \widetilde{(\nabla_X S)_\sigma}, \\
[\Phi_\pm^{(n)} X^h, \tilde{S}]_\sigma &= \left(\widetilde{\nabla_{\varphi_\pm^\sigma X_p} S} \right)_\sigma \\
&\quad - \frac{1}{2 \pm 1} \{ \pm 3 [g(X_p, \xi_\sigma) \tilde{S}_\sigma(\xi_\sigma) - g(\tilde{S}_\sigma(\xi_\sigma), X_p) \xi_\sigma] + [1 \pm (-1)] \tilde{S}_\sigma X \}_\sigma^h, \\
[\Phi_\pm^{(n)} X^h, \Phi_\pm^{(n)} \tilde{S}]_\sigma &= \Phi_\pm^{(n)} \left(\widetilde{\nabla_{\varphi_\pm^\sigma X_p} S} \right)_\sigma \\
&\quad - \frac{1}{2 \pm 1} \{ \pm 3 [g(X_p, \xi_\sigma) (\Phi_\pm^{(n)} \tilde{S}_\sigma)(\xi_\sigma) - g((\Phi_\pm^{(n)} \tilde{S}_\sigma)(\xi_\sigma), X_p) \xi_\sigma] \\
&\quad \quad \quad + [1 \pm (-1)] (\Phi_\pm^{(n)} \tilde{S}_\sigma) X \}_\sigma^h, \\
[(\Phi_\pm^{(n)})^2 X^h, \tilde{S}]_\sigma &= \left(\widetilde{\nabla_{(\varphi_\pm^\sigma)^2 X_p} S} \right)_\sigma \\
&\quad - \frac{1}{2 \pm 1} \{ \pm 3 [g(X_p, \xi_\sigma) \varphi_\pm^\sigma(\tilde{S}_\sigma(\xi_\sigma)) + g(\varphi_\pm^\sigma(\tilde{S}_\sigma(\xi_\sigma)), X_p) \xi_\sigma] \\
&\quad \quad \quad + [1 \pm (-1)] [\varphi_\pm^\sigma(\tilde{S}_\sigma X) + \tilde{S}_\sigma(\varphi_\pm^\sigma X)] \}_\sigma^h, \\
[(\Phi_\pm^{(n)})^2 X^h, \Phi_\pm^{(n)} \tilde{S}]_\sigma &= \Phi_\pm^{(n)} \left(\widetilde{\nabla_{(\varphi_\pm^\sigma)^2 X_p} S} \right)_\sigma \\
&\quad - \frac{1}{2 \pm 1} \{ \pm 3 [g(X_p, \xi_\sigma) \varphi_\pm^\sigma((\Phi_\pm^{(n)} \tilde{S}_\sigma)(\xi_\sigma)) + g(\varphi_\pm^\sigma((\Phi_\pm^{(n)} \tilde{S}_\sigma)(\xi_\sigma)), X_p) \xi_\sigma] \\
&\quad \quad \quad + [1 \pm (-1)] [\varphi_\pm^\sigma((\Phi_\pm^{(n)} \tilde{S}_\sigma) X) + (\Phi_\pm^{(n)} \tilde{S}_\sigma)(\varphi_\pm^\sigma X)] \}_\sigma^h
\end{aligned}$$

Proof. In the local coordinates of $\mathbb{T}$ introduced above,

$$\tilde{S} = \sum_{i=1}^3 \tilde{S}_i \frac{\partial}{\partial y_i} \quad \text{where} \quad \tilde{S}_i = \frac{1}{5} (g(S, \kappa_i) \circ \pi - y_i \sum_{j=1}^3 y_j g(S, \kappa_j) \circ \pi).$$

Note that, since $\nabla \kappa_i|_p = 0$, $i = 1, 2, 3$, we have

$$(19) \quad X_\sigma^h = \sum_{\alpha=1}^5 X^\alpha(p) \left(\frac{\partial}{\partial \tilde{x}_\alpha} \right)_\sigma, \quad [X^h, \frac{\partial}{\partial y_i}]_\sigma = 0, \quad i = 1, 2, 3,$$

$$(\nabla_X S)_p = \frac{1}{5} \sum_{i=1}^3 X_p(g(S, \kappa_i))(\kappa_i)_p.$$

Note also that, setting $S_i = \frac{1}{5} g(S, \kappa_i) \circ \pi$, we have

$$(20) \quad \Phi_\pm^{(n)} \tilde{S} = (y_2 S_3 - y_3 S_2) \frac{\partial}{\partial y_1} + (y_3 S_1 - y_1 S_3) \frac{\partial}{\partial y_2} + (y_1 S_2 - y_2 S_1) \frac{\partial}{\partial y_3}.$$

Now the first and the second formulas of the lemma follow from (14) and (20).

In view of (16), (18) and (19), we have

$$\begin{aligned}
 [\Phi_{\pm}^{(n)} X^h, \tilde{S}]_{\sigma} &= \sum_{\alpha\beta=1}^5 \sum_{i=1}^3 g_p(X, E_{\alpha}) f_{\alpha\beta}^{\pm}(\sigma) (E_{\beta})_{\sigma}^h (\tilde{S}_i) \left(\frac{\partial}{\partial y_i} \right)_{\sigma} \\
 &\quad - \frac{1}{2 \pm 1} \{ \tilde{S}_1(\sigma) [\pm 3g_p(\sqrt{3}E_1 \wedge E_5, X \wedge E_{\beta}) + [1 \pm (-1)]g_p(\kappa_1, X \wedge E_{\beta})] \\
 &\quad + \tilde{S}_2(\sigma) [\pm 3g_p(\sqrt{3}E_1 \wedge E_3, X \wedge E_{\beta}) + [1 \pm (-1)]g_p(\kappa_2, X \wedge E_{\beta})] \} (E_{\beta})_{\sigma}^h \\
 &= \left(\widetilde{\nabla_{\varphi_{\pm}^{\sigma} X_p} S} \right)_{\sigma} \\
 &\quad - \frac{1}{2 \pm 1} \{ \pm 3g_p(\xi_{\sigma} \wedge \tilde{S}_{\sigma}(\xi_{\sigma}), X \wedge E_{\beta}) + [1 \pm (-1)]g_p(\tilde{S}_{\sigma}, X \wedge E_{\beta}) \} (E_{\beta})_{\sigma}^h
 \end{aligned}$$

since $(E_1)_p = -\xi_{\sigma}$ and $\sqrt{3}E_5 = \kappa_1(E_1)$, $\sqrt{3}E_3 = \kappa_2(E_1)$. This implies the third formula. Using (16) and (20) we obtain the forth formula by a similar computation. Identities (16) and (18) imply that

$$\begin{aligned}
 [(\Phi_{\pm}^{(n)})^2 X^h, \tilde{S}]_{\sigma} &= (D(\widetilde{(\varphi_{\pm}^{\sigma})^2 X_p} S))_{\sigma} - \frac{1}{2 \pm 1} \{ \pm 3[g_p(\xi_{\sigma} \wedge \tilde{S}_{\sigma}(\xi_{\sigma}), X \wedge E_{\beta}) \\
 &\quad + [1 \pm (-1)]g_p(\tilde{S}_{\sigma}, X \wedge E_{\beta})]g_p(\varphi_{\pm}^{\sigma} E_{\beta}, E_{\gamma}) \\
 &\quad + [\pm 3g_p(\xi_{\sigma} \wedge \tilde{S}_{\sigma}(\xi_{\sigma}), E_{\beta} \wedge E_{\gamma}) \\
 &\quad + [1 \pm (-1)]g_p(\tilde{S}_{\sigma}, E_{\beta} \wedge E_{\gamma})]g_p(\varphi_{\pm}^{\sigma} X, E_{\beta}) \} (E_{\gamma})_{\sigma}^h.
 \end{aligned}$$

This gives the fifth formula of the lemma. The last one can be obtained by a similar computation. $\square$

Proposition 1. The contact distribution of $(\Phi_{\pm}^{(n)}, \chi, h_t)$ is not integrable.

Proof. The section κ_1 of $\Lambda^2 TM$ yields the vertical vector field $\tilde{\kappa}_1$ on $\mathbb{T}$ given by $\tilde{\kappa}_1 = (1 - y_1^2) \frac{\partial}{\partial y_1} - y_1 y_2 \frac{\partial}{\partial y_2} - y_1 y_3 \frac{\partial}{\partial y_3}$. Put $A = \sqrt{3}y_1 y_3 E_1^h - (\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - y_3^2)E_3^h$. Then $\tilde{\kappa}_1$ and A are vector fields on $\mathbb{T}$ perpendicular to the characteristic vector field χ . Bearing in mind identities (19) and the fact that $\sigma = (\kappa_3)_{\pi(\sigma)}$, we easily see that $[\tilde{\kappa}_1, A]_{\sigma} = \sqrt{3}(E_1)_{\sigma}^h = -\sqrt{3}\chi_{\sigma}$. Thus $\tilde{\kappa}_1$ and A are sections of the contact distributions whose Lie bracket is not a section of it. $\square$

Lemma 4. Let $\sigma \in \mathbb{T}$ and let X, Y be vector fields near the point $p = \pi(\sigma)$ such that $\nabla X|_p = 0$, $\nabla Y|_p = 0$. Then, for any $k, l = 0, 1, 2$,

$$[(\Phi_{\pm}^{(n)})^k X^h, (\Phi_{\pm}^{(n)})^l Y^h]_{\sigma} = -(T_p((\varphi_{\pm}^{\sigma})^k X, (\varphi_{\pm}^{\sigma})^l Y))_{\sigma}^h + R_p((\varphi_{\pm}^{\sigma})^k X, (\varphi_{\pm}^{\sigma})^l Y)_{\sigma},$$

$n = 1, 2$.

Proof. We have $\nabla \kappa_i|_p = \nabla(*\kappa_i)|_p = 0$, $i = 1, 2, 3$, since $\nabla E_{\alpha}|_p = 0$, $1 \leq \alpha \leq 5$. Then it is clear from (18) and (19) that $Z_{\sigma}^h(f_{\alpha\beta}) = 0$ for every $Z \in T_p M$. In view of the assumption $\nabla X|_p = \nabla Y|_p = 0$, an easy computation using (15) and (16) gives the lemma. $\square$

Denote the Levi-Civita connection of the metric h_t by D . Let ∇^{LC} be the Levi-Civita connection of the metric g on M . Then the Koszul formula and the fact that

the Lie bracket of a vertical and a horizontal vector field is a vertical vector field imply

$$(D_{X^h}Y^h)_\sigma = (\nabla_X^{LC}Y)_\sigma^h + \frac{1}{2}R(X, Y)\sigma.$$

We have $\nabla_X Y = \nabla_X^{LC}Y + \frac{1}{2}T(X, Y)$ since ∇ is a metric connection with skew-symmetric torsion T . Thus

$$(21) \quad (D_{X^h}Y^h)_\sigma = (\nabla_X Y - \frac{1}{2}T(X, Y))_\sigma^h + \frac{1}{2}R(X, Y)\sigma.$$

Let V be a vertical vector field in a neighbourhood of a point $\sigma \in \mathbb{T}$. Take a section S of $\Lambda_3^2 TM$ near the point $p = \pi(\sigma)$ such that $S_p = V_\sigma$ and $\nabla S|_p = 0$. We can also find sections S_1, S_2 such that $\nabla S_1|_p = \nabla S_2|_p = 0$ and the corresponding vertical vector fields $\tilde{S}_1, \tilde{S}_2$ constitute a frame of the vertical bundle $\mathcal{V}$ in a neighbourhood of σ . Then the Koszul formula and the first identity of Lemma 3 imply that $(D_{\tilde{S}}\tilde{S}_k)_\sigma \in \mathcal{V}_\sigma$. It follows that, for every vertical vector field W , $D_V W$ is a vertical vector field. Thus the fibres of $\mathbb{T}$ are totally geodesic submanifolds. This also follows from the Vilms theorem [9] (or [2]). Then $D_V X^h$ is orthogonal to every vertical vector field, thus $D_V X^h$ is a horizontal vector field. Hence $D_V X^h = \mathcal{H}D_{X^h}V$ since $[V, X^h]$ is vertical. Therefore

$$(22) \quad g(D_V X^h, Y^h)_\sigma = g(D_{X^h}V, Y^h)_\sigma = -h_t(V, D_{X^h}Y^h)_\sigma = -\frac{t}{2}g(R(X, Y)\sigma, V).$$

Lemma 5. Let $V \in \mathcal{V}_\sigma$ and $X \in T_{\pi(\sigma)}M$. Then

$$(D_{X^h}\chi)_\sigma = -\frac{1}{2}(T(X, \xi_\sigma))_\sigma^h + \frac{1}{2}R(X, \xi_\sigma)\sigma, \quad (D_V\chi)_\sigma \in \mathcal{H}_\sigma,$$

$$h_t(D_V\chi, X^h)_\sigma = g((\mathcal{J}V)(\xi_\sigma), X) + \frac{t}{2}g(R(X, \xi_\sigma)\sigma, V).$$

Proof. We have

(23)

$$\chi = \left(\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - y_3^2\right)E_1^h - \sqrt{3}y_1y_2E_2^h + \sqrt{3}y_1y_3E_3^h - \frac{\sqrt{3}}{2}(y_1^2 - y_2^2)E_4^h - \sqrt{3}y_2y_3E_5^h$$

in view of (17). Identities (19), (23) and (21) imply

$$(D_{X^h}\chi)_\sigma = -D_{X^h}E_1^h = \frac{1}{2}(T(X, E_1))_\sigma^h - \frac{1}{2}R(X, E_1)\sigma =$$

$$-\frac{1}{2}(T(X, \xi_\sigma))_\sigma^h + \frac{1}{2}R(X, \xi_\sigma)\sigma.$$

Let $V = v_1\left(\frac{\partial}{\partial y_1}\right)_\sigma + v_2\left(\frac{\partial}{\partial y_2}\right)_\sigma$. Then, by (23),

$$(D_V\chi)_\sigma = -(D_VE_1^h)_\sigma + (\sqrt{3}v_1E_3 - \sqrt{3}v_2E_5)_\sigma^h =$$

$$-(D_VE_1^h)_\sigma + ((v_1\kappa_2 - v_2\kappa_1)(E_1))_\sigma^h = -(D_VE_1^h)_\sigma + ((\sigma \times V)(E_1))_\sigma^h.$$

This and (22) imply the second identity of the lemma. Note also that $(D_VE_1^h)_\sigma \in \mathcal{H}_\sigma$, so $(D_V\chi, X^h)_\sigma \in \mathcal{H}_\sigma$. $\square$

Set

$$\eta_t(A) = h_t(A, \chi), \quad \Omega_{\pm t}^{(n)}(A, B) = h_t(A, \Phi_{\pm}^{(n)} B), \quad A, B \in T\mathbb{T}.$$

Corollary 1. If $V, W \in \mathcal{V}_\sigma$ and $X, Y \in T_{\pi(\sigma)}M$, then

$$\begin{aligned} d\eta_t(X_\sigma^h + V, Y_\sigma^h + W) = & -\frac{1}{2}[g(T(X, \xi_\sigma), Y) - g(T(Y, \xi_\sigma), X)] \\ & \pm[g(\varphi_\pm^\sigma X, W(\xi_\sigma)) - g(\varphi_\pm^\sigma Y, V(\xi_\sigma))]. \end{aligned}$$

In particular, $d\eta_t \neq 0$ at every point and the almost contact structure $(\Phi_\pm^{(n)}, \chi, h_t)$ is not contact, i.e. $d\eta_t \neq \Omega_{\pm t}^{(n)}$.

Proof. Note first that, for every $V \in \mathcal{V}_\sigma$,

$$(24) \quad (\mathcal{J}_\pm^{(n)} V)(\xi_\sigma) = \pm(-1)^{n+1} J_\pm^\sigma(V(\xi_\sigma)), \quad n = 1, 2.$$

Indeed, it is easy to check this identity for $V = (\kappa_1)_p$ and $V = (\kappa_2)_p$ taking into account the fact that, since $\sigma = (\kappa_3)_p$, we have $\xi_\sigma = -(E_1)_p$, $H^\sigma = \text{span}\{E_2, E_4, E_3, E_5\}$, $J_\pm^\sigma E_2 = E_4$, $J_\pm^\sigma E_3 = \pm E_5$. By (24), we have

$$(\mathcal{J}V)(\xi_\sigma) = (-1)^{n+1}(\Phi_\pm^n V)(\xi_\sigma) = (-1)^{n+1}(\mathcal{J}_\pm^{(n)} V)(\xi_\sigma) = \pm J_\pm^\sigma(V(\xi_\sigma)) = \pm \varphi_\pm^\sigma(V(\xi_\sigma)).$$

Now Lemma 5 implies the formula for $d\eta_t$ stated in the corollary.

For $\sigma = \kappa_3$, $X = E_3$, $W = \kappa_1$, we have $d\eta_t(X, W) = \sqrt{3}$. Thus $d\eta_t(X_\sigma^h, W) = \pm g(E_5, -\sqrt{3}E_5) = \mp\sqrt{3}$ while $\Omega_{\pm t}^{(n)}(X_\sigma^h, W) = 0$. $\square$

Corollary 2. Every integral curve of the characteristic vector field χ is a geodesic.

Recall that a metric connection is said to be *characteristic* if its torsion $T(X, Y, Z) = g(T(X, Y), Z)$ is totally skew-symmetric. According to [4, Theorem 5.5] a $SO(3)$ -connection ∇ on M is characteristic if and only the tensor Υ satisfies the identity $(\nabla_v^{LC} \Upsilon)(v, v, v) = 0$ for every $v \in TM$ where ∇^{LC} is the Levi-Civita connection of (M, g) . In this case we say that the $SO(3)$ -structure is *nearly integrable* by analogy with the case of a nearly Kähler structure. The characteristic connection of such a structure is unique [ibid]. We refer to [1, 4, 6] for interesting example of nearly integrable structures.

Corollary 3. If ∇ is the characteristic connection, then:

- (i) The vector field χ is conformally Killing only when it is Killing.
- (ii) χ is Killing if and only if ∇ is of constant curvature t^{-1} .

Proof. Suppose that $\mathcal{L}_\chi h_t = \lambda h_t$ for some constant λ , $\mathcal{L}$ being the Lie derivative. According to Lemma 5, this identity is equivalent to the identities $g(T(X, \xi_\sigma), Y) + g(T(Y, \xi_\sigma), X) = -2\lambda g(X, Y)$ and $tg(R(X, \xi_\sigma)\sigma, V) + g((\mathcal{J}V)(\xi_\sigma), X) = 0$ for every $\sigma \in \mathbb{T}$, $X, Y \in T_{\pi(\sigma)}M$, $V \in \mathcal{V}_\sigma$. The first of these identities holds if and only if $\lambda = 0$ since the torsion is skew-symmetric. Take an adapted basis $(a_1, \dots, a_5)$ of $T_{\pi(\sigma)}M$. Then, in view of (17), the second identity is equivalent to

$$(25) \quad tg(R(X, a_\alpha)\sigma, V) + g(\mathcal{J}V(a_\alpha), X) = 0, \quad \alpha = 1, \dots, 5.$$

Using the fact that R is the curvature tensor of a metric connection, one can check that, for every $\sigma, \tau \in \Lambda_3^2 T_p M$,

$$(26) \quad g(R(X, Y)\sigma, \tau) = -g(\mathcal{R}(X \wedge Y), \sigma \times \tau), \quad X, Y \in T_p M.$$

It follows that identity (25) is equivalent to $tg(\mathcal{R}(X \wedge Y), \mathcal{J}V) = g(\mathcal{J}V, X \wedge Y)$. The latter identity holds if and only if $tg(\mathcal{R}(X \wedge Y), \tau) = g(X \wedge Y, \tau)$ for every $\tau \in \Lambda_3^2 T_{\pi(\sigma)} M$, hence $t\mathcal{R} = \mathcal{P}$ since $\mathcal{R}$ takes its values in $\Lambda_3^2 TM$. $\square$

Lemma 3, identity (24) and Corollary 1 imply the following.

Proposition 2. If $V \in \mathcal{V}_\sigma$, $\sigma \in \mathbb{T}$, and $X \in T_{\pi(\sigma)} M$, then

$$\begin{aligned} N_+^{(n)}(X_\sigma^h, V) &= \{[(-1)^{n+1} - 1]g(X, \xi_\sigma)(J_+^{(n)}V)(\xi_\sigma) + \\ &\quad [(-1)^n + 1]g((J_+^{(n)}V)(\xi_\sigma), X)\xi_\sigma\}_\sigma^h; \\ N_-^{(n)}(X_\sigma^h, V) &= \{3[(-1)^n + 1]g(X, \xi_\sigma)(J_-^{(n)}V)(\xi_\sigma) + \\ &\quad [(-1)^n + 3]g((J_-^{(n)}V)(\xi_\sigma), X)\xi_\sigma + 2(J_-^{(n)}V)(X) - 2\varphi_-^\sigma(VX)\}_\sigma^h. \end{aligned}$$

Lemma 4 and Corollary 1 imply

Proposition 3. Let $\sigma \in \mathbb{T}$ and $X, Y \in T_{\pi(\sigma)} M$. Then

$$\begin{aligned} N_\pm^{(n)}(X_\sigma^h, Y_\sigma^h) &= \{T(X, Y) - T(\varphi_\pm^\sigma X, \varphi_\pm^\sigma Y) + \varphi_\pm^\sigma T(\varphi_\pm^\sigma X, Y) + \varphi_\pm^\sigma T(X, \varphi_\pm^\sigma Y)\}_\sigma^h \\ &\quad - R(X, Y)\sigma + R(\varphi_\pm^\sigma X, \varphi_\pm^\sigma Y)\sigma - \mathcal{J}_\pm^{(n)}R(\varphi_\pm^\sigma X, Y)\sigma - \mathcal{J}_\pm^{(n)}R(X, \varphi_\pm^\sigma Y)\sigma. \end{aligned}$$

Proposition 4. The almost contact structures $(\Phi_-^{(1)}, \chi, h_t)$ and $(\Phi_\pm^{(2)}, \chi, h_t)$ are not normal.

Proof. Take $\sigma = \kappa_3$ for an adapted basis $\{E_1, \dots, E_5\}$ and set $V = (\kappa_1)_{\pi(\sigma)}$, $X = E_3$. We have $\xi_\sigma = -E_1$, $\mathcal{J}_\pm^{(n)}V = (-1)^n(\kappa_2)_{\pi(\sigma)}$, $\varphi_-^\sigma E_2 = E_4$, $\varphi_-^\sigma E_3 = -E_5$. Then, by the Proposition 2, $N_-^{(1)}(E_3^h, \kappa_1)_\sigma = 2(E_4)_\sigma^h$, $N_+^{(2)}(E_3^h, \kappa_1)_\sigma = 2\sqrt{3}(E_1)_\sigma^h$, $N_-^{(2)}(E_3^h, \kappa_1)_\sigma = 2(\sqrt{3}E_1 + E_4)_\sigma^h$. $\square$

In order to analyze the normality condition $N_+^{(1)} = 0$, we need, in view of Proposition 3, some preliminaries on the curvature tensor R of ∇ .

4.3. A curvature decomposition.

Since ∇ is a $SO(3)$ -connection, the curvature operator at any point $p \in M$ takes its values in $\Lambda_3^2 T_p M$, i.e. $\mathcal{R} \in \text{Hom}(\Lambda^2 TM, \Lambda_3^2 TM)$.

The irreducible components of $\text{Hom}(\Lambda^2, \Lambda_3^2)$, where $\Lambda^2 = \Lambda^2 \mathbb{R}^5$, under the action of $SO(3)$ have been described in [4, Proposition 5.8]. To state the result of [4] we introduced the following notation. Given a map $\mathcal{K} \in \text{Hom}(\Lambda^2, \Lambda^2)$, we denote:

- the corresponding 4-tensor by K , i.e.

$$K(X, Y, Z, U) = g(\mathcal{K}(X \wedge Y), Z \wedge U), \quad X, Y, Z, U \in \mathbb{R}^5,$$

where g stands for the metric on Λ^2 induced by the standard metric of $\mathbb{R}^5$;

- the Ricci tensor of K - by ρ_K ,

$$\rho_K(X, Y) = \text{Trace}\{(Z, U) \rightarrow K(X, Z, Y, U)\};$$

- the symmetric and skew-symmetric parts of ρ_K - by ρ_K^+ and ρ_K^- ,

$$\rho_K^\pm(X, Y) = \frac{1}{2}(\rho_K(X, Y) \pm \rho_K(Y, X));$$

- the scalar curvature of K - by s_K , $s_K = \text{Trace } \rho_K$;

- the anti-symmetrization of K - by A_K .

Then we have the following.

Proposition 5. ([4]) The map

$$\mathcal{K} \rightarrow (A_K, \rho_K^-, s_K g, \rho_K^+ - \frac{1}{5} s_K g)$$

is an equivariant isomorphism

$$\Psi : \text{Hom}(\Lambda^2, \Lambda_3^2) \rightarrow \Lambda^4 \oplus (\Lambda_3^2 \oplus \Lambda_7^2) \oplus \odot_1^2 \oplus (\odot_5^2 \oplus \odot_9^2)$$

Note that the summands on the right-hand side are irreducible under the $SO(3)$ -action. We shall give a description of their images in $\text{Hom}(\Lambda^2, \Lambda_3^2)$ under the inverse isomorphism Ψ^{-1} .

Denote by $\mathcal{P} : \Lambda^2 = \Lambda_3^2 \oplus \Lambda_7^2 \rightarrow \Lambda_3^2$ the orthogonal projection onto Λ_3^2 . Given a form $\nu \in \Lambda^4$, let $\mathcal{N} : \Lambda^2 \rightarrow \Lambda^2$ be the associated symmetric linear map defined by $g(\mathcal{N}(X \wedge Y), Z \wedge U) = \nu(X \wedge Y \wedge Z \wedge U)$. Set

$$\mathcal{K}_\nu = \frac{10}{3} \mathcal{P} \circ \mathcal{N}.$$

Lemma 6. $\Psi(\mathcal{K}_\nu) = \nu$.

Proof. Let $\{E_1, \dots, E_5\}$ be an adapted basis of $\mathbb{R}^5$ and let $\{\kappa_1, \kappa_2, \kappa_3\}$ be the corresponding basis of Λ_3^2 defined via (3).

Taking into account the skew-symmetry of ν , one can easily see by a straightforward computation that

$$\rho_{K_\nu}(X, Z) = \frac{2}{3} \sum_{\alpha=1}^5 \sum_{i=1}^3 \nu(X \wedge E_\alpha \wedge \kappa_i) g(\kappa_i, Z \wedge E_\alpha) = 0.$$

For the anti-symmetrization of K_ν , we have

$$\begin{aligned} A_{K_\nu}(X, Y; Z, U) = & \frac{1}{9} \sum_{\alpha=1}^5 \sum_{i=1}^3 [\nu(X \wedge Y \wedge \kappa_i) g(\kappa_i, Z \wedge U) + \nu(Z \wedge U \wedge \kappa_i) g(\kappa_i, X \wedge Y) \\ & + \nu(Y \wedge Z \wedge \kappa_i) g(\kappa_i, X \wedge U) + \nu(X \wedge U \wedge \kappa_i) g(\kappa_i, Y \wedge Z) \\ & + \nu(Z \wedge X \wedge \kappa_i) g(\kappa_i, Y \wedge U) + \nu(Y \wedge U \wedge \kappa_i) g(\kappa_i, Z \wedge X)]. \end{aligned}$$

It is easy to check by means of this formula that $(AK_\nu) = \nu$ on the standard basis of Λ^4 yielded by $\{E_1, \dots, E_5\}$. Thus $(AK_\nu) = \nu$ on Λ^4 and the lemma is proved. $\square$

Now, let $\eta \in \otimes^2 \mathbb{R}^5$ and denote again by η the endomorphism of $\mathbb{R}^5$ determined by the form η via $g(\eta(X), Y) = \eta(X, Y)$. The form η is the Ricci tensor of the operator $\Lambda^2 \rightarrow \Lambda^2$ defined by $X \wedge Y \rightarrow \eta(X) \wedge Y + X \wedge \eta(Y)$. Since we need an operator with values in Λ_3^2 , we consider $\mathcal{P}(\eta(X) \wedge Y + X \wedge \eta(Y))$. It is not hard to check that the anti-symmetrization of the corresponding 4-tensor vanishes on the standard basis of Λ^4 yielded by an adapted basis of $\mathbb{R}^5$. To find the Ricci tensor, we note first that a direct computation gives

$$\sum_{i=1}^3 g(\kappa_i X, \kappa_i Y) = \sum_{i=1}^3 \sum_{\alpha=1}^5 g(X, \kappa_i E_\alpha) g(Y, \kappa_i E_\alpha) = 6g(X, Y).$$

Then

$$\begin{aligned} \sum_{\alpha=1}^5 \sum_{i=1}^3 g(\eta(X) \wedge E_{\alpha}, \kappa_i) g(\kappa_i, Z \wedge E_{\alpha}) &= \sum_{i=1}^3 g(\kappa_i(\eta(X)), \kappa_i(Z)) \\ &= 6g(\eta(X), Z) = 6\eta(X, Z). \end{aligned}$$

Moreover, in the cases when η is either symmetric or skew-symmetric, we have

$$\begin{aligned} \sum_{\alpha=1}^5 \sum_{i=1}^3 g(X \wedge \eta(E_{\alpha}), \kappa_i) g(\kappa_i, Z \wedge E_{\alpha}) \\ = \sum_{\alpha=1}^5 \sum_{i=1}^3 g(\kappa_i X, \eta(E_{\alpha}) g(\kappa_i Z, E_{\alpha})) = \pm \sum_{i=1}^3 \eta(\kappa_i X, \kappa_i Z) \end{aligned}$$

where the plus sign corresponds to the case when η is symmetric and the minus sign– to the case of a skew-symmetric form η . This suggests to define a (symmetric, respectively, skew-symmetric) form setting

$$(27) \quad \eta'(X, Z) = \text{Trace}\{\Lambda_3^2 \ni \sigma \rightarrow \eta(\sigma X, \sigma Z)\}.$$

Set

$$\mathcal{K}_{\eta}^{-}(X \wedge Y) = \frac{5}{6} \mathcal{P}(\eta(X) \wedge Y + X \wedge \eta(Y) + \eta'(X) \wedge Y + X \wedge \eta'(Y)).$$

Lemma 7. If $\eta \in \Lambda^2$, then $\Psi(\mathcal{K}_{\eta}^{-}) = \eta$.

Proof. Since η is skew-symmetric, $\eta'(X \wedge Z) = \eta(\sum_{i=1}^3 \kappa_i X \wedge \kappa_i Z)$ and one can see by a direct computation that

$$\sum_{i=1}^2 \kappa_i X \wedge \kappa_i Z = \sum_{i=1}^3 g(\kappa_i X, Z) \kappa_i = \sum_{i=1}^3 g(X \wedge Z, \kappa_i) \kappa_i = 5\mathcal{P}(X \wedge Z).$$

Thus, $\eta' = 5\eta \circ \mathcal{P}$. Then the anti-symmetrization of the 4-tensor K_{η}^{-} vanishes and its Ricci tensor is equal to η . $\square$

Now set

$$\mathcal{K}_{\eta}^{+}(X \wedge Y) = \frac{5}{18} \mathcal{P}(5\eta(X) \wedge Y + 5X \wedge \eta(Y) - \eta'(X) \wedge Y - X \wedge \eta'(Y))$$

Lemma 8. In the notation (2), if $\eta \in \odot_5^2 \oplus \odot_3^2$, then $\Psi(\mathcal{K}_{\eta}^{+}) = \eta$.

Proof. We set $\eta'' = (\eta')'$ for any symmetric form η . Then a simple (but tedious) computation shows that

$$\eta'(X, Z) + \eta''(X, Z) = 12\eta(X, Y) + 6\text{Trace} \eta \cdot g(X, Z).$$

Noting that the last term vanishes when η is orthogonal to g , we get the lemma. $\square$

The preceding considerations give also the following.

Lemma 9. $\Psi(\frac{5}{6}\mathcal{P}) = g$.

4.4. Conditions for normality of $(\Phi_{+}^{(1)}, \chi, h_t)$.

Now, in order to give necessary and sufficient conditions for normality of the almost contact metric structure $(\Phi_{+}^{(1)}, \chi, h_t)$, we define a tensor on M as follows. The definition $\xi_{\sigma} = \frac{1}{4} * (\sigma \wedge \sigma)$ makes sense for every $\sigma \in \Lambda_3^2 TM$. Note that we have $g(\xi_{\sigma}, \xi_{\sigma}) = |\sigma|^4$. We extend also the definition of the operator φ_{+}^{σ} setting $\varphi_{+}^{\sigma} X = \frac{1}{3} \iota_X(\sigma|\sigma|^2 + \iota_{\xi_{\sigma}}(*\sigma))$ for $\sigma \in \Lambda_3^2 TM$ and $X \in T_{\pi(\sigma)}M$. As above, let A_R

be the anti-symmetrization of the curvature 4-tensor R and ρ_R^+ the symmetric part of the Ricci tensor. Set

$$\eta = \rho_R^+ - \frac{1}{5}s_R g$$

and

$$(28) \quad \begin{aligned} Q(\sigma, X) &= 12[A_R(\xi_\sigma \wedge \varphi_+^\sigma X \wedge \sigma) - A_R(\xi_\sigma \wedge \iota_X \sigma \wedge \sigma)|\sigma|^2] \\ &+ 5[g(X, \xi_\sigma)\eta(\xi_\sigma, \xi_\sigma) - \eta(X, \xi_\sigma)g(\xi_\sigma, \xi_\sigma) - \eta(\varphi_+^\sigma(\iota_X \sigma), \xi_\sigma)]. \end{aligned}$$

Let $(\kappa_1, \kappa_2, \kappa_3)$ be the frame of $\Lambda_3^2 TM$ defined by means of an adapted frame $(a_1, \dots, a_5)$ of TM via formulas (3). Then, for $\sigma = y_1\kappa_1 + y_2\kappa_2 + y_3\kappa_3$, the coefficients of ξ_σ in the basis $(a_1, \dots, a_5)$ are homogeneous polynomials of degree 2 in y_1, y_2, y_3 while the coefficients of $\varphi_+^\sigma X$, X -fixed, in the basis $a_i \wedge a_j$ are homogeneous polynomials in y_1, y_2, y_3 of degree 3. Thus $Q(\sigma, X)$ is a homogeneous polynomial of degree 6 in σ and is linear in X . Therefore $Q(\sigma, X)$ determined a section of the bundle $\odot^6 \Lambda_3^2 TM \otimes TM$. The latter is a subbundle of $\otimes^{13} TM$, so Q can be considered as a section of $\otimes^{13} TM$, i.e. as a tensor on M . This tensor will again be denoted by Q .

We need the following.

Lemma 10. We have:

$$Q(\sigma, \xi_\sigma) = 0,$$

$$Q(\sigma, X) = 0 \text{ for every } X \in \mathcal{V}(\xi_\sigma) = \{V(\xi_\sigma) : V \in \mathcal{V}_\sigma\},$$

$$Q(\sigma, X) = -6A_R(\xi_\sigma \wedge \iota_X \sigma \wedge \sigma)|\sigma| + 5\eta(X, \xi_\sigma)g(\xi_\sigma, \xi_\sigma) \text{ for } X \in (\mathbb{R}\xi_\sigma \oplus \mathcal{V}(\xi_\sigma))^\perp.$$

Proof. The identity $Q(\sigma, \xi_\sigma) = 0$ is obvious in view of the identities $\varphi_+^\sigma(\xi_\sigma) = 0$ and $\iota_{\xi_\sigma} \sigma = 0$. Since $Q(\sigma, X)$ is homogeneous in σ , to prove the remaining identities of the lemma, we may assume that $|\sigma|^2 = 5$. Then, by Lemma 2, we can take an adapted basis $a = (a_1, \dots, a_5)$ of $T_{\pi(\sigma)}M$ such that $\sigma = \kappa_3(a)$. Recall that in this case $\xi_\sigma = -a_1$, $H^\sigma = \text{span}\{a_2, a_4, a_3, a_5\}$, $\varphi_+^\sigma a_2 = a_4$, $\varphi_+^\sigma a_3 = a_5$. Moreover, $\mathcal{V}_\sigma = \text{span}\{\kappa_1, \kappa_2\}$, so $\mathcal{V}(\xi_\sigma) = \text{span}\{a_3, a_5\}$ and $(\mathbb{R}\xi_\sigma \oplus \mathcal{V}(\xi_\sigma))^\perp = \text{span}\{a_2, a_4\}$. Thus $\iota_X \sigma = \varphi_+^\sigma X$ for $X \in \mathcal{V}(\xi_\sigma)$, $\iota_X \sigma = 2\varphi_+^\sigma X$ for $X \in (\mathbb{R}\xi_\sigma \oplus \mathcal{V}(\xi_\sigma))^\perp$, and the result follows. $\square$

Theorem 1. Suppose that the $SO(3)$ -structure on M is nearly integrable. Then the almost contact structure $(\Phi_+^{(1)}, \chi, h_t)$ is normal if and only if $*T \in \Lambda_3^2 TM$, the tensor Q and the $\odot_9^2$, and Λ_7^2 -components of $\mathcal{R}$ vanishes.

Proof. Let V, W be vertical vectors at a point $\sigma \in \mathbb{T}$ and $X, Y \in T_{\pi(\sigma)}M$. The restriction of $\Phi_+^{(1)}$ to the vertical bundle is a complex structure, so, in view of Corollary 1, $N_+^{(1)}(V, W) = 0$. We also have $N_+^{(1)}(X_\sigma^h, V) = 0$ by Proposition 2. Therefore, in view of Proposition 3, the almost contact structure $(\Phi_+^{(1)}, \chi, h_t)$ is normal if and only if

$$(29) \quad T(X, Y) - T(\varphi_+^\sigma X, \varphi_+^\sigma Y) + \varphi_+^\sigma T(\varphi_+^\sigma X, Y) + \varphi_+^\sigma T(X, \varphi_+^\sigma Y) = 0$$

and

$$R(X, Y)\sigma - R(\varphi_+^\sigma X, \varphi_+^\sigma Y)\sigma + \mathcal{J}_+^{(1)}R(\varphi_+^\sigma X, Y)\sigma + \mathcal{J}_+^{(1)}R(X, \varphi_+^\sigma Y)\sigma = 0.$$

In view of (26), the latter identity is equivalent to

$$(30) \quad g(\mathcal{R}(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{R}(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) = 0$$

for every $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and every $X, Y \in T_{\pi(\sigma)}M$.

Identity (29) is equivalent to vanishing of the following tensor

$$\begin{aligned} \tilde{T}(X, Y, Z) &= g(T(X, Y), Z) - g(T(\varphi_+^\sigma X, \varphi_+^\sigma Y), Z) \\ &\quad - g(T(\varphi_+^\sigma X, Y), \varphi_+^\sigma Z) - g(T(X, \varphi_+^\sigma Y), \varphi_+^\sigma Z) \end{aligned}$$

Take an adapted basis $(a_1, \dots, a_5)$ of $T_p M$, $p = \pi(\sigma)$, such that $\sigma = \kappa_3$, so $\xi_\sigma = -a_1$, $\varphi_+^\sigma a_2 = a_4$, $\varphi_+^\sigma a_3 = a_5$. Since the torsion is totally skew-symmetric, the tensor $\tilde{T}$ is also skew-symmetric, hence $\tilde{T} = 0$ exactly when $\tilde{T}(a_i, a_j, a_k) = 0$ for $1 \leq i < j < k \leq 5$. Direct computation shows that these identities hold if and only if $g(T(a_1, a_2), a_3) - g(T(a_1, a_4), a_5) = 0$ and $g(T(a_1, a_2), a_5) + g(T(a_1, a_4), a_2) = 0$. The latter identities are equivalent to

$$(31) \quad g(T(J_+^\sigma X, J_+^\sigma Y), \xi_\sigma) = g(T(X, Y), \xi_\sigma)$$

for every $X, Y \in H^\sigma$.

Suppose that the torsion T satisfies this condition. Take a point $p \in M$ and an adapted basis $\{a_1, \dots, a_5\}$ of $T_p M$. Define $\{\kappa_1, \kappa_2, \kappa_3\}$ by means of (3). Set $T_{ijk} = g(T(a_i, a_j), a_k)$, $i, j, k = 1, \dots, 5$. For $\sigma = \kappa_1$, we have $\xi_\sigma = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_4$ by (17), hence $\frac{2}{3}\sigma_+ = (\frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4) \wedge a_5 + a_2 \wedge a_3$ by (12). Thus, in this case, $J_+^\sigma(\frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4) = a_5$ and $J_+^\sigma a_2 = a_3$. Then identity (31) with $X = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4$, $Y = a_2$ and $Z = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4$, $Y = a_3$ gives

$$(32) \quad T_{135} + \sqrt{3}T_{345} = 2T_{124}, \quad T_{125} + \sqrt{3}T_{245} = -2T_{134}.$$

If $\sigma = \kappa_2$, then $\xi_\sigma = \frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_4$ and $\frac{2}{3}\sigma_+ = (\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_4) \wedge a_3 + a_2 \wedge a_5$. It follows from (31) that

$$(33) \quad T_{135} - \sqrt{3}T_{345} = 2T_{124}, \quad T_{123} + \sqrt{3}T_{234} = -2T_{145}.$$

For $\sigma = \kappa_3$, we have $\xi_\sigma = -a_1$, $\frac{2}{3}\sigma_+ = a_2 \wedge a_4 + a_3 \wedge a_5$ and by (31)

$$(34) \quad T_{145} = T_{123}, \quad T_{134} = T_{125}.$$

Moreover, for $\sigma = \frac{1}{\sqrt{2}}(\kappa_1 + \kappa_2)$, we have $\xi_\sigma = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2$, hence $\frac{2}{3}\sigma_+ = (\frac{\sqrt{6}}{4}a_1 + \frac{\sqrt{2}}{4}a_2 + \frac{2\sqrt{2}}{4}a_4) \wedge a_5 + (\frac{\sqrt{6}}{4}a_1 + \frac{\sqrt{2}}{4}a_2 - \frac{2\sqrt{2}}{4}a_4) \wedge a_3$. In this case identity (31) gives

$$(35) \quad 2T_{124} = T_{135} - \sqrt{3}T_{235}.$$

The first identities of (32) and (33) imply

$$(36) \quad T_{345} = 0, \quad 2T_{124} - T_{135} = 0.$$

This and (35) give

$$(37) \quad T_{235} = 0.$$

The second identities of (32) and (33), and identity (34) imply

$$(38) \quad \sqrt{3}T_{134} + T_{245} = 0, \quad \sqrt{3}T_{145} + T_{234} = 0.$$

Thus, according to (34), (36) – (38), the 2-form $*T$ vanishes on the following basis of $(\Lambda_3^2 T_p M)^\perp$: $a_2 \wedge a_3 - a_4 \wedge a_5$, $a_2 \wedge a_5 - a_3 \wedge a_4$, $a_1 \wedge a_2$, $a_2 \wedge a_4 - 2a_3 \wedge a_5$, $a_1 \wedge a_4$, $a_1 \wedge a_3 - \sqrt{3}a_2 \wedge a_5$, $a_1 \wedge a_5 - \sqrt{3}a_2 \wedge a_3$. Therefore $*T \in \Lambda_3^2 T_p M$.

Conversely, assume that the latter condition holds for every $p \in M$. Take $\sigma \in \mathbb{T}$ and choose an adapted basis $\{a_1, \dots, a_5\}$ of $T_p M$, $p = \pi(\sigma)$, such that $\sigma = \kappa_3$.

Then, at the point p , identity (31) is equivalent to identities (34). Obviously, these identities are satisfied since $*T$ vanishes on $a_2 \wedge a_3 - a_4 \wedge a_5$ and $a_2 \wedge a_5 - a_3 \wedge a_4$.

Now we address the problem when identity (30) holds for every $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and every $X, Y \in H^\sigma$.

In the notation of Proposition 5, denote the component of $\mathcal{R}$ in $\Psi^{-1}(\Lambda^4)$, $\Psi^{-1}(\Lambda_3^2 \oplus \Lambda_7^2)$, $\Psi^{-1}(\odot_5^2 \oplus \odot_9^2)$ by $\mathcal{A}$, $\mathcal{B}^-$, $\mathcal{B}^+$, respectively. Then, in view of Lemma 9,

$$(39) \quad \mathcal{R} = \frac{5s_R}{6}\mathcal{P} + \mathcal{A} + \mathcal{B}^- + \mathcal{B}^+.$$

Take $\sigma \in \mathbb{T}$ and choose an adapted basis $\{a_1, \dots, a_5\}$ of $T_p M$, $p = \pi(\sigma)$, such that $\sigma = \kappa_3$. Then $H^\sigma = \text{span}\{a_2, a_4, a_3, a_5\}$ and $\varphi_+^\sigma a_2 = a_4$, $\varphi_+^\sigma a_3 = a_5$.

It is easy to check that for every $X, Y \in H^\sigma$ and $\tau \in \mathbb{T}$, $\tau \perp \sigma$, we have $-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y \perp \tau$. This implies

$$g(\mathcal{P}(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{P}(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) = 0.$$

Next, in view of Lemma 6 (with $\nu = A_R$), it is easy to check that

$$g(\mathcal{A}(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{A}(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) = 0.$$

Now, let η be a bilinear form on $T_p M$. For $X, Y \in H^\sigma$ and $\tau \in \mathbb{T}$ with $\pi(\tau) = \pi(\sigma)$, set

$$(40) \quad \begin{aligned} \mathcal{E}_\eta(X, Y, \sigma, \tau) = & g(-\eta(X) \wedge Y - X \wedge \eta(Y) + \eta(\varphi_+^\sigma X) \wedge \varphi_+^\sigma Y + \varphi_+^\sigma X \wedge \eta(\varphi_+^\sigma Y), \sigma \times \tau) \\ & + g(\eta(\varphi_+^\sigma X) \wedge Y + \varphi_+^\sigma X \wedge \eta(Y) + \eta(X) \wedge \varphi_+^\sigma Y + X \wedge \eta(\varphi_+^\sigma Y), \tau). \end{aligned}$$

We have

$$(41) \quad \begin{aligned} \mathcal{E}_\eta(X, Y, \sigma, \tau) &= -\mathcal{E}_\eta(Y, X, \sigma, \tau), \quad \mathcal{E}_\eta(\varphi_+^\sigma X, \varphi_+^\sigma Y, \sigma, \tau) = -\mathcal{E}_\eta(X, Y, \sigma, \tau), \\ \mathcal{E}_\eta(X, \varphi_+^\sigma X, \sigma, \tau) &= 0, \end{aligned}$$

$$\mathcal{E}_\eta(a_2, a_3, \kappa_3, \kappa_1) = \mathcal{E}_\eta(a_2, a_5, \kappa_3, \kappa_2) = 2(\eta(a_2, a_4) + \eta(a_4, a_2)),$$

$$\mathcal{E}_\eta(a_2, a_3, \kappa_3, \kappa_2) = -\mathcal{E}_\eta(a_2, a_5, \kappa_3, \kappa_1) = 2(\eta(a_2, a_2) - \eta(a_4, a_4)).$$

In view of Lemma 7 (with $\eta = \rho_R^-$), this implies that, for $\tau \perp \sigma$, $X, Y \in H^\sigma$,

$$g(\mathcal{B}^-(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{B}^-(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) = 0.$$

If η is symmetric, then

$$\eta(a_2, a_4) = -4\eta'(a_2, a_4), \quad \eta(a_2, a_2) - \eta(a_4, a_4) = -4(\eta'(a_2, a_2) - \eta'(a_4, a_4)),$$

where, as above, $\eta'(X, Z) = \text{Trace}\{\Lambda_3^2 \ni \sigma \rightarrow \eta(\sigma X, \sigma Z)\}$. The latter identities,

(41) and Lemma 8 imply that for $\eta \in \odot_5^2 T_p M \oplus \odot_9^2 T_p M$

$$(42) \quad \begin{aligned} g(\mathcal{K}_\eta^+(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{K}_\eta^+(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) \\ = 5\mathcal{E}_\eta(X, Y, \sigma, \tau). \end{aligned}$$

It follows that identity (30) holds for every $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and for every $X, Y \in H^\sigma$ if and only if

$$\mathcal{E}_\eta(X, Y, \sigma, \tau) = 0$$

where $\eta = \rho_R^+ - \frac{1}{5}s_R g$.

Suppose identity (30) holds for $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and $X, Y \in H^\sigma$. Let $p \in M$ and let $\{a_1, \dots, a_5\}$ be an adapted basis of $T_p M$. Define $\{\kappa_1, \kappa_2, \kappa_3\}$ by

means of this basis. Set $\eta_{\alpha\beta} = \eta(a_\alpha, a_\beta)$ for $\eta = \rho_R^+ - \frac{1}{5}s_Rg$, $\alpha, \beta = 1, \dots, 5$. Then (41) gives

$$\eta_{24} = 0, \quad \eta_{22} - \eta_{44} = 0.$$

If $\sigma = \kappa_2$, then $\varphi_+^\sigma(\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_4) = a_3$, $\varphi_+^\sigma a_2 = a_5$ and (40) implies

$$\sqrt{3}\eta_{14} - \frac{3}{2}\eta_{11} + 2\eta_{33} - \frac{1}{2}\eta_{44} = 0, \quad \sqrt{3}\eta_{13} - \eta_{34} = 0.$$

For $\sigma = \kappa_1$, we have $\varphi_+^\sigma(\frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4) = a_5$, $\varphi_+^\sigma a_2 = a_3$ and we obtain from (40) that

$$\sqrt{3}\eta_{14} + \frac{3}{2}\eta_{11} - 2\eta_{55} + \frac{1}{2}\eta_{44} = 0, \quad \sqrt{3}\eta_{15} + \eta_{45} = 0.$$

Every adapted basis $(a_1, \dots, a_5)$ can be used to obtain a new adapted basis $(a'_1, \dots, a'_5)$ by means of formulas (10). According to the preceding considerations, $\sqrt{3}\eta(a'_1, a'_3) - \eta(a'_3, a'_4) = 0$. In view of (10), this identity can be written as

$$\begin{aligned} & (\sqrt{3}\eta_{13} + 2\eta_{25} \sin^2 \psi) \cos \psi + (\sqrt{3}\eta_{15} + 2\eta_{23} \cos^2 \psi) \sin \psi \\ & - (\eta_{34} \cos \psi + \eta_{45} \sin \psi) \cos 2\psi = 0. \end{aligned}$$

Taking $\psi = \frac{\pi}{4}$ and $\psi = \frac{3\pi}{4}$, we obtain

$$(\sqrt{3}\eta_{13} + \eta_{25}) + (\sqrt{3}\eta_{15} + \eta_{23}) = 0, \quad -(\sqrt{3}\eta_{13} + \eta_{25}) + (\sqrt{3}\eta_{15} + \eta_{23}) = 0$$

Thus, we have

$$\sqrt{3}\eta_{13} + \eta_{25} = 0, \quad \sqrt{3}\eta_{15} + \eta_{23} = 0.$$

Finally, consider the adapted basis $\{c_1, \dots, c_5\}$ obtained from $\{a_1, \dots, a_5\}$ by means of formulas (8). Then $\eta(c_2, c_2) = \eta(c_4, c_4)$ and we obtain the identity

$$\begin{aligned} & -\frac{3}{4}\eta_{11} - \frac{1}{4}\eta_{22} \sin^2 2\theta + \eta_{33} \cos^2 \theta + \eta_{55} \sin^2 \theta \\ & - (\frac{\sqrt{3}}{2}\eta_{12} - \eta_{35}) \sin 2\theta - (\frac{\sqrt{3}}{2}\eta_{14} + \frac{1}{2}\eta_{24} \sin 2\theta - \frac{1}{4}\eta_{44} \cos 2\theta) \cos 2\theta = 0. \end{aligned}$$

Taking $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, we get

$$-\frac{3}{4}\eta_{11} - \frac{1}{4}\eta_{22} + \frac{1}{2}\eta_{33} + \frac{1}{2}\eta_{55} = 0, \quad \sqrt{3}\eta_{12} - 2\eta_{35} = 0.$$

Thus, the form $\eta = \rho_R^+ - \frac{1}{5}s_Rg$ vanishes on the vectors $a_2 \odot a_4$, $a_2 \odot a_2 - a_4 \odot a_4$, $2\sqrt{3}a_1 \odot a_4 - 3a_1 \odot a_1 + 4a_3 \odot a_3 - a_4 \odot a_4$, $\sqrt{3}a_1 \odot a_3 - a_3 \odot a_4$, $2\sqrt{3}a_1 \odot a_4 + 3a_1 \odot a_1 - 4a_5 \odot a_5 + a_4 \odot a_4$, $\sqrt{3}a_1 \odot a_5 + a_4 \odot a_5$, $\sqrt{3}a_1 \odot a_3 + a_2 \odot a_5$, $\sqrt{3}a_1 \odot a_5 + a_2 \odot a_3$, $3a_1 \odot a_1 + a_2 \odot a_2 - 2a_3 \odot a_3 - 2a_5 \odot a_5$, $\sqrt{3}a_1 \odot a_2 - 2a_3 \odot a_5$. These vectors constitute a basis of $\odot_9^2 T_p M$. It follows that the $\odot_9^2$ -component of $\mathcal{R}$ vanishes.

Conversely, suppose that $\odot_9^2$ -component of $\mathcal{R}$ vanishes. Let $\sigma \in \mathbb{T}$ and take an adapted basis $\{a_1, \dots, a_5\}$ of $T_p M$, $p = \pi(\sigma)$, such that $\sigma = \kappa_3$. By assumption, the form $\eta = \rho_R^+ - \frac{1}{5}s_Rg$ vanishes on the vectors $a_2 \odot a_4$ and $a_2 \odot a_2 - a_4 \odot a_4$. Hence, by (41), $\mathcal{E}_\eta(X, Y, \sigma, \tau) = 0$ for every $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and $X, Y \in H^\sigma$. Then, by (42),

$$g(\mathcal{K}_\eta^+(-X \wedge Y + \varphi_+^\sigma X \wedge \varphi_+^\sigma Y), \sigma \times \tau) + g(\mathcal{K}_\eta^+(\varphi_+^\sigma X \wedge Y + X \wedge \varphi_+^\sigma Y), \tau) = 0.$$

Now, it follows from the preceding considerations that identity (30) holds for $\sigma, \tau \in \mathbb{T}$ with $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$ and $X, Y \in H^\sigma$.

Next, note that, in view of (26), identity (30) holds for $\sigma \in \mathbb{T}$, $X \in H^\sigma$ and $Y = \xi_\sigma$ if and only if

$$(43) \quad g(\mathcal{R}(X \wedge \xi_\sigma), \sigma \times \tau) - g(\mathcal{R}(\varphi_+^\sigma X \wedge \xi_\sigma), \tau) = 0$$

for every $\sigma \in \mathbb{T}$, $\tau \in \mathcal{V}_\sigma$, $X \in H^\sigma$. Let $p \in M$, $\sigma \in \Lambda_3^2 T_p M$, $|\sigma|^2 = 5$. Taking an adapted basis $a = (a_1, \dots, a_5)$ of $T_p M$ such that $\sigma = \kappa_3(a)$, it is easy to check that the operator $\mathcal{P}$ satisfies identity (43). Suppose that this identity holds. Let $a = (a_1, \dots, a_5)$ be an arbitrary adapted basis of $T_p M$. Set $\rho_{\alpha\beta}^- = \rho_R^-(a_\alpha, a_\beta)$, $\eta = \rho_R^+ - \frac{1}{5}s_R g$ and $\eta_{\alpha\beta} = \eta(a_\alpha, a_\beta)$, $\alpha, \beta = 1, \dots, 5$. Recall that for $\sigma = \kappa_1(a)$, we have $\xi_\sigma = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_4$, $X = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4 \in H^\sigma$ and $\varphi_+^\sigma X = a_5$. Set $\tau = \kappa_2(a)$. Then, by a computation using (39), we get from (43) that

$$3\sqrt{3}((\sqrt{3}\eta_{12} - 2\eta_{35}) + \eta_{24}) + 3\rho_{12}^- - \sqrt{3}(\rho_{24}^- - 2\rho_{35}^-) = 0.$$

We have $\sqrt{3}\eta_{12} - 2\eta_{35} = 0$ and $\eta_{24} = 0$ since the $\odot_9^2$ -component of $\mathcal{R}$ vanishes, i.e. η vanishes on $\odot_9^2 TM$. Hence

$$(44) \quad 3\rho_{12}^- - \sqrt{3}(\rho_{24}^- - 2\rho_{35}^-) = 0.$$

Similarly, applying (43) with $\sigma = \kappa_1(a)$, $\tau = \kappa_3(a)$, $X = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4$ and taking into account (39) and the fact that $\eta|_{\odot_9^2 TM} = 0$, we get

$$(45) \quad (\rho_{13}^- - \sqrt{3}\rho_{34}^-) + 2(\rho_{13}^- - \sqrt{3}\rho_{25}^-) = 0.$$

If $\sigma = \kappa_2(a)$, then $\xi_\sigma = \frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_4$, $X = \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_4 \in H^\sigma$ and $\varphi_+^\sigma X = a_3$. Then, putting $\tau = \kappa_1(a)$ and $\tau = \kappa_3(a)$ in (43), we obtain the identities

$$(46) \quad 3\rho_{12}^- + \sqrt{3}(\rho_{24}^- - 2\rho_{35}^-) = 0, \quad (\rho_{15}^- - \sqrt{3}\rho_{45}^-) + 2(\rho_{15}^- - \sqrt{3}\rho_{23}^-) = 0.$$

For $\sigma = \kappa_3(a)$, $X = a_2$, identity (43) with $\tau = \kappa_1(a)$ and $\tau = \kappa_2(a)$ gives

$$(47) \quad \rho_{23}^- - \rho_{45}^- = 0, \quad \rho_{25}^- - \rho_{34}^- = 0.$$

We also set $\sigma = \frac{1}{\sqrt{2}}(\kappa_1(a) + \kappa_2(a))$. Then $\xi_\sigma = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2$, $X = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_2 \in H^\sigma$ and $\varphi_+^\sigma X = \frac{1}{\sqrt{2}}(a_3 + a_5)$. Setting $\tau = \kappa_1(a) - \kappa_2(a)$ in (43), we obtain

$$(48) \quad \sqrt{3}\rho_{14}^- + (\rho_{24}^- - 2\rho_{35}^-) = 0.$$

Identities (44) - (48) imply that ρ_R^- vanishes on the bi-vectors $a_1 \wedge a_2$, $a_2 \wedge a_4 - 2a_3 \wedge a_5$, $a_1 \wedge a_4$, $a_2 \wedge a_3 - a_4 \wedge a_5$, $a_2 \wedge a_5 - a_3 \wedge a_4$, $a_1 \wedge a_5 - \sqrt{3}a_4 \wedge a_5$, $a_1 \wedge a_3 - \sqrt{3}a_3 \wedge a_4$. These bi-vectors form a basis of the space $\Lambda_7^2 T_p M$. Hence $\rho_R^-|_{\Lambda_7^2 T_p M} = 0$. This means that Λ_7^2 -component of $\mathcal{R}$ vanishes. Now, let $\sigma \in \mathbb{T}$ be arbitrary. Take an adapted basis $a = (a_1, \dots, a_5)$ of $T_{\pi(\sigma)} M$ such that $\sigma = \kappa_3(a)$ and apply (43) with $X = a_3$ and $\tau = \kappa_1(a)$. Then a computation making use of Lemmas 6-8 gives

$$24A_{1235} + \rho_{14}^- + (-4\sqrt{3}\eta_{33} + 4\sqrt{3}\eta_{55} - 2\eta_{14}) = 0$$

where $A_{1235} = A_R(a_1 \wedge a_2 \wedge a_3 \wedge a_5)$. We have $\rho_{14}^- = 0$ since $\rho_R^-|_{\Lambda_7^2 TM} = 0$ and $-4\sqrt{3}\eta_{33} + 4\sqrt{3}\eta_{55} - 2\eta_{14} = 10\eta_{14}$ since $\eta|_{\odot_9^2 TM} = 0$. Thus

$$12A_R(a_1 \wedge a_2 \wedge a_3 \wedge a_5) + 5\eta(a_1, a_4) = 0$$

Applying (43) with $X = a_3$ and $\tau = \kappa_2(a)$ we get by a similar computation that

$$12A_R(a_1 \wedge a_3 \wedge a_4 \wedge a_5) + 5\eta(a_1, a_2) = 0.$$

The left-hand sides of the last two identities are equal to $-Q(\kappa_3, a_4)$ and $-Q(\kappa_3, a_2)$, respectively. It follows that $Q(\sigma, X) = 0$ for $X \perp (\mathcal{V}(\xi_\sigma) \oplus \mathbb{R}\xi_\sigma)$. This and Lemma 10 imply that $Q(\sigma, X) = 0$ for every $\sigma \in \mathbb{T}$, $X \in T_{\pi(\sigma)}M$. It follows that $Q = 0$ in view of the homogeneity of $Q(\sigma; X)$ in σ .

Conversely, suppose that $*T \in \Lambda_3^2 TM$, $Q = 0$ and the $\odot_9^2$, and Λ_7^2 -components of $\mathcal{R}$ vanish. Fix an arbitrary $\sigma \in \mathbb{T}$ and take an adapted basis of $T_{\pi(\sigma)}M$ for which $\sigma = \kappa_3$. Then it is easy to check that identities (29) and (30) are fulfilled. $\square$

The next statement, combined with Lemma 10, might be useful to check the condition $Q = 0$.

Lemma 11. Let A be a skew-symmetric 4-form and η a symmetric bilinear form on $\mathbb{R}^5$ such that $\eta|_{\odot_9^2} = 0$. Define a tensor Q by (28) with $A_R = A$.

Suppose that $Q(q_i, X) = 0$, $i = 1, 2, 3$, for an orthogonal basis q_1, q_2, q_3 of Λ_3^2 with $|q_1|^2 = |q_2|^2 = |q_3|^2 = 5$ and every $X \in \mathbb{R}^5$.

Then $Q(\sigma, X) = 0$ for every $\sigma \in \Lambda_3^2$ and every $X \in \mathbb{R}^5$.

Proof. It is enough to show that $Q(\sigma, X) = 0$ in the case when $|\sigma| = 5$. We may assume that the basis q_1, q_2, q_3 yields the orientation of Λ_3^2 . Then, by Lemma 2, there are adapted bases $a = (a_1, \dots, a_5)$ and $a' = (a'_1, \dots, a'_5)$ of $\mathbb{R}^5$ such that $q_1 = \kappa_1(a)$, $q_2 = \kappa_2(a)$, $q_3 = \kappa_3(a)$ and $\sigma = \kappa_1(a')$. Since the group $SO(3)$ acts transitively on the set of adapted bases, there is a matrix h in $SO(3)$ such that $\iota(h)$ sends the basis a to the basis a' . We shall use the notation in the proof of Lemma 1. First we show that $Q(\tau, X) = 0$ for $\tau = \kappa_1(b), \kappa_2(b), \kappa_3(b)$ where $\kappa_i(b)$, $i = 1, 2, 3$, are given by (7) and the adapted basis $b = (b_1, \dots, b_5)$ is defined by (6). If $\tau = \kappa_1(b)$, then $\xi_\tau = \frac{1}{2}b_1 - \frac{\sqrt{3}}{2}b_4 = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_4 \sin 2\varphi + \frac{\sqrt{3}}{2}a_4 \cos 2\varphi$ and $(\mathbb{R}\xi_\tau \oplus \mathcal{V}(\xi_\tau))^\perp = \text{span}\{\frac{\sqrt{3}}{2}b_1 + \frac{1}{2}b_4, b_5\}$ where $\frac{\sqrt{3}}{2}b_1 + \frac{1}{2}b_4 = -\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_2 \sin 2\varphi + \frac{1}{2}a_4 \cos 2\varphi$ and $b_5 = a_3 \cos \varphi + a_5 \sin \varphi$. Set $A_{\alpha\beta\gamma\delta} = A(a_\alpha \wedge a_\beta \wedge a_\gamma \wedge a_\delta)$ and $\eta_{\alpha\beta} = \eta(a_\alpha, a_\beta)$, $\alpha, \beta, \gamma, \delta = 1, \dots, 5$. Then, applying Lemma 10, we obtain

$$\begin{aligned} Q(\kappa_1(b), \frac{\sqrt{3}}{2}b_1 + \frac{1}{2}b_4) &= \frac{\sin 2\varphi}{2}(12A_{1345} + 5\eta_{12}) - \frac{\cos 2\varphi}{2}(12A_{1235} + 5\eta_{14}) \\ &\quad - \sqrt{3}(6A_{2345} + \frac{5}{4}\eta_{11} - \frac{5}{4}\eta_{22} \sin^2 2\varphi - \frac{5}{4}\eta_{44} \cos^2 2\varphi) - \frac{5\sqrt{3}}{4}\eta_{24} \sin 4\varphi. \end{aligned}$$

We also have

$$12A_{1345} + 5\eta_{12} = -Q(\kappa_3(a), a_2) = 0,$$

$$12A_{1235} + 5\eta_{14} = -Q(\kappa_3(a), a_4) = 0,$$

$$-6A_{1235} + 6\sqrt{3}A_{2345} + \frac{5}{4}(\sqrt{3}\eta_{11} - 2\eta_{14} - \sqrt{3}\eta_{44}) = Q(\kappa_1(a), \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_4) = 0.$$

The last two identities imply

$$6A_{2345} + \frac{5}{4}(\eta_{11} - \eta_{44}) = 0.$$

We have $\eta_{22} = \eta_{44}$ and $\eta_{24} = 0$ since $\eta|_{\odot_9^2} = 0$. It follows that

$$Q(\kappa_1(b), \frac{\sqrt{3}}{2}b_1 + \frac{1}{2}b_4) = 0.$$

It is easy to compute that

$$\begin{aligned} Q(\kappa_1(b), b_5) &= Q(\kappa_1(a), a_5) \sin \varphi + Q(\kappa_2(a), a_3) \cos \varphi \\ &+ \frac{5\sqrt{3}}{2}(\eta_{45} - \eta_{23}) \sin 2\varphi \cos \varphi - \frac{5\sqrt{3}}{2}(\eta_{25} + \eta_{34}) \sin 2\varphi \sin \varphi. \end{aligned}$$

By assumption, $Q(\kappa_1(a), a_5) = Q(\kappa_2(a), a_3) = 0$ and $\eta_{45} - \eta_{23} = 0$, $\eta_{25} + \eta_{34} = 0$. Thus,

$$Q(\kappa_1(b), b_5) = 0.$$

Similar computations show also that

$$\begin{aligned} Q(\kappa_2(b), \frac{\sqrt{3}}{2}b_1 - \frac{1}{2}b_4) &= Q(\kappa_2(b), b_3) = 0, \\ Q(\kappa_3(b), b_2) &= Q(\kappa_3(b), b_4) = 0. \end{aligned}$$

Now Lemma 10 implies that $Q(\tau, X) = 0$ for $\tau = \kappa_1(b), \kappa_2(b), \kappa_3(b)$ and $X \in \mathbb{R}^5$. Using this, (8) and (9) we see in a similar way that $Q(\tau, X) = 0$ for $\tau = \kappa_1(c), \kappa_2(c), \kappa_3(c)$. The latter identity, (10) and (11) imply $Q(\kappa_1(a'), X) = 0$. $\square$

Examples. 1. According to [4, Theorem 4.7] every 5-manifold M with an irreducible $SO(3)$ -structure whose characteristic connection coincides with the Levi-Civita connection is locally isometric to one of the following symmetric space $\mathbb{R}^5$, $SU(3)/SO(3)$, $SL(3, \mathbb{R})/SO(3)$. The Riemannian metric of these spaces is Einstein. It is flat only for $M = \mathbb{R}^5$; in the other two cases it is not even conformally flat [ibid.]. Clearly we have $\rho_R^- = 0$, $\rho_R^+ - \frac{1}{5}s_R g = 0$. Also, by the algebraic Bianchi identity, $A_R = 0$, hence $Q = 0$. Thus, by Theorem 1, the almost contact structure of the twistor space of M is normal.

2. Consider $SO(2)$ as a subgroup of $SO(3)$ via the standard diagonal imbedding $A \rightarrow (1, A)$. Then $SO(2)$ can also be considered as a subgroup of $SO(3) \times SO(1, 2)$ by means of the map $SO(2) \ni A \rightarrow (A^2, A)$. It is shown in the proof of [4, Proposition 6.3] that for every $t \in \mathbb{R} \setminus \{0\}$ there exists linearly independent invariant 1-forms $\tilde{\theta}_1, \dots, \tilde{\theta}_5$ such that:

(1) the tensors

$$\tilde{g} = \tilde{\theta}_1^2 + \dots + \tilde{\theta}_5^2, \quad \tilde{\Upsilon} = \frac{1}{2}\tilde{\theta}_1(6\tilde{\theta}_2^2 + 6\tilde{\theta}_4^2 - 2\tilde{\theta}_1^2 - 3\tilde{\theta}_3^2 - 3\tilde{\theta}_5^2) + \frac{3\sqrt{3}}{2}\tilde{\theta}_4(\tilde{\theta}_5^2 - \tilde{\theta}_3^2) + 3\sqrt{3}\tilde{\theta}_2\tilde{\theta}_3\tilde{\theta}_5$$

descend to a nearly integrable irreducible $SO(3)$ -structure (g, Υ) on the 5-dimensional manifold $M = (SO(3) \times SO(1, 2))/SO(2)$,

(2) if $\theta_1, \dots, \theta_5$ is the co-frame on M induced by the forms $\tilde{\theta}_1, \dots, \tilde{\theta}_5$, the torsion 3-form T and the curvature tensor R of the characteristic connection are given by

$$T = t(\theta_1 \wedge \theta_2 \wedge \theta_4 + 2\theta_1 \wedge \theta_3 \wedge \theta_5),$$

$$g(R(X, Y)Z, U) = 2t^2 g(X \wedge Y, \kappa_3) g(\kappa_3, Z \wedge U) \quad X, Y, Z, U \in TM,$$

where κ_3 is defined by means of the dual frame $E_1, \dots, E_5$ of $\theta_1, \dots, \theta_5$ (note: the curvature tensor used here differs by a sign from that used in [4]).

We refer also to [1] for more information about the irreducible $SO(3)$ -structure on $M = (SO(3) \times SO(1, 2))/SO(2)$.

Clearly $*T = -t(2\theta_2 \wedge \theta_4 + \theta_3 \wedge \theta_5) \in \Lambda_3^2 T^*M$. For the Ricci tensor ρ we have the following identities

$$\begin{aligned}\rho(E_1, E_\alpha) &= 0, & \rho(E_2, E_\alpha) &= 8t^2\delta_{2\alpha}, & \rho(E_3, E_\alpha) &= 2t^2\delta_{3\alpha}, \\ \rho(E_4, E_\alpha) &= 8t^2\delta_{4\alpha}, & \rho(E_5, E_\alpha) &= 2t^2\delta_{5\alpha}, & \alpha &= 1, \dots, 5.\end{aligned}$$

Hence $\rho^- = 0$, in particular the Λ_7^2 -component of $\mathcal{R}$ vanishes. Set $\eta = \rho^+ - \frac{1}{5}sg$ where s is the scalar curvature. Then

$$\begin{aligned}\eta(E_1, E_\alpha) &= -4t^2\delta_{1\alpha}, & \eta(E_2, E_\alpha) &= 4t^2\delta_{2\alpha}, & \eta(E_3, E_\alpha) &= -2t^2\delta_{3\alpha}, \\ \eta(E_4, E_\alpha) &= 4t^2\delta_{4\alpha}, & \eta(E_5, E_\alpha) &= -2t^2\delta_{5\alpha}, & \alpha &= 1, \dots, 5.\end{aligned}$$

This implies that η vanishes on the basis of $\odot_9^2 TM$ we have used in the proof of Theorem 1, thus the $\odot_9^2$ -component of $\mathcal{R}$ vanishes. If A_R is the anti-symmetrization of the curvature tensor, set $A_{ijkl} = A_R(E_i \wedge E_j \wedge E_k \wedge E_l)$. Then $A_{1235} = 0$, $A_{2345} = -\frac{4t^2}{3}$, thus

$$\begin{aligned}Q(\kappa_1, \frac{\sqrt{3}}{2}E_1 + \frac{1}{2}E_4) &= \\ -6A_{1235} + 6\sqrt{3}A_{2345} + \frac{5}{4}(\sqrt{3}\eta_{11} - 2\eta_{14} - \sqrt{3}\eta_{44}) &= -18\sqrt{3}t^2 \neq 0.\end{aligned}$$

Therefore, by Theorem 1, the almost contact metric structure $(\Phi_+^{(1)}, \chi, h_t)$ is not normal.

5. ADDENDUM

In this section we give a new proof of the integrability result of [4] for the almost CR -structures $(\mathcal{D}, \mathcal{J}_\pm^{(n)})$ on the manifold T defined at the beginning of Section 3.

Recall that an almost Cauchy-Riemann (CR) structure on a manifold N is a pair $(\mathcal{D}, \mathcal{J})$ of a subbundle $\mathcal{D}$ of the tangent bundle TN and an almost complex structure $\mathcal{J}$ of the bundle $\mathcal{D}$. For any two sections X, Y of $\mathcal{D}$, the value of $[X, Y] \bmod \mathcal{D}$ at a point $p \in N$ depends only on the values of X and Y at p , so we have a skew-symmetric bilinear form $\omega : \mathcal{D} \times \mathcal{D} \rightarrow TN/\mathcal{D}$ defined by $\omega(X, Y) = [X, Y] \bmod \mathcal{D}$ and called the Levi form of the almost CR -structure $(\mathcal{D}, \mathcal{J})$. If the Levi form is $\mathcal{J}$ -invariant, we can define the Nijenhuis tensor of the almost CR -structure $(\mathcal{D}, \mathcal{J})$ by

$$N^{CR}(X, Y) = -[X, Y] + [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}([\mathcal{J}X, Y] + [X, \mathcal{J}Y]);$$

its value at a point $p \in N$ lies in $\mathcal{D}$ and depends only on the values of the sections X, Y at p . An almost CR -structure is said to be *integrable* if its Levi form is $\mathcal{J}$ -invariant and the Nijenhuis tensor vanishes. An integrable almost CR -structure is called simply a CR -structure. Let $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$ be the decomposition of the complexification of $\mathcal{D}$ into $(1, 0)$ and $(0, 1)$ parts with respect to $\mathcal{J}$. If the almost CR -structure $(\mathcal{D}, \mathcal{J})$ is integrable, then the bundle $\mathcal{D}^{1,0}$ satisfies the following two conditions:

$$\mathcal{D}^{1,0} \cap \overline{\mathcal{D}^{1,0}} = 0, \quad [\Gamma(\mathcal{D}^{1,0}), \Gamma(\mathcal{D}^{1,0})] \subset \Gamma(\mathcal{D}^{1,0})$$

where $\Gamma(\mathcal{D}^{1,0})$ stands for the space of smooth sections of $\mathcal{D}^{1,0}$. Conversely, suppose we are given a complex subbundle $\mathcal{E}$ of the complexified tangent bundle $T^{\mathbb{C}}N$ such that $\mathcal{E} \cap \overline{\mathcal{E}} = 0$ and $[\Gamma(\mathcal{E}), \Gamma(\mathcal{E})] \subset \Gamma(\mathcal{E})$ (a bundle with these properties is often also called a " CR -structure"). Set $\mathcal{D} = \{X \in TN : X = Z + \bar{Z} \text{ for some (unique) } Z \in \mathcal{E}\}$ and put $\mathcal{J}X = -Im Z$ for $X \in \mathcal{D}$. Then $(\mathcal{D}, \mathcal{J})$ is an integrable almost CR -structure such that $\mathcal{D}^{1,0} = \mathcal{E}$.

5.1. The Levi form of the almost CR -structures on the twistor space.

Lemma 12. Let $A, B \in \mathcal{D}_\sigma$ be horizontal vectors and $V, W \in \mathcal{D}_\sigma$ vertical ones at a point $\sigma \in \mathbb{T}$. Then the Levi form $\omega_\pm^{(n)}$ of the almost CR -structure $(\mathcal{D}, \mathcal{J}_\pm^{(n)})$ is given by:

$$\omega_\pm^{(n)}(A, B) = -g_p(T(\pi_*A, \pi_*B), \xi_\sigma)(\xi_\sigma)_\sigma^h,$$

$$\omega_\pm^{(n)}(V, W) = 0,$$

$$\omega_\pm^{(n)}(A, V) = \pm g_p(\pi_*A, J_\pm^\sigma(V(\xi_\sigma))(\xi_\sigma)_\sigma^h).$$

Proof. Take vector fields X and Y near the point $p = \pi(\sigma)$ such that $X_p = \pi_*A$, $Y_p = \pi_*B$ and $\nabla X|_p = \nabla Y|_p = 0$. Then $-(\Phi_\pm^{(n)})^2 X^h$ and $-(\Phi_\pm^{(n)})^2 Y^h$ are sections of $\mathcal{D}$ with values A and B at the point σ , so

$$\omega_\pm^{(n)}(A, B) = ((\Phi_\pm^{(n)})^2 + Id)[(\Phi_\pm^{(n)})^2 X^h, (\Phi_\pm^{(n)})^2 Y^h]_\sigma.$$

The tangent vectors X_p, Y_p are orthogonal to ξ_σ and Lemma 4 implies

$$\omega_\pm^{(n)}(A, B) = -((\varphi_\pm^\sigma)^2(T_p(X, Y)))_\sigma^h - (T_p(X, Y))_\sigma^h = -g_p(T(X, Y), \xi_\sigma)(\xi_\sigma)_\sigma^h.$$

Extend V and W to vertical vector fields of $\mathbb{T}$ on a neighbourhood of σ . These vector fields are section of $\mathcal{D}$, whose Lie bracket is a vertical vector field, hence a section of $\mathcal{D}$. Therefore

$$\omega_\pm^{(n)}(V, W) = 0.$$

Finally, take a section S of $\Lambda_3^2 TM$ near the point p such that $S_p = V$ and $\nabla S|_p = 0$. Then, by Lemma 3, we have

$$\begin{aligned} \omega_\pm^{(n)}(A, V) &= ((\Phi_\pm^{(n)})^2 + Id)[(-\Phi_\pm^{(n)})^2 X^h, \tilde{S}]_\sigma = \\ &= \frac{1}{2 \pm 1} \{ \pm 3g_p(\varphi_\pm^\sigma(V(\xi_\sigma)), X)(\xi_\sigma)_\sigma^h \\ &\quad + [1 \pm (-1)][(\varphi_\pm^\sigma)^2(V \circ \varphi_\pm^\sigma(X)) + V \circ \varphi_\pm^\sigma(X)]_\sigma^h \} = \\ &= \frac{1}{2 \pm 1} \{ \pm 3g_p(\varphi_\pm^\sigma \circ V(\xi_\sigma), X) + [1 \pm (-1)]g_p(V \circ \varphi_\pm^\sigma(X), \xi_\sigma) \} (\xi_\sigma)_\sigma^h = \\ &= \frac{1}{2 \pm 1} \{ \pm 3 + [1 \pm (-1)] \} g_p(\varphi_\pm^\sigma \circ V(\xi_\sigma), X)(\xi_\sigma)_\sigma^h. \end{aligned}$$

Note that $V(\xi_\sigma)$ is orthogonal to ξ_σ since the map V is skew-symmetric. Hence $\varphi_\pm^\sigma(V(\xi_\sigma)) = J_\pm^\sigma(V(\xi_\sigma))$. This implies the third formula of the lemma. $\square$

Corollary 4. (i) The Levi form $\omega_+^{(1)}$, respectively $\omega_-^{(2)}$, is $\mathcal{J}_+^{(1)}$ -invariant, respectively $\mathcal{J}_-^{(2)}$ -invariant, if and only if

$$g(T(J_\pm^\sigma X, J_\pm^\sigma Y), \xi_\sigma) = g(T(X, Y), \xi_\sigma)$$

for every $\sigma \in \mathbb{T}$ and $X, Y \in T_{\pi(\sigma)}M$, $X, Y \perp \xi_\sigma$.

(ii) The Levi form $\omega_-^{(1)}$ is not $\mathcal{J}_-^{(1)}$ -invariant and $\omega_+^{(2)}$ is not $\mathcal{J}_+^{(2)}$ -invariant.

Proof. By Lemma 12 and identity (24), if $A \in \mathcal{D}_\sigma$ is a horizontal vector and $V \in \mathcal{V}_\sigma$,

$$\omega_\pm^{(n)}(\mathcal{J}_\pm^{(n)}A, \mathcal{J}_\pm^{(n)}V) = \pm(-1)^{n+1}\omega_\pm^{(n)}(A, V).$$

This and Lemma 12 imply (i). Also, $\omega_+^{(2)}$ is $\mathcal{J}_+^{(2)}$ -invariant if and only if $\omega_\pm^{(2)}(A, V) = 0$ for every A and V . By Lemma 12, we have $\omega_+^{(2)}((E_3)_\sigma^h, (\kappa_1)_p) = -\sqrt{3}$. Thus $\omega_+^{(2)}$ is not $\mathcal{J}_+^{(2)}$ -invariant. Similarly, $\omega_-^{(1)}$ is not $\mathcal{J}_-^{(1)}$ -invariant.

□

Corollary 5. ([4]) The almost CR -structures $(\mathcal{D}, \mathcal{J}_-^{(1)})$, $(\mathcal{D}, \mathcal{J}_\pm^{(2)})$ are not integrable.

Proof. According to Corollary 4, the almost CR -structures $(\mathcal{D}, \mathcal{J}_-^{(1)})$ and $(\mathcal{D}, \mathcal{J}_+^{(2)})$ are not integrable. Denote by $N_-^{(2)}$ be the Nijenhuis tensor of the CR -structure $(\mathcal{D}, \mathcal{J}_-^{(2)})$. Then, using Lemma 3, one easily computes that $N_-^{(2)}((E_2)_\sigma^h, (\kappa_1)_p) = -4(E_5)_\sigma^h \neq 0$, so $(\mathcal{D}, \mathcal{J}_-^{(2)})$ is not integrable.

□

Proposition 6. If ∇ is the characteristic connection, the Levi form $\omega_+^{(1)}$ is $\mathcal{J}_+^{(1)}$ -invariant if and only if $*T \in \Lambda_3^2 TM$.

Proof. By Corollary 4, $\omega_+^{(1)}$ is $\mathcal{J}_+^{(1)}$ -invariant if and only if

$$g(T(J_+^\sigma X, J_+^\sigma Y), \xi_\sigma) = g(T(X, Y), \xi_\sigma)$$

for $\sigma \in \mathbb{T}$, $X, Y \in T_{\pi(\sigma)}M$, $X, Y \perp \xi_\sigma$.

□

This, in fact, is identity (31). As we have seen in the proof of Theorem 1 it is equivalent to $*T \in \Lambda_3^2 T_p M$.

5.2. The Nijenhuis tensor of the almost CR -structure $(\mathcal{D}, \mathcal{J}_+^{(1)})$ on the twistor space.

Denote the Nijenhuis tensor of the CR -structure $(\mathcal{D}, \mathcal{J}_+^{(1)})$ by N^{CR} . Then we have the following.

Lemma 13. Let $A, B \in \mathcal{D}_\sigma$ be horizontal vectors and $V, W \in \mathcal{D}_\sigma$ vertical ones at a point $\sigma \in \mathbb{T}$. Let $X = \pi_* A, Y = \pi_* B$. Suppose that the Levi form $\omega_+^{(1)}$ is $\mathcal{J}$ -invariant. Then:

$$N^{CR}(A, B) = -R(X, Y)\sigma + R(J_+^\sigma X, J_+^\sigma Y)\sigma - \mathcal{J}_+^{(1)}(R(J_+^\sigma X, Y)\sigma + R(X, J_+^\sigma Y)\sigma).$$

$$N^{CR}(A, V) = 0, \quad N^{CR}(V, W) = 0.$$

Proof. Take vector fields X and Y near the point $p = \pi(\sigma)$ such that $X_p = \pi_* A$, $Y_p = \pi_* B$ and $\nabla X|_p = \nabla Y|_p = 0$. Then $-(\Phi_+^{(1)})^2 X^h$ and $-(\Phi_+^{(1)})^2 Y^h$ are sections of $\mathcal{D}$ with values A and B at the point σ , so $N^{CR}(A, B) = N_\sigma^{CR}((\Phi_+^{(1)})^2 X^h, (\Phi_+^{(1)})^2 Y^h)$. Then Lemma 4 and Corollary 4 imply the first identity of the lemma.

Take a section S of $\Lambda_3^2 TM$ near the point p such that $S_p = V$ and $DS|_p = 0$. The vertical vector field $\tilde{S}$ determined by S takes value V at the point σ . Then Lemma 3 and identity (24) give

$$N^{CR}(A, V) = -N^{CR}((\Phi_+^{(1)})^2 X^h, \tilde{S})_\sigma = 0.$$

The restriction of $\mathcal{J}_+^{(1)}$ to any vertical space is the complex structure of the corresponding fibre of $\mathbb{T}$, hence $N^{CR}(V, W) = 0$.

□

Proposition 7. The Nijenhuis tensor N^{CR} vanishes if and only if the $\odot_9^2$ -component of $\mathcal{R}$ vanishes.

Proof. By Lemma 13 and Proposition 3, the condition $N^{CR} = 0$ is the same as $\mathcal{V}N_+^{(1)}(X_\sigma^h, Y_\sigma^h) = 0$ for every $\sigma \in \mathbb{T}$ and $X, Y \in H^\sigma$. We have seen in the proof of Theorem 1 that the latter condition is equivalent to vanishing of the $\odot_9^2$ -component of $\mathcal{R}$. $\square$

5.3. Integrability of the almost CR -structure $(\mathcal{D}, \mathcal{J}_+^{(1)})$ on the twistor space.

Propositions 6 and 7 imply the following.

Theorem 2. ([4]) Suppose that the $SO(3)$ -structure on M is nearly integrable. Then the almost CR structure $(\mathcal{D}, \mathcal{J}_+^{(1)})$ on the twistor space $\mathbb{T}$ defined by means of the characteristic connection is integrable if and only if $*T \in \Lambda_3^2 TM$ and the $\odot_9^2$ -component of the curvature vanishes.

Remark. It is a result of Inus [8] (see [3, Theorem 6.57]) that any normal almost contact metric structure induces an integrable almost CR -structure. It is well-known that the converse is not true. The almost contact metric structure $(\Phi_+^{(1)}, \chi, h_t)$ in Example 2 above also gives a counterexample. As we have seen as an application of Theorem 1, it is not normal, although the induced CR structure $(\mathcal{D}, \mathcal{J}_+^{(1)})$ is integrable by Theorem 2.

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