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MANGASARIAN-TYPE SUFFICIENT OPTIMALITY CONDITIONS FOR AGE-STRUCTURED CONTROL PROBLEMS WITH STATE CONSTRAINTS. AN APPLICATION TO INVESTMENT IN VINTAGE CAPITAL

Vladimir Krastev

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Abstract. We consider a class of age-structured control problems with state constraints, nonlocal dynamics and boundary conditions, defined on finite time intervals. For these problems we suggest Mangasarian-type sufficient conditions for the optimality of the control. As an application we consider a model with a state constraint of optimal investment in vintage capital goods. To solve this model we suggest a numerical method and we prove that this method converges to an optimal solution.

1. Introduction and the general problem. The age (vintage) plays an important role in the statement of many problems which arise in biology, economics, demography and other sciences. Such problems, known as age-structured optimal control problems, are considered for example in [1], [2], [5],

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[6], [9], [10], [12] and [16]. The finding of an optimal solution to a control problem is usually connected with implementation of some optimality condition. It turns out that the most applicable optimality condition for the age-structured control problems without state constraints is the Pontryagin's type necessary condition which is obtained in [7]. The use of state constraints is an intrinsic feature of the economic models. As we know there have not been obtained optimality conditions for the control problems which correspond to these models. In the present paper we suggest Mangasarian type sufficient conditions for optimality for a class of age-structured optimal control problems with state constraints. These conditions are analogous to the sufficient optimality conditions for optimal control problems governed by ODEs suggested in [17] and [18].

We consider the following general control problem:

$$\begin{aligned} J(u,v,w) = &\int_0^T\int_0^\omega f_0\Big(t,a,y(t,a),p(t,a),q(t),u(t,a)\Big)\,da\,dt+ \\ &+\int_0^\omega \Big(\psi_0(a,w(a))+l(a,y(T,a))\Big)\,da+\int_0^T \varphi_0\Big(t,q(t),v(t)\Big)\,dt \longrightarrow \max, \end{aligned} \tag{1}$$

subject to the dynamic equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)y(t,a) = f\Big(t,a,y(t,a),p(t,a),q(t),u(t,a)\Big), \tag{2}$$

$$p(t,a) = \int_0^\omega g\Big(t,a,a',y(t,a'),u(t,a')\Big)\,da', \tag{3}$$

$$q(t) = \int_0^\omega h\Big(t,a,y(t,a),p(t,a),q(t),u(t,a)\Big)\,da, \tag{4}$$

the initial condition

$$y(0,a) = \psi(a,w(a)) \qquad \text{for } a\in[0,\omega], \tag{5}$$

the boundary condition

$$y(t,0) = \varphi(t,q(t),v(t)) \qquad \text{for } t\in[0,T], \tag{6}$$

the control variable constraints

$$u(t,a) \in U, \quad v(t) \in V, \quad w(a) \in W \qquad \text{for } t \in [0,T] \text{ and } a \in [0,\omega], \tag{7}$$

the mixed state constraints

$$\begin{gathered} \Pi\Big(t,a,y(t,a),p(t,a),q(t),u(t,a)\Big) \geq 0, \quad \Phi\Big(t,q(t),v(t)\Big) \geq 0, \\ \Psi\Big(a,w(a)\Big) \geq 0 \end{gathered} \tag{8}$$

for $t \in [0,T]$ and $a \in [0,\omega]$, and the pure local state constraints

$$\pi\Big(t,a,y(t,a)\Big) \geq 0 \qquad \text{for } t \in [0,T] \text{ and } a \in [0,\omega]. \tag{9}$$

The third group of inequalities of (8) is not actually a group of state constraints, but we are introducing it to obtain symmetry in the results. In the optimization problems the sets U, V and W are usually specified by inequalities. These inequalities might be incorporated into (8), so there might be various allocations of these sets. It is known from the optimal control theory for ODEs, that the pure state constraints (that is the constraints in which control variables are not involved) are more difficult to handle. Therefore we separate here the inequalities (9) which do not include control and nonlocal (p and q) variables. We will call to these inequalities pure local state constraints. They might be included into the constraints (8) by substitution of the function Π with the function (Π, π). But analogously to the optimal control theory for ODEs, we will suggest relaxed sufficient conditions for the age-structured problems with this kind of constraints.

In the paper we denote by t the time, running in the interval $[0,T]$. The scalar valued variable $a \in [0,\omega]$ is usually interpreted as the age of the controlled objects in the applications. Sometimes it is convenient to transform the original (t,a) variables (coordinates) to the characteristic (t,x) coordinates by the transformation $x = t - a$, $a = t - x$. Then the variable $x \in [-\omega, T]$ is interpreted as the time of birth.

We denote by (s,c) any pair of state (phase) variable $s \stackrel{\text{def}}{=} (y,p,q)$ and control variable $c \stackrel{\text{def}}{=} (u,v,w)$. Besides y is the local state variable, (p,q) are nonlocal state variables, and the variables u, v and w are distributed, boundary and initial control respectively. The strict formulation of these variables and of the functions used in the problem is given in the next section.

The present paper may be considered as a continuation of the paper [14]. The problem considered here is a modification of the main control problems which have been considered in [7] and [14]. The difference with the paper [14] is the presence of state constraints given by the inequalities (8) and (9). Unlike the main problem considered in [7], in [14] and in the present paper the objective functional (1) is divided additively.

The paper is organized as follows. In the next section we give the formulation of the main problem in details and define the corresponding Hamiltonians, Lagrangians, adjoint variables and Lagrange multipliers. In section 3 we suggest and prove sufficient conditions for optimality for the main problem. As application of the suggested sufficient conditions in the last section 4 we consider a model with pure local state constraint of a problem for investment in vintage capital goods. In order to solve this model we suggest a numerical algorithm based on the shooting method. We show that this algorithm converges to an optimal solution.

2. Basic definitions and assumptions. Let us denote by $Q \stackrel{\text{def}}{=} [0,T] \times [0,\omega]$ the domain in which we consider the main problem. We refer to the vector $\vec{e} = (1,1)$ as the (characteristic) direction of the differential operator $\Big(\partial/\partial t + \partial/\partial a\Big)$, and to the last operator we refer as to directional derivative along the direction $\vec{e}$, that is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t,a) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \frac{y(t+\varepsilon, a+\varepsilon) - y(t,a)}{\varepsilon}$$

The control variables are $u : Q \to R^{k_1}$, $v : [0,T] \to R^{k_2}$ and $w : [0,\omega] \to R^{k_3}$, and they are restricted in the sets $U \subset R^{k_1}$, $V \subset R^{k_2}$ and $W \subset R^{k_3}$ respectively. The state variables are $y : Q \to R^m$, $p : Q \to R^n$ and $q : [0,T] \to R^r$. The functions used in the main problem f_0, ψ_0, l and φ_0 are scalar valued. The other functions are vector valued: $f \in R^m$, $g \in R^n$, $h \in R^r$, $\psi \in R^m$, $\varphi \in R^m$, $\Pi \in R^{l_1}$, $\Phi \in R^{l_2}$, $\Psi \in R^{l_3}$ and $\pi \in R^{l_4}$. As in [14] the functions used f_0, f, h, g, ψ_0, ψ, l, φ_0, φ, Π, Φ, Ψ and π are Carathéodory (measurable with respect to t, a, a' and continuous with respect to the rest of the variables), essentially bounded, and differentiable with respect to (y,p,q,u,v,w). Their partial derivatives are also Carathéodory (measurable with respect to t, a, a' and continuous with respect to the rest of the variables) and essentially bounded.

An admissible control is any triplet $c(\cdot,\cdot) = (u(\cdot,\cdot), v(\cdot), w(\cdot))$ where $u(\cdot,\cdot)$, $v(\cdot)$ and $w(\cdot)$ are measurable in Q, $[0,T]$ and $[0,\omega]$ respectively and the constraints (7) are satisfied. If $c(\cdot,\cdot)$ is a fixed admissible control, we use the notion of solution $y \in L^\infty(Q; R^m)$, $p \in L^\infty(Q; R^n)$ and $q \in L^\infty([0,T]; R^r)$ to the dynamic system

(2)–(4), which is given in the definition 1 of [7]. Besides we'll assume that for any admissible control there exists a unique solution on Q to this dynamic system. If this solution satisfies (5), (6), (8) and (9) then we will call this solution admissible state (phase) trajectory.

For the main problem we use the distributed, the boundary and the initial Hamiltonians

$$\begin{aligned} &H(t,a,y,p,q,u,\xi,\eta_t(\cdot),\zeta) \stackrel{\mathrm{def}}{=} f_0(t,a,y,p,q,u) + \xi f(t,a,y,p,q,u) + \\ &+ \int_0^\omega \eta_t(a') g(t,a',a,y,u)\,da' + \zeta h(t,a,y,p,q,u), \end{aligned} \tag{10}$$

$$H_b(t,q,v,\xi_t(\cdot)) \stackrel{\mathrm{def}}{=} \varphi_0(t,q,v) + \xi_t(0)\varphi(t,q,v), \tag{11}$$

$$H_0(a,w,\xi_a(\cdot)) \stackrel{\mathrm{def}}{=} \psi_0(a,w) + \xi_a(0)\psi(a,w), \tag{12}$$

which are functionals of the functions $\eta_t(\cdot)$, $\xi_t(\cdot)$ and $\xi_a(\cdot)$. We will use these Hamiltonians together with the adjoint functions (variables) $\xi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$ and $\zeta(\cdot)$ which will be defined below. Besides, for the functions referred to as arguments of the Hamiltonians, we will always use $\eta_t(\cdot) = \eta(t,\cdot)$, $\xi_t(\cdot) = \xi(t,\cdot)$ and $\xi_a(\cdot) = \xi(\cdot,a)$.

In order to handle the state constraints, we define the following distributed, boundary and initial Lagrangians:

$$\begin{aligned} &L(t,a,y,p,q,u,\xi,\eta_t(\cdot),\zeta,\lambda,\lambda') \stackrel{\mathrm{def}}{=} \\ &\stackrel{\mathrm{def}}{=} H(t,a,y,p,q,u,\xi,\eta_t(\cdot),\zeta) + \lambda\,\Pi(t,a,y,p,q,u) + \lambda'\pi(t,a,y), \end{aligned} \tag{13}$$

$$L_b(t,q,v,\xi_t(\cdot),\mu) \stackrel{\mathrm{def}}{=} H_b(t,q,v,\xi_t(\cdot)) + \mu\Phi(t,q,v), \tag{14}$$

$$L_0(a,w,\xi_a(\cdot),\nu) \stackrel{\mathrm{def}}{=} H_0(a,w,\xi_a(\cdot)) + \nu\Psi(a,w). \tag{15}$$

Here we assume that the Lagrange multipliers are $\lambda \in L^1(Q;R^{l_1})$, $\lambda' \in L^1(Q;R^{l_4})$, $\mu \in L^1\left([0,T]\,;R^{l_2}\right)$ and $\nu \in L^1\left([0,\omega];R^{l_3}\right)$. Of course the Lagrangians are also functionals of $\eta_t(\cdot)$, $\xi_t(\cdot)$ and $\xi_a(\cdot)$, and we will always use the same functions for these arguments.

In the definitions (10)–(12) of the Hamiltonians we have used the adjoint functions $\xi \in L^{\infty}(Q;R^m)$, $\eta \in L^{\infty}(Q;R^n)$ and $\zeta \in L^{\infty}([0,T];R^r)$, which are a solution to the following adjoint system of equations:

$$\begin{aligned}(16)\qquad &\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\xi(t,a)=-\frac{\partial}{\partial y}L\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t),\lambda,\lambda'\right)=\\ &=-\frac{\partial}{\partial y}H\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t)\right)-\\ &\quad-\lambda\frac{\partial}{\partial y}\Pi\left(t,a,y,p,q,u\right)-\lambda'\frac{\partial}{\partial y}\pi\left(t,a,y\right),\end{aligned}$$

$$\begin{aligned}(17)\qquad &\eta(t,a)=\frac{\partial}{\partial p}L\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t),\lambda,\lambda'\right)=\\ &=\frac{\partial}{\partial p}H\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t)\right)+\lambda\frac{\partial}{\partial p}\Pi\left(t,a,y,p,q,u\right),\end{aligned}$$

$$\begin{aligned}(18)\qquad &\zeta(t)=\frac{\partial}{\partial q}L_b\left(t,q,v,\xi(t,0),\mu\right)+\\ &+\int_0^{\omega}\frac{\partial}{\partial q}L\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t),\lambda,\lambda'\right)da=\\ &=\frac{\partial}{\partial q}H_b\left(t,q,v,\xi(t,0)\right)+\mu\frac{\partial}{\partial q}\Phi(t,q,v)+\\ &+\int_0^{\omega}\frac{\partial}{\partial q}H\left(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t)\right)da+\int_0^{\omega}\lambda\frac{\partial}{\partial q}\Pi\left(t,a,y,p,q,u\right)da.\end{aligned}$$

The definition of a solution to this adjoint system is analogous to the definition of the solution to the dynamic system (2)–(4). We omit the arguments t and a of the control and state variables as well as the arguments (t,a) of the Lagrange multipliers λ and λ', the argument t of the multiplier μ and a of the multiplier ν. Further, for the sake of brevity we will continue to omit the arguments t and a. For example, instead of $f\left(t,a,y(t,a),p(t,a),q(t),u(t,a)\right)$ we will write as $f\left(t,a,y,p,q,u\right)$. The control and state variables, which we'll test for optimality, we will denote by "hats". Moreover, we will use abbreviations, such as $f^{\wedge}[t,a] \stackrel{\text{def}}{=} f\left(t,a,\hat{y},\hat{p},\hat{q},\hat{u}\right)$ and $f[t,a] \stackrel{\text{def}}{=} f\left(t,a,y,p,q,u\right)$. For the Hamiltonians of the variables denoted by "hats" these abbreviations written in detail are $H^{\wedge}[t,a]=H\left(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),\hat{u}(t,a),\xi(t,a),\eta(t,\cdot),\zeta(t)\right)$,

$H_0^\wedge[a] = H_0\left(a, \hat{w}(a), \xi(0,a)\right)$ and $H_b^\wedge[t] = H_b\left(t, \hat{q}(t), \hat{v}(t), \xi(t,0)\right)$. The abbreviations are similar for the Hamiltonians of variables without "hats" and for the Lagrangians. We will omit the sign for transposition in the dot products of vectors.

3. Sufficient conditions for optimality. In this section we consider the problems (1)–(8) and (1)–(9) which are problems with and without pure local state constraints, respectively. In order to prove the main results we need the following:

Lemma 1. *Let the function $f(t,a) \in L^1(Q)$ be absolutely continuous along the characteristic direction and let $(\partial/\partial t + \partial/\partial a)\, f(t,a) \in L^1(Q)$.*

Then the following equality holds:

$$\int_0^T \int_0^\omega \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) f(t,a)\, da\, dt \;=\; \int_0^T f(t,a)\Big|_0^\omega dt \;+\; \int_0^\omega f(t,a)\Big|_0^T da \tag{19}$$

This lemma is proven in [14] and it is actually a reformulation of Lemma 2 from the paper [7]. Therefore we omit the proof here.

Further we use the following well-known notion:

Quasiconcave function. *Let $f : S \to R$ be a real-valued function defined on a convex subset S of a real vector space. The function f is said to be quasiconcave if for each $x_1, x_2 \in S$ the following inequality holds:*

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left\{f\left(x_1\right), f\left(x_2\right)\right\} \quad \textit{for each } \lambda \in [0,1].$$

In other words f is said to be quasiconcave if $-f$ is quasiconvex.

In the following theorem we suggest a sufficient condition for optimality for the age-structured control problem with mixed state constraints (1)–(8).

Theorem 1. *Let $(\hat{s}, \hat{c}) = (\hat{y}, \hat{p}, \hat{q}, \hat{u}, \hat{v}, \hat{w})$ be an admissible pair of state and control variables for the problem (1)–(8). Let there exist Lagrange multipliers λ, μ and ν, and an absolutely continuous along the characteristic direction $\vec{e}$ solution (ξ, η, ζ) of the adjoint system (16)–(18), which corresponds to the considered pair and to the Lagrange multipliers. Besides let the following six assumptions hold:*

1. *The necessary conditions for local maximum of the Lagrangians:*

$$(20)\quad \frac{\partial}{\partial u} L^{\wedge}[t,a]\Big(u-\hat{u}(t,a)\Big) = \\ = \left\{\frac{\partial}{\partial u} H^{\wedge}[t,a] + \lambda(t,a)\frac{\partial}{\partial u}\Pi^{\wedge}[t,a]\right\}\Big(u-\hat{u}(t,a)\Big) \le 0$$

for a.e. $(t,a) \in Q$ and a.e. $u \in U$;

$$(21)\quad \frac{\partial}{\partial v} L_b^{\wedge}[t]\Big(v-\hat{v}(t)\Big) = \left\{\frac{\partial}{\partial v} H_b^{\wedge}[t] + \mu(t)\frac{\partial}{\partial v}\Phi^{\wedge}[t]\right\}\Big(v-\hat{v}(t)\Big) \le 0$$

for a.e. $t \in [0,T]$ and a.e. $v \in V$;

$$(22)\quad \frac{\partial}{\partial w} L_0^{\wedge}[a]\Big(w-\hat{w}(a)\Big) = \left\{\frac{\partial}{\partial w} H_0^{\wedge}[a] + \nu(a)\frac{\partial}{\partial w}\Psi^{\wedge}[a]\right\}\Big(w-\hat{w}(a)\Big) \le 0$$

for a.e. $a \in [0,\omega]$ and a.e. $w \in W$.

2. *The conditions for non-negativity of the Lagrange multipliers and the complementary slackness conditions:*

$$(23)\quad \begin{array}{l} \lambda(t,a) \ge 0, \quad \lambda(t,a)\Pi^{\wedge}[t,a] = 0; \quad \mu(t) \ge 0, \quad \mu(t)\Phi^{\wedge}[t] = 0; \\ \nu(a) \ge 0, \qquad \nu(a)\Psi^{\wedge}[a] = 0 \end{array}$$

for a.e. $(t,a) \in Q$, $t \in [0,T]$ and $a \in [0,\omega]$ respectively.

3. *The transversality conditions:*

$$(24)\quad \xi(T,a) = \frac{\partial}{\partial y} l\Big(a, \hat{y}(T,a)\Big) \qquad \text{for a.e. } a \in [0,\omega],$$

$$(25)\quad \xi(t,\omega) = 0 \qquad \text{for a.e. } t \in [0,T].$$

4. *The function $\Pi(t,a,y,p,q,u)$ is quasiconcave with respect to (y,p,q,u), $\Phi(t,q,v)$ is quasiconcave with respect to (q,v) and $\Psi(a,w)$ is quasiconcave with respect to w. The sets U, V and W are convex.*

5. *For the given adjoint functions ξ, η and ζ the distributed Hamiltonian $H(t,a,y,p,q,u,\xi(t,a),\eta(t,\cdot),\zeta(t))$ is concave with respect to (y,p,q,u), the boundary Hamiltonian $H_b(t,q,v,\xi(t,\cdot))$ is concave with respect to (q,v) and the initial Hamiltonian $H_0(a,w,\xi(\cdot,a))$ is concave with respect to w.*

6. *The function $l(a,y)$ is concave with respect to y.*

Then the pair $(\hat{s},\hat{c})=(\hat{y},\hat{p},\hat{q},\hat{u},\hat{v},\hat{w})$ is optimal for the problem (1)–(8).

Proof. The ideas are the same as in [17] and [18]. Let us note first, that from the concavity of the Hamiltonians (assumption 5 of the theorem) the following inequalities hold (see in [3, p. 103]):

$$
\begin{aligned}
&H^\wedge[t,a]-H[t,a]\geq\frac{\partial}{\partial y}H^\wedge[t,a]\,(\hat{y}-y)+\\
&+\frac{\partial}{\partial p}H^\wedge[t,a]\,(\hat{p}-p)+\frac{\partial}{\partial q}H^\wedge[t,a]\,(\hat{q}-q)+\frac{\partial}{\partial u}H^\wedge[t,a]\,(\hat{u}-u)\,,
\end{aligned}
\tag{26}
$$

$$
H_b^\wedge[t]-H_b[t]\geq\frac{\partial}{\partial q}H_b^\wedge[t]\,(\hat{q}-q)+\frac{\partial}{\partial v}H_b^\wedge[t]\,(\hat{v}-v)\,,
\tag{27}
$$

$$
H_0^\wedge[a]-H_0[a]\geq\frac{\partial}{\partial w}H_0^\wedge[a]\,(\hat{w}-w)\,.
\tag{28}
$$

According to [3, p. 115, Theorem 3.5.4] and to the standing assumptions of the present paper for differentiability, the functions $\Pi(t,a,y,p,q,u)$, $\Phi(t,q,v)$ and $\Psi(a,w)$ are quasiconcave with respect to (y,p,q,u), (q,v) and w respectively if and only if the statement

$$
\begin{aligned}
&\text{if}\quad \Pi[t,a]\geq\Pi^\wedge[t,a]\quad\text{then}\\
&\frac{\partial\Pi^\wedge[t,a]}{\partial y}\,(y-\hat{y})+\frac{\partial\Pi^\wedge[t,a]}{\partial p}\,(p-\hat{p})+\\
&\qquad\qquad+\frac{\partial\Pi^\wedge[t,a]}{\partial q}\,(q-\hat{q})+\frac{\partial\Pi^\wedge[t,a]}{\partial u}\,(u-\hat{u})\geq 0,
\end{aligned}
\tag{29}
$$

$$
\text{if}\quad \Phi[t]\geq\Phi^\wedge[t]\quad\text{then}\quad\frac{\partial\Phi^\wedge[t]}{\partial q}\,(q-\hat{q})+\frac{\partial\Phi^\wedge[t]}{\partial v}\,(v-\hat{v})\geq 0,
\tag{30}
$$

$$
\text{if}\quad \Psi[a]\geq\Psi^\wedge[a]\quad\text{then}\quad\frac{\partial\Psi^\wedge[a]}{\partial w}\,(w-\hat{w})\geq 0
\tag{31}
$$

holds true respectively.

Let (s,c) be an arbitrary admissible pair of state and control variables. Let us denote by $\Delta \overset{\text{def}}{=} J(\hat{u},\hat{v},\hat{w}) - J(u,v,w)$ the difference between the values of the objective functional for $\hat{c}$ and c. We must prove that $\Delta \geq 0$. From (1) we see that the difference

$$
\begin{aligned}
\Delta = &\int_0^T \int_0^\omega \Big(f_0(t,a,\hat{y},\hat{p},\hat{q},\hat{u}) - f_0(t,a,y,p,q,u) \Big) da dt + \\
(32) \qquad &+ \int_0^\omega \Big(\psi_0(a,\hat{w}) + l(a,\hat{y}) - \psi_0(a,w) - l(a,y) \Big) da + \\
&+ \int_0^T \Big(\varphi_0(t,\hat{q},\hat{v}) - \varphi_0(t,q,v) \Big) dt
\end{aligned}
$$

is a sum of three integrals, which we denote by I_1, I_2 and I_3.

We have to estimate each of these integrals. First, using the definition of the distributed Hamiltonian, we estimate I_1 consecutively:

$$
\begin{aligned}
I_1 = &\int_0^T \int_0^\omega \Big(H^\wedge[t,a] - H[t,a] \Big) da dt - \int_0^T \int_0^\omega \xi \Big(f^\wedge[t,a] - f[t,a] \Big) da dt - \\
(33) \quad &- \int_0^T \int_0^\omega \int_0^\omega \eta(t,a') \Big(g(t,a',a,\hat{y},\hat{u}) - g(t,a',a,y,u) \Big) da' da dt - \\
&- \int_0^T \int_0^\omega \zeta(t) \Big(h^\wedge[t,a] - h[t,a] \Big) da dt \geq
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_0^T \int_0^\omega \left(\frac{\partial H^\wedge[t,a]}{\partial y}(\hat{y}-y) + \frac{\partial H^\wedge[t,a]}{\partial p}(\hat{p}-p) + \frac{\partial H^\wedge[t,a]}{\partial q}(\hat{q}-q) \right) da dt + \\
(33a) \quad &+ \int_0^T \int_0^\omega \frac{\partial H^\wedge[t,a]}{\partial u}(\hat{u}-u) da dt - \int_0^T \int_0^\omega \xi \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) (\hat{y}-y) da dt - \\
&- \int_0^T \int_0^\omega \eta(t,a') \Big(\hat{p}(t,a') - p(t,a') \Big) da' dt - \int_0^T \zeta(\hat{q}-q) dt \geq
\end{aligned}
$$

(33b)
$$\geq \int_0^T\int_0^\omega \left\{\left[\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\xi + \lambda\frac{\partial \Pi^\wedge[t,a]}{\partial y}\right](y-\hat{y}) + \left(\lambda\frac{\partial \Pi^\wedge[t,a]}{\partial p}-\eta\right)(p-\hat{p})\right\} dadt$$
$$+\int_0^T\int_0^\omega \lambda\frac{\partial \Pi^\wedge[t,a]}{\partial u}(u-\hat{u})\, dadt+$$
$$+\int_0^T \left\{\frac{\partial H_b^\wedge[t,a]}{\partial q} + \mu\frac{\partial \Phi^\wedge[t]}{\partial q} + \int_0^\omega \lambda\frac{\partial \Pi^\wedge[t,a]}{\partial q}da - \zeta(t)\right\}(q-\hat{q})\, dt+$$
$$+\int_0^T\int_0^\omega \xi\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)(y-\hat{y})\, dadt + \int_0^T\int_0^\omega \eta\,(p-\hat{p})\, dadt + \int_0^T \zeta\,(q-\hat{q})\, dt =$$

(33c)
$$= \int_0^T\int_0^\omega \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\Big[\xi\,(y-\hat{y})\Big] dadt + \int_0^T \left\{\frac{\partial H_b^\wedge[t]}{\partial q} + \mu\frac{\partial \Phi^\wedge[t]}{\partial q}\right\}(q-\hat{q})\, dt+$$
$$+\int_0^T\int_0^\omega \lambda\left\{\frac{\partial \Pi^\wedge[t,a]}{\partial y}(y-\hat{y}) + \frac{\partial \Pi^\wedge[t,a]}{\partial p}(p-\hat{p}) + \frac{\partial \Pi^\wedge[t,a]}{\partial q}(q-\hat{q})\right\} dadt+$$
$$+\int_0^T\int_0^\omega \lambda\frac{\partial \Pi^\wedge[t,a]}{\partial u}(u-\hat{u})\, dadt \geq$$

(33d)
$$\geq \int_0^T\int_0^\omega \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\Big[\xi\,(y-\hat{y})\Big] dadt + \int_0^T \left\{\frac{\partial H_b^\wedge[t]}{\partial q} + \mu\frac{\partial \Phi^\wedge[t]}{\partial q}\right\}(q-\hat{q})\, dt =$$

(33e)
$$= \int_0^\omega \xi(t,a)\Big(y(t,a)-\hat{y}(t,a)\Big)\Big|_0^T da + \int_0^T \xi(t,a)\Big(y(t,a)-\hat{y}(t,a)\Big)\Big|_0^\omega dt+$$
$$+\int_0^T \left\{\frac{\partial H_b^\wedge[t]}{\partial q} + \mu\frac{\partial \Phi^\wedge[t]}{\partial q}\right\}(q-\hat{q})\, dt =$$

$$
\begin{aligned}
&= \int_0^\omega \xi\,(T,a)\Big(y(T,a)-\hat{y}(T,a)\Big)da - \int_0^\omega \xi(0,a)\Big(\psi(a,w)-\psi\,(a,\hat{w})\Big)da + \\
&+ \int_0^T \xi(t,\omega)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)dt - \int_0^T \xi(t,0)\Big(\varphi(t,q,v)-\varphi\,(t,\hat{q},\hat{v})\Big)dt + \\
&+ \int_0^T \left\{\frac{\partial H_b^{\wedge}[t]}{\partial q} + \mu \frac{\partial \Phi^{\wedge}[t]}{\partial q}\right\}(q-\hat{q})\,dt
\end{aligned} \tag{33f}
$$

This series of estimations is obtained: a) from the inequality (26) and from the dynamic equations (2), (3) and (4); b) from the adjoint system (16), (17), (18) and from the condition (20) for local maximum of the distributed Lagrangian; c) from the derivative of the product of functions rule and by canceling of opposite terms; d) from the conditions (23) for non-negativity and complementary slackness and from the quasiconcavity of the function Π with respect to (y,p,q,u) (if $\lambda(t,a)>0$ then $\Pi^{\wedge}[t,a]=0$, therefore $\Pi[t,a]\geq \Pi^{\wedge}[t,a]$ and the inequality of the statement (29) holds); e) from the lemma; f) from the initial condition (5) and the boundary condition (6).

Now we estimate the integral I_2:

$$
I_2 = \int_0^\omega \Big(\psi_0\,(a,\hat{w})-\psi_0(a,w)\Big)da + \int_0^\omega \Big(l\,(a,\hat{y}(T,a)) - l\,(a,y(T,a))\Big)da \geq \tag{34}
$$

$$
\begin{aligned}
&\geq \int_0^\omega \Big(H_0^{\wedge}[a]-H_0[a]\Big)da - \int_0^\omega \xi(0,a)\Big(\psi\,(a,\hat{w})-\psi(a,w)\Big)da + \\
&+ \int_0^\omega \frac{\partial}{\partial y}\, l\Big(a,\hat{y}(T,a)\Big)\Big(\hat{y}(T,a)-y(T,a)\Big)da \;\geq
\end{aligned} \tag{34a}
$$

$$
\begin{aligned}
&\geq \int_0^\omega \frac{\partial}{\partial w} H_0^{\wedge}[a]\,(\hat{w}-w)\,da - \int_0^\omega \xi(0,a)\Big(\psi\,(a,\hat{w})-\psi(a,w)\Big)da + \\
&+ \int_0^\omega \frac{\partial}{\partial y}\, l\Big(a,\hat{y}(T,a)\Big)\Big(\hat{y}(T,a)-y(T,a)\Big)da \;\geq
\end{aligned} \tag{34b}
$$

(34c)
$$\geq \int_0^\omega \nu(a)\frac{\partial}{\partial w}\Psi^\wedge[a]\,(w-\hat{w})\,da - \int_0^\omega \xi(0,a)\Big(\psi\,(a,\hat{w}) - \psi(a,w)\Big)\,da + \\ + \int_0^\omega \frac{\partial}{\partial y}\,l\Big(a,\hat{y}(T,a)\Big)\Big(\hat{y}(T,a) - y(T,a)\Big)\,da\;\geq$$

(34d)
$$\geq \int_0^\omega \xi(0,a)\Big(\psi\,(a,w) - \psi\,(a,\hat{w})\,\Big)da + \int_0^\omega \frac{\partial}{\partial y} l\Big(a,\hat{y}(T,a)\Big)\Big(\hat{y}(T,a) - y(T,a)\Big)da$$

We have obtained this estimation using: a) the definition of the initial Hamiltonian (12) and the concavity of the function $l(a,y)$ with respect to y; b) the inequality (28); c) the condition (22) for local maximum of the initial Lagrangian L_0; d) the condition (23) and the quasiconcavity of the function $\Psi(a,w)$ with respect to w (as in the estimation of I_1 if $\nu(a) > 0$ then $\Psi^\wedge[a] = 0$, therefore $\Psi[a] \geq \Psi^\wedge[a]$ and the inequality of (31) holds).

Similarly we estimate the integral I_3:

(35)
$$I_3 = \int_0^T \Big(\varphi_0\,(t,\hat{q},\hat{v}) - \varphi_0(t,q,v)\Big)\,dt =$$

(35a)
$$= \int_0^T \Big(H_b^\wedge[t] - H_b[t]\Big)\,dt - \int_0^T \xi(t,0)\Big(\varphi\,(t,\hat{q},\hat{v}) - \varphi(t,q,v)\Big)\,dt\;\geq$$

(35b)
$$\geq \int_0^T \frac{\partial}{\partial q} H_b^\wedge[t]\,(\hat{q}-q)\,dt + \int_0^T \frac{\partial}{\partial v} H_b^\wedge[t]\,(\hat{v}-v)\,dt - \\ - \int_0^T \xi(t,0)\Big(\varphi\,(t,\hat{q},\hat{v}) - \varphi(t,q,v)\Big)\,dt\;\geq$$

$$\begin{aligned}&\geq \int_0^T \frac{\partial}{\partial q} H_b^\wedge[t]\,(\hat{q}-q)\,dt + \int_0^T \mu(t)\frac{\partial}{\partial v}\Phi^\wedge[t]\,(v-\hat{v})\,dt - \\ &-\int_0^T \xi(t,0)\Big(\varphi\,(t,\hat{q},\hat{v}) - \varphi(t,q,v)\Big)\,dt\end{aligned} \tag{35c}$$

This series of evaluations we have obtained by using: a) the definition of the boundary Hamiltonian (11); b) the inequality (27); c) the necessary condition for local maximum of the boundary Lagrangian (21).

Adding the right-hand sides of the inequalities (33), (34) and (35), and canceling the opposite terms we obtain the following evaluation for the difference Δ:

$$\begin{aligned}\Delta \geq &\int_0^\omega \left[\xi(T,a) - \frac{\partial}{\partial y}\, l\Big(a,\hat{y}(T,a)\Big)\right]\Big(y(T,a)-\hat{y}(T,a)\Big)\,da + \\ &+\int_0^T \xi(t,\omega)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)\,dt + \\ &+\int_0^T \mu(t)\left\{\frac{\partial}{\partial q}\Phi^\wedge[t]\,(q-\hat{q}) + \frac{\partial}{\partial v}\Phi^\wedge[t]\,(v-\hat{v})\right\}dt\end{aligned} \tag{36}$$

From the transversality conditions (24) and (25) it follows that the first two integrals in the right-hand side of the above inequality are non-negative. From the conditions (23) and from the quasiconcavity of the function Φ with respect to (q,v) it follows that the third integral is also non-negative (if $\mu(t)>0$ then $\Phi^\wedge[t]=0$, $\Phi[t]\geq\Phi^\wedge[t]$ and therefore the inequality of (30) holds). Therefore $\Delta\geq 0$ and the theorem is proven. $\square$

It turns out that if the age-structured control problem includes pure local state constraints, as the separated in the inequalities (9) ones, we may have to allow the adjoint variable $\xi(t,a)$ to jump at the end points $t=T$ and $a=\omega$. Therefore we suggest the following sufficient condition for the problem (1)–(9):

Theorem 2. *Let* $(\hat{s},\hat{c}) = (\hat{y},\hat{p},\hat{q},\hat{u},\hat{v},\hat{w})$ *be an admissible pair for the problem* (1)–(9). *Let there exist Lagrange multipliers* λ, λ', μ *and* ν, *and an ab-*

solutely continuous along the characteristic direction solution (ξ, η, ζ) *of the adjoint system* (16)–(18), *which correspond to* $(\hat{s}, \hat{c})$ *and to the Lagrange multipliers. Furthermore let there exist functions* $\alpha \in L^1([0,T]; R^m)$ *and* $\beta \in L^1([0,\omega]; R^m)$.

Let the six assumptions of the Theorem 1 hold with λ *replaced by* (λ, λ') *and* Π *replaced by* (Π, π), *and in addition the next three assumptions be satisfied:*

1. *The function* $\pi(t, a, y)$ *is quasiconcave with respect to* y*;*
2. *The adjoint variable* $\xi(t, a)$ *could be discontinuous at the end points* $a = \omega$ *in which case the following jump condition is satisfied*

$$\xi\left(t, \omega^-\right) - \xi(t, \omega) \;=\; \alpha(t)\,\frac{\partial}{\partial y}\,\pi\Big(t, \omega, \hat{y}(t, \omega)\Big) \qquad \textit{for a.e. } t \in [0,T] \tag{37}$$

together with the conditions for non-negativity of the function α *and the complementary slackness*

$$\alpha(t) \geq 0, \quad \alpha(t)\,\pi\Big(t, \omega, \hat{y}(t, \omega)\Big) = 0 \qquad \textit{for a.e. } t \in [0,T]. \tag{38}$$

3. *The variable* $\xi(t, a)$ *could have jump discontinuities at* $t = T$ *in which case*

$$\xi\left(T^-, a\right) - \xi(T, a) \;=\; \beta(a)\,\frac{\partial}{\partial y}\,\pi\Big(T, a, \hat{y}(T, a)\Big) \quad \textit{for a.e. } a \in [0,\omega] \tag{39}$$

together with

$$\beta(a) \geq 0, \quad \beta(a)\,\pi\Big(T, a, \hat{y}(T, a)\Big) = 0 \qquad \textit{for a.e. } a \in [0,\omega]. \tag{40}$$

Then the pair $(\hat{s}, \hat{c})$ *is optimal.*

In the assumptions of this theorem we use the denotations

$$\xi\left(T^-, a\right) \stackrel{\text{def}}{=} \lim_{\substack{\varepsilon \to 0\\ \varepsilon > 0}} \xi(T - \varepsilon, a - \varepsilon) \quad \text{and} \quad \xi\left(t, \omega^-\right) \stackrel{\text{def}}{=} \lim_{\substack{\varepsilon \to 0\\ \varepsilon > 0}} \xi(t - \varepsilon, \omega - \varepsilon).$$

Proof. The only difference between the proofs of the previous and the present theorems is in the estimation of the integral I_1. In the same way here we receive the estimation (33f), but $\xi(T,a)$ and $\xi(t,\omega)$ are replaced by $\xi(T^-,a)$ and $\xi(t,\omega^-)$ respectively. Using the equations (37)–(40) and the quasiconcavity of the function π we estimate the sum of the first and the third terms of (33f) with the above mentioned replacements:

$$\int_0^\omega \xi\left(T^-,a\right)\Big(y(T,a)-\hat{y}(T,a)\Big)da+\int_0^T \xi\left(t,\omega^-\right)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)dt =$$

$$=\int_0^\omega \xi\left(T,a\right)\Big(y(T,a)-\hat{y}(T,a)\Big)da+\int_0^\omega \beta(a)\frac{\partial}{\partial y}\pi\Big(T,a,\hat{y}(T,a)\Big)\Big(y(T,a)-\hat{y}(T,a)\Big)da+$$

$$+\int_0^T \xi(t,\omega)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)dt+\int_0^T \alpha(t)\frac{\partial}{\partial y}\pi\Big(t,\omega,\hat{y}(t,\omega)\Big)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)dt\geq$$

$$\geq\int_0^\omega \xi(T,a)\Big(y(T,a)-\hat{y}(T,a)\Big)da+\int_0^T \xi(t,\omega)\Big(y(t,\omega)-\hat{y}(t,\omega)\Big)dt.$$

So the estimation for I_1 from the proof of the previous theorem holds and this proves the present theorem. $\square$

Remarks. The above theorem is a relaxation of Theorem 1. It is possible to generalize this relaxation by proving sufficient conditions in which to allow the adjoint variable $\xi(t,a)$ to have jumps on curves having parts which are internal to the domain Q. In this paper we will not prove such a generalization.

In the formulation of the Assumptions 2 and 3 of Theorem 2 instead of the jump conditions (37) and (39) we can assume that the adjoint variable $\xi(t,a)$ is continuous along the characteristic direction at the end points $a=\omega$ and $t=T$, and instead of the transversality conditions (24) and (25) from Theorem 1 the following transversality conditions hold:

$$(37')\quad \xi(t,\omega)=\alpha(t)\frac{\partial}{\partial y}\pi\Big(t,\omega,\hat{y}(t,\omega)\Big)\qquad \text{for a.e. } t\in[0,T],$$

$$(39')\quad \xi(T,a)=\beta(a)\frac{\partial}{\partial y}\pi\Big(T,a,\hat{y}(T,a)\Big)+\frac{\partial}{\partial y}l\Big(a,\hat{y}(T,a)\Big)\quad \text{for a.e. } a\in[0,\omega].$$

Let the assumptions of the Theorem 1 or the Theorem 2 hold. Obviously if any one of the Hamiltonians is strictly concave, then the pair $(\hat{s},\hat{c})$ is the unique optimal solution to the control problem. If we are looking for the minimum of the objective functional, we must replace all requirements for concavity and quasiconcavity of the functions with requirements for convexity and quasiconvexity respectively, the relations "$\leq$" in (20)–(22) with the opposite ones, and the conditions for non-negativity of the functions α, β and the Lagrange multipliers with the conditions for non-positivity.

4. An application to investment in vintage capital goods. We consider here an age-structured model of a newly established firm which produces a single product by means of capital goods (e.g. machines). The last can be of various vintages. In order to ensure the manufacturing process the firm has to invest in these capital goods being able to choose between a continuum of generations of them. The model is as follows:

$$\begin{aligned} J(I, I_0, K_0) = &\int_0^T \int_0^\omega e^{-rt} \left(p(t-a)K(t,a) - b(t,a)I(t,a) - \frac{c}{2} I^2(t,a) \right) da dt - \\ &- \int_0^\omega \left(b(0,a)K_0(a) + \frac{c}{2} K_0^2(a) \right) da - \int_0^T e^{-rt} \left(b(t,0)I_0(t) + \frac{c}{2} I_0^2(t) \right) dt \to \max \end{aligned} \tag{41}$$

subject to the dynamics

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) K(t,a) = I(t,a) - \delta K(t,a) \qquad t \in [0,T], \;\; a \in [0,\omega], \tag{42}$$

the boundary condition

$$K(t,0) = I_0(t) \qquad t \in [0,T], \tag{43}$$

the initial condition

$$K(0,a) = K_0(a) \qquad a \in [0,\omega], \tag{44}$$

and the state constraint

$$K(t,a) \geq 0 \quad \text{for each } (t,a) \in Q = [0,T] \times [0,\omega]. \tag{45}$$

In this model t denotes the time and a the age of a capital good. The vintage of the last is the difference $x = t - a$. The capital goods can be used no more than ω years after the year in which they are produced, so $a \in [0, \omega]$. The firm wants to determine its optimal investment strategy for a period of T years, hence $t \in [0, T]$. We shall assume that this time period is sufficiently long, so $T > \omega$. The stock (which is the number) of capital goods of age a at time t is the state variable in the model and is denoted by $K(t, a)$. In order to represent the fact that the stock can not be negative, we introduce the state constraint (45). The distributed control variable $I(t, a)$ is the investment in capital goods of age a at time t and it is the number of units purchased (if $I(t, a) > 0$) or sold (if $I(t, a) < 0$). Since the firm can buy and sell the capital goods, the model does not include any direct constraints on this control variable. Note however that the state constraint (45) restricts this variable indirectly: the firm can sell only if it has a positive stock of capital goods. Apart from the distributed control, we consider the investments in new capital goods $I_0(t)$ as a boundary control variable and the choice of the stock of capital goods at the moment of establishment of the firm $K_0(a)$ as an initial control variable. We assume that the other functions used in the model are sufficiently smooth and the constants used are positive.

In order to find the optimal investment strategy, the firm has to maximize the objective functional (41), which is the discounted profit stream. The integrand of the first term of (41) represents the discounted profit of the capital goods (machines) of age a at time t via the difference between the revenues from their exploitation $p(t-a)K(t, a)$ and the investment costs $b(t, a)I(t, a) + \dfrac{c}{2}I^2(t, a)$. The latter are divided into acquisition costs $b(t, a)I(t, a)$ and implementation costs $\frac{c}{2}I^2(t, a)$ (for example installation costs and uninstallation costs). In fact, the first term is revenue when the firm sells. The descriptions of the other integrands of (41) are analogous: $b(0, a)K_0(a) + \dfrac{c}{2}K_0^2(a)$ are the costs for initial capital of age a and $b(t, 0)I_0(t) + \dfrac{c}{2}I_0^2(t)$ are the investment costs at the moment t for the newest generation of the capital goods.

Due to technological progress the productivity of the capital goods is increasing with respect to their vintage, therefore $p'(x) \geq 0$. For the same reason the price of each fixed generation of capital goods (the capital goods with fixed vintage, i.e. $t - a = \text{const}$) must decline with age, hence $\dfrac{\partial}{\partial a}b(x + a, a) < 0$. We assume that anticipating the future technological progress the firm can estimate the future productivity and prices of the capital goods. Besides we assume that the price of each fixed generation of capital goods can be presented as a declining exponent of the age:

$$b(t, a) = b_0(t - a)e^{-\Delta(t-a)a} \tag{46}$$

where $b_0(x)$ and $\Delta(x)$ are positive valued functions. Most of the capital goods are durable, and according to Bayus [4] generally declining price is optimal for each generation of goods. This has been supported in many empirical studies. Having high coefficient of determination, the exponential trends fit to the optimal price paths and also fit to the empirical examples. The rate of decline of the prices $\Delta(x)$ increases for subsequent generations, therefore $\Delta'(x) > 0$. But according to [4], in some cases, penetration pricing strategy is optimal for the older generations of the durable good. We stress that in these cases our assumption that prices decline fails to hold.

The model is related to Feichtinger et al. [9]. Similar models have been investigated in [2] and in [10]. The evolution law of the capital described by (42)–(44) is the same as that one, described in the models of the cited papers. The full description of this evolution law can be found in [2]. Our model does not represent some features, represented in [9], such as the increase of the depreciation rate δ with the age, the effect of the experience on the firm's expenditures and the saturation of the market of the firm's product. However, through the introduction of the state constraint we guarantee that the optimal solution for the capital cannot be negative. As Feichtinger et al. have noted in [10], if the state constraint fails to hold for the optimal solution, the last could not have economic meaning.

Let us continue with solving the model (41)–(45). We will suggest a numerical algorithm which is based on the shooting method (see for example in [19] p. 502). Then we will show that the algorithm generates a sequence of pairs of control and state variables, and this sequence converges to an optimal pair, because the last satisfies the conditions of Theorem 2. First let us introduce the Hamiltonians and the Lagrangians. For the adjoint variable $\xi(t,a)$, which we'll introduce later, the distributed, the boundary and the initial Hamiltonians are

$$H(t,a,K,I,\xi) = e^{-rt}\left(p(t-a)K - b(t,a)I - \frac{c}{2}I^2\right) + \xi\,(I - \delta K)\,, \tag{47}$$

$$H_b\,(t, I_0, \xi(t,\cdot)) = -e^{-rt}\left(b(t,0)I_0 + \frac{c}{2}I_0^2\right) + \xi(t,0)I_0, \tag{48}$$

$$H_0\,(a, K_0, \xi(\cdot,a)) = -\left(b(0,a)K_0 + \frac{c}{2}K_0^2\right) + \xi(0,a)K_0, \tag{49}$$

respectively.

Obviously the conditions for concavity of Theorem 1 are satisfied for these Hamiltonians. Since the pure local state constraint (45) is the only state constraint in the model, the side Lagrangians coincide with the corresponding Hamiltonians, that is $L_b \equiv H_b$ and $L_0 \equiv H_0$, and the distributed Lagrangian is

$$L(t, a, K, I, \xi, \lambda) = H(t, a, K, I, \xi) + \lambda K. \tag{50}$$

Furthermore, it should be noted that the assumption for quasiconcavity of Theorem 2 (assumption 1) is satisfied for the state constraint (45).

The Hamiltonians and the Lagrangians introduced here are regular, which means that their maximizers with respect to the corresponding control variables are unique. These maximizers of the Lagrangians are

$$\hat{I}(t, a) = \frac{e^{rt}\xi(t, a) - b(t, a)}{c}, \tag{51}$$

$$\hat{I}_0(t) = \frac{e^{rt}\xi(t, 0) - b(t, 0)}{c}, \tag{52}$$

$$\hat{K}_0(a) = \frac{\xi(0, a) - b(0, a)}{c}. \tag{53}$$

From the above equations we see that $\hat{I}_0(t) = \hat{I}(t, 0)$ for each $t \in [0, T]$ and $\hat{K}_0(a) = \hat{I}(0, a)$ for each $a \in [0, \omega]$. It is known in the optimal control theory for ODEs that if the Hamiltonian is regular and some constraint qualifications are satisfied, then the adjoint variable is continuous, that is the adjoint variable has no jumps in the interior of time interval (see in [13] or in [8, Chapter 6]). That is what tells us to check the applicability of Theorem 2 for our age-structured model. For this purpose we must find an absolutely continuous with respect to the characteristic direction $\vec{e}$ solution for the adjoint variable $\xi(t, a)$. In other words the adjoint variable would not have jumps along the characteristic direction $\vec{e}$ at points which are internal to the domain Q. This variable could have such jumps only on the endpoints $a = \omega$ and $t = T$ of the characteristic segments.

According to the definitions in the section 2 the adjoint variable $\xi(t, a)$ must be a solution of the adjoint equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\xi(t, a) = -e^{-rt}p(t - a) + \delta\xi(t, a) - \lambda(t, a) \tag{54}$$

for almost each $(t, a) \in Q$. To apply Theorem 2 and the remarks which are made afterwards, we impose the following terminal conditions on the adjoint variable:

$$\xi(t, \omega) = \alpha(t), \qquad t \in [0, T], \tag{55}$$

$$\xi(T, a) = \beta(a), \qquad a \in [0, \omega] \tag{56}$$

for some functions $\alpha \in L^1([0, T])$ and $\beta \in L^1([0, \omega])$, for which the following conditions must hold:

$$\alpha(t) \geq 0, \quad \beta(a) \geq 0, \quad \alpha(t) K(t, \omega) = \beta(a) K(T, a) = 0. \tag{57}$$

The Lagrange multiplier $\lambda(t, a)$ must be integrable on the domain Q. Moreover the following conditions for nonnegativity and for comlementary slackness must hold:

$$\lambda(t, a) \geq 0, \quad \lambda(t, a) K(t, a) = 0 \quad \text{for a.e. } (t, a) \in Q. \tag{58}$$

Summarizing the above considerations we see that the conditions of Theorem 2 would be satisfied if we prove the existence of an absolutely continuous along the characteristic direction solution for the pair (K, ξ) of state and adjoint variables for the boundary value problem of the differential equations (42) and (54) subject to the conditions (43)–(45), (51)–(53), (55)–(58). It turns out that for each fixed vintage $x = t - a$, this problem is a two point boundary value problem for ODEs. In the characteristic coordinates (t, x) this problem is:

$$\dot{\boldsymbol{K}}(t) = \boldsymbol{I}(t) - \delta \boldsymbol{K}(t), \tag{59}$$

$$\dot{\boldsymbol{\xi}}(t) = \delta \boldsymbol{\xi}(t) - \boldsymbol{p} e^{-rt} - \boldsymbol{\lambda}(t), \tag{60}$$

$$\boldsymbol{I}(t) = \frac{1}{c} \left(e^{rt} \boldsymbol{\xi}(t) - \boldsymbol{b_0} e^{-\boldsymbol{\Delta} t} \right), \tag{61}$$

$$\boldsymbol{K}(t) \geq 0, \quad \boldsymbol{K}(\boldsymbol{t_0}) = \boldsymbol{I}(\boldsymbol{t_0}), \tag{62}$$

$$\boldsymbol{\gamma} \stackrel{\text{def}}{=} \boldsymbol{\xi}(\boldsymbol{t_1}) \geq 0, \quad \boldsymbol{\gamma} \boldsymbol{K}(\boldsymbol{t_1}) = 0, \tag{63}$$

$$\boldsymbol{\lambda}(t) \geq 0, \quad \boldsymbol{\lambda}(t) \boldsymbol{K}(t) = 0 \tag{64}$$

for some Lagrange multiplier $\boldsymbol{\lambda}(t) = \lambda(t, t-x) \in L^1(Q)$ and some function $\boldsymbol{\gamma} = \gamma(x) \in L^1([-\omega, T])$. Besides the solution for the adjoint variable $\boldsymbol{\xi}(t) = \xi(t, t-x)$ must be measurable and bounded on Q.

We have denoted the functions which depend on x by bold font in the above system (59)–(64): $\boldsymbol{K}(t) = K(t, t-x)$, $\boldsymbol{I}(t) = I(t, t-x)$, $\boldsymbol{\xi}(t) = \xi(t, t-x)$, $\boldsymbol{\lambda}(t) = \lambda(t, t-x)$, $\boldsymbol{p} = p(x)$, $\boldsymbol{\Delta} = \Delta(x)$ and $\boldsymbol{b}(t) = \boldsymbol{b_0} e^{-\boldsymbol{\Delta} t} = b_0(x) e^{\Delta(x) x} e^{-\Delta(x) t}$. By $\boldsymbol{\gamma}$ we have denoted the function $\boldsymbol{\gamma} = \alpha(\boldsymbol{t_1}) = \alpha(x+\omega)$ when $x \le T - \omega$ and the function $\boldsymbol{\gamma} = \beta(\boldsymbol{t_1} - x) = \beta(T - x)$ when $x \ge T - \omega$ in the formulas (63). By $\boldsymbol{t_0} = \max\{x, 0\}$ and $\boldsymbol{t_1} = \min\{x + \omega, T\}$ we have denoted the endpoints of the characteristic segment of the domain Q which correspond to each fixed $x \in [-\omega, T]$.

Before proving the existence of a solution and suggesting a numerical method for finding it, we explore the properties of the solution to this two point boundary value problem.

Proposition 1. *Let $(\boldsymbol{K}, \boldsymbol{\xi})$ be a solution to the problem* (59)–(64) *which corresponds to a fixed $x \in [-\omega, T]$ and let $\boldsymbol{I}$ be the control variable which corresponds to $\boldsymbol{\xi}$. Let us define the function*

$$\boldsymbol{\Lambda}(t) \stackrel{\text{def}}{=} (r + \delta + \boldsymbol{\Delta}) \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})t} - \boldsymbol{p} e^{-rt}. \tag{65}$$

The unique zero $\boldsymbol{l}$ of the above function is

$$\boldsymbol{l} \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{\Delta}} \Big(\ln \boldsymbol{b_0} + \ln(r + \delta + \boldsymbol{\Delta}) - \ln \boldsymbol{p} \Big). \tag{66}$$

The following assertions hold:

1. *The values of the function $\boldsymbol{\gamma}$ introduced in* (63) *are bounded within the interval $\left[0, \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})\boldsymbol{t_1}}\right]$;*

2. *If $[\tau_1, \tau_2]$ is a boundary interval with positive length, then the Lagrange multiplier is $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t)$ in the interior (τ_1, τ_2) of this interval;*

3. *The control variable $\boldsymbol{I}(t)$ is an absolutely continuous solution to the differential equation*

$$\dot{\boldsymbol{I}}(t) = (r + \delta) \boldsymbol{I}(t) + \frac{e^{rt}}{c} \Big(\boldsymbol{\Lambda}(t) - \boldsymbol{\lambda}(t) \Big) \tag{67}$$

for some terminal condition $\boldsymbol{I}(\boldsymbol{t_1}) \in \left[-\frac{\boldsymbol{b_0}}{c} e^{-\boldsymbol{\Delta} \boldsymbol{t_1}}, 0 \right]$. Besides, if $\boldsymbol{K}(\boldsymbol{t_1}) > 0$, then the terminal condition must be $\boldsymbol{I}(\boldsymbol{t_1}) = -\frac{\boldsymbol{b_0}}{c} e^{-\boldsymbol{\Delta} \boldsymbol{t_1}}$.

Proof.

1. From the complementary slackness condition (63) it follows that if $\boldsymbol{K}(\boldsymbol{t_1}) > 0$ then $\boldsymbol{\gamma}$ must vanish. Let us suppose that $\boldsymbol{K}(\boldsymbol{t_1}) = 0$. Then from the continuity of the control variable $\boldsymbol{I}(t)$ and in order the state constraint (62) to be retained, the inequality $\boldsymbol{I}(\boldsymbol{t_1}) \le 0$ must hold. From (61) we obtain that $\boldsymbol{\gamma} \le \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})\boldsymbol{t_1}}$.

2. Let us suppose that there exists a boundary interval $[\tau_1, \tau_2]$ of positive length. Then $\dot{\boldsymbol{K}}(t) \equiv 0$ in the interior of this interval and therefore $\boldsymbol{I}(t) \equiv 0$, but then it follows from (61) that $\boldsymbol{\xi}(t) = e^{-rt}\boldsymbol{b}(t)$. Hence we obtain that the function $e^{-rt}\boldsymbol{b}(t)$ is a solution to the adjoint equation (60) in the open interval (τ_1, τ_2). From this adjoint equation we also obtain that the Lagrange multiplier $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t)$ within the interval (τ_1, τ_2).

3. From the representation of the control variable $\boldsymbol{I}(t)$ by the equality (61) we can see that $\boldsymbol{I}(t)$ is absolutely continuous and moreover $\boldsymbol{I}(t)$ is differentiable at time t if and only if $\boldsymbol{\xi}(t)$ is differentiable at the time t. Let these functions be differentiable at time t. Differentiating (61) and using (60) we obtain (67) immediately. Since $\boldsymbol{\gamma} = \boldsymbol{\xi}(\boldsymbol{t_1}) \in \left[0, \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})\boldsymbol{t_1}}\right]$ we obtain from (61) that $\boldsymbol{I}(\boldsymbol{t_1}) \in \left[-\left(\boldsymbol{b_0} e^{-\boldsymbol{\Delta t_1}}\right)/c,\ 0\right]$. Furthermore, from the complementary slackness condition (63) it follows that if $\boldsymbol{K}(\boldsymbol{t_1}) > 0$ then $\boldsymbol{\gamma}$ must vanish. But from (61) we see that the last holds true if and only if $\boldsymbol{I}(\boldsymbol{t_1}) = -\left(\boldsymbol{b_0} e^{-\boldsymbol{\Delta t_1}}\right)/c$. □

We see that the problem (59)–(64) is equivalent to the problem (59), (61)–(64), (67). On the other hand it turns out that instead of looking for a solution $(\boldsymbol{K}, \boldsymbol{I})$ to the second boundary value problem it is more convenient to look for the following "accumulated at interest" state variable and "discounted" control variable:

$$k(t) \stackrel{\text{def}}{=} e^{\delta t}\boldsymbol{K}(t), \quad i(t) \stackrel{\text{def}}{=} e^{-(r+\delta)t}\boldsymbol{I}(t). \tag{68}$$

For these variables the boundary value problem (59), (61)–(64), (67) becomes:

$$\dot{k}(t) = e^{(r+2\delta)t} i(t), \qquad k(\boldsymbol{t_0}) = e^{(r+2\delta)\boldsymbol{t_0}}\, i(\boldsymbol{t_0}), \tag{69}$$

$$\dot{i}(t) = \frac{e^{-\delta t}}{c}\Big(\boldsymbol{\Lambda}(t) - \boldsymbol{\lambda}(t)\Big), \qquad \eta \stackrel{\text{def}}{=} i(\boldsymbol{t_1}) \in \left[-\frac{\boldsymbol{b_0}}{c} e^{-(r+\delta+\boldsymbol{\Delta})\boldsymbol{t_1}}, 0\right], \tag{70}$$

$$k(t) \ge 0, \quad \boldsymbol{\lambda}(t) \ge 0, \quad \boldsymbol{\lambda}(t)k(t) = 0. \tag{71}$$

Besides, if $k(\boldsymbol{t_1}) > 0$ then $i(\boldsymbol{t_1}) = \eta_{\min} \stackrel{\text{def}}{=} -\dfrac{\boldsymbol{b_0}}{c} e^{-(r+\delta+\boldsymbol{\Delta})\boldsymbol{t_1}}$ must hold true.

Proposition 2.

1. *If* $\boldsymbol{t_1} \leq \boldsymbol{l}$ *then* $\boldsymbol{K}(t) \equiv \boldsymbol{I}(t) \equiv 0$, $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t)$ *is a solution to the two point boundary value problem* (59), (61)–(64), (67)*;*

2. *If* $\boldsymbol{t_1} > \boldsymbol{l}$ *then the solution to the problem* (59), (61)–(64), (67) *has no boundary intervals of positive lengths in the interval* $[\boldsymbol{l}, \boldsymbol{t_1}]$ *and we can assume that the solution has no more than one boundary interval in* $[\boldsymbol{t_0}, \boldsymbol{l}]$*. Besides, we can assume that the entry time of this boundary interval is* $\boldsymbol{t_0}$*.*

P r o o f. Let us note first that the values of the function $\boldsymbol{\Lambda}(t)$ are positive for $t < \boldsymbol{l}$ and negative for $t > \boldsymbol{l}$, $\boldsymbol{\Lambda}'(\boldsymbol{l}) < 0$, $\lim\limits_{t\to-\infty} \boldsymbol{\Lambda}(t) = +\infty$ and $\lim\limits_{t\to+\infty} \boldsymbol{\Lambda}(t) = 0$.

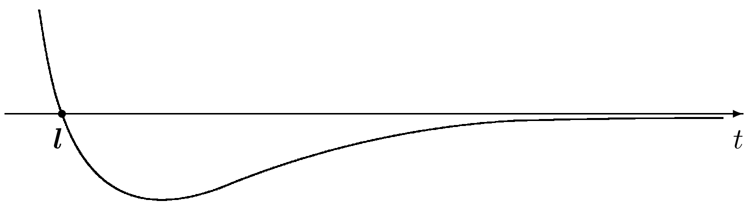

Fig. 1. The graph of the function $\boldsymbol{\Lambda}(t)$

1. According to the assertions proven in the previous proposition if $\boldsymbol{K}(t) \equiv \boldsymbol{I}(t) \equiv 0$ then $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t) \geq 0$. We obtain from (61) that $\boldsymbol{\xi}(t) = \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})t}$. Through direct verification we find out that $\boldsymbol{\xi}(t)$ and $\boldsymbol{\lambda}(t)$ satisfy the equation (60) and that $\boldsymbol{\gamma} = \boldsymbol{\xi}\,(\boldsymbol{t_1}) = \boldsymbol{b_0} e^{-(r+\boldsymbol{\Delta})\boldsymbol{t_1}}$. Thus we see that all conditions of the problem (59), (61)–(64), (67) are satisfied.

2. We have proven in the previous proposition that if $[\tau_1, \tau_2]$ is a boundary interval of positive length then in the interior of the interval $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t)$. Since $\boldsymbol{\Lambda}(t) < 0$ in $(\boldsymbol{l}, \boldsymbol{t_1}]$ and the Lagrange multiplier cannot be negative then the solution of the problem (59), (61)–(64), (67) has no boundary intervals of positive length in $[\boldsymbol{l}, \boldsymbol{t_1}]$.

Let us suppose that $\tau \in (\boldsymbol{t_0}, \boldsymbol{l}]$ belongs to a boundary interval or it is a contact time. Then $\boldsymbol{K}(\tau) = 0$. From the continuity of the control variable it follows that $\boldsymbol{I}(\tau) = 0$. We see that the conditions (59), (61)–(64), (67) would not be violated if we assume that $\boldsymbol{K}(t) \equiv 0$ within the interval $[\boldsymbol{t_0}, \tau]$. $\square$

Having found a solution to the problem (59)–(64) for these characteristics for which $\boldsymbol{t_1} \leq \boldsymbol{l}$ we must find the solution in the remaining cases, that is in the cases when $\boldsymbol{t_1} \geq \boldsymbol{l}$. As we have mentioned, by suggesting a numerical algorithm based on the shooting method we will prove that the boundary problem (59)–(64) has a solution in the remaining cases. For that purpose let us first define the notion of shot.

Definition of shot. *We choose a value for* $\eta \in \left[-\frac{\boldsymbol{b_0}}{c}e^{-(r+\delta+\boldsymbol{\Delta})\boldsymbol{t_1}},\ 0\right]$. *For the chosen value for* η *we find a "discounted" control* $i(t)$ *as a solution to the equation* (70) *with* $\boldsymbol{\lambda}(t) \equiv 0$ *and the terminal condition* $i(\boldsymbol{t_1}) = \eta$.

Let us introduce the curve Γ *as the line* $\Gamma \stackrel{\text{def}}{=} \{(\boldsymbol{t_0}, v) : v \in R\}$ *in the case* $\boldsymbol{l} \leq \boldsymbol{t_0}$ *and as the curve* $\Gamma \stackrel{\text{def}}{=} \{(t,v)\} = \{(\boldsymbol{t_0}, v) : v \geq 0\} \cup \{(t,0) : \boldsymbol{t_0} \leq t \leq \boldsymbol{l}\} \cup \{(\boldsymbol{l}, v) : v \leq 0\}$ *in the case* $\boldsymbol{t_0} < \boldsymbol{l} \leq \boldsymbol{t_1}$. *The graph of the "discounted" control* $i(\cdot)$ (*with* $\boldsymbol{\lambda}(t) \equiv 0$ *and* $i(\boldsymbol{t_1}) = \eta$) *intersects* Γ *at the point* $P(t_p, v_p)$.

After the determination of the point $P(t_p, v_p)$ *we find the "accumulated at interest" state variable* $k(t)$ *for the time interval* $[t_p, \boldsymbol{t_1}]$ *as a solution to the equation* (69) *with the initial condition* $k(t_p) = e^{(r+2\delta)t_p}v_p$.

We will associate the notion of shot with the function $\Sigma(\eta) \stackrel{\text{def}}{=} k(\boldsymbol{t_1})$. *Thus, we will say that for each value* $\eta \in \left[-\frac{\boldsymbol{b_0}}{c}e^{-(r+\delta+\boldsymbol{\Delta})\boldsymbol{t_1}}, 0\right]$ *there is a corresponding value of the shot* $\Sigma(\eta)$.

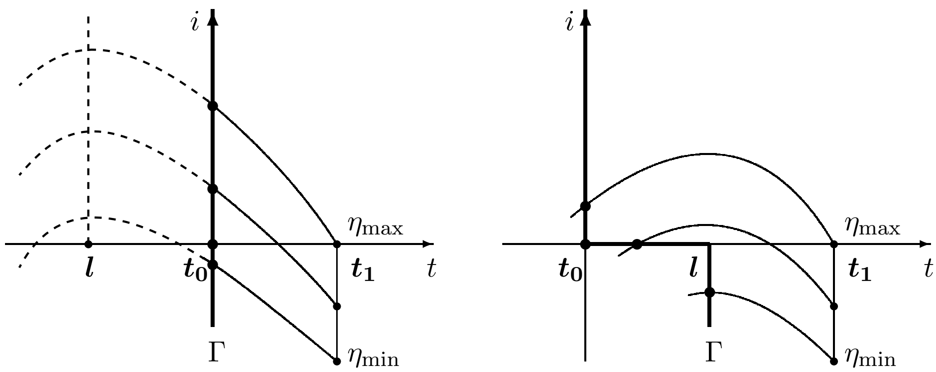

Fig. 2. The graphs of the "discounted" control $i(t)$ for different shots: in the case $\boldsymbol{l} \leq \boldsymbol{t_0}$ (left); in the case $\boldsymbol{t_0} < \boldsymbol{l} \leq \boldsymbol{t_1}$ (right)

Proposition 3. *The function* $\Sigma(\eta)$ *defined above as a shot is continuous and strictly increasing.*

P r o o f. Let us show the correctness of the above definition first, before proving the properties of the function $\Sigma(\eta)$. The function $\mathbf{\Lambda}(t)$ is defined and continuous on R, therefore its values are bounded for each bounded interval $[\boldsymbol{t_0}, \boldsymbol{t_1}]$ of values of its argument. Therefore, the equation (70) with $\boldsymbol{\lambda}(t) \equiv 0$ has an unique solution for each particular terminal value η, and the integral curve of the solution intersects Γ. It is obvious that the point of intersection with Γ is unique in the case $\boldsymbol{l} \leq \boldsymbol{t_0}$. We see from the equation (70) with $\boldsymbol{\lambda}(t) \equiv 0$ and from the properties of the function $\mathbf{\Lambda}(t)$ that $i(t)$ is strictly increasing on $t < \boldsymbol{l}$. Therefore in the case $\boldsymbol{t_0} < \boldsymbol{l} \leq \boldsymbol{t_1}$ the integral curve of the solution $i(t)$ cannot intersect twice the curve Γ.

Let us return to the proof of the properties of the function $\Sigma(\eta)$. To prove that the function is strictly increasing let us consider two shots with the arguments η_1 and η_2, and let $\eta_1 < \eta_2$. These two shots determine the control variables $i_1(t)$ and $i_2(t)$, the points $P_1\left(t_{p1}, v_{p1}\right)$ and $P_2\left(t_{p2}, v_{p2}\right)$, and the state variables $k_1(t)$ and $k_2(t)$ respectively. Since $i(t) = \eta + \Pi(t)$ for some primitive integral $\Pi(t)$ of the right hand side of (70) it is clear that $i_1(t) < i_2(t)$ for each t. Therefore if $t_{p_1} = t_{p_2}$, which holds for example in the case $\boldsymbol{l} \leq \boldsymbol{t_0}$, then the inequality $k_2\left(t_{p_1}\right) > k_1\left(t_{p_1}\right)$ holds true for the initial values of the state variables. Let us consider the other case, that is the case in which $t_{p_1} \neq t_{p_2}$. This case is possible when $\boldsymbol{t_0} < \boldsymbol{l} \leq \boldsymbol{t_1}$. From the inequality $i_1\left(t_{p_1}\right) < i_2\left(t_{p_1}\right)$ and from the fact that $i_2(t)$ is an increasing function on $t \leq t_{p_1}$, it follows that $t_{p_2} < t_{p_1} \leq \boldsymbol{l}$. We see from the equation (69) that the state variable $k_2(t)$ is strictly increasing on the interval $\left(t_{p_2}, t_{p_1}\right)$, therefore in this case the inequality $k_2\left(t_{p_1}\right) > k_1\left(t_{p_1}\right)$ also holds true. Again from (69) we see that the phase speed of $k_2(t)$ is greater than the phase speed of $k_1(t)$. Then $k_2(t) > k_1(t)$ for each $t \geq t_{p_1}$, and therefore $\Sigma(\eta_2) > \Sigma(\eta_1)$.

Let us continue with the proof of the continuity of the function $\Sigma(\eta)$. Here we will denote by $i(t;\eta)$ the solution of the Cauchy problem (70) with $\boldsymbol{\lambda}(t) \equiv 0$ and terminal condition $i\left(\boldsymbol{t_1}\right) = \eta$. By $k(t;\eta)$ we will denote the solution of the Cauchy problem (69) which corresponds to $i(t;\eta)$. By $t_p(\eta)$ and $v_p(\eta)$ we will denote the coordinates of the intersection point of the integral curve of $i(t;\eta)$ with the curve Γ. First we will show that the coordinates of the point $P\left(t_p(\eta), v_p(\eta)\right)$ depend continuously on η. This is obvious for the ordinate $v_p(\eta)$ because the solution of the equation (70) with $\boldsymbol{\lambda}(t) \equiv 0$ depends continuously on the terminal condition. The abscissa $t_p(\eta)$ is determined implicitly by the formula

$$
(72) \qquad t_p(\eta) = \begin{cases} \boldsymbol{t_0}, & \text{if} \quad \boldsymbol{l} \le \boldsymbol{t_0}, \\ \boldsymbol{l}, & \text{if} \quad \boldsymbol{t_0} < \boldsymbol{l} \ \text{ and } \ v_p(\eta) < 0, \\ \max\left\{\boldsymbol{t_0}, \{t : i(t;\eta) = 0, t \le \boldsymbol{l}\}\right\} & \text{otherwise.} \end{cases}
$$

The continuity of $t_p(\eta)$ is obvious in the first two cases of (72). Let us start with the consideration of the third case of (72) when $i\left(t_p\left(\bar{\eta}\right);\bar{\eta}\right) > 0$. In this case $t_p\left(\bar{\eta}\right) = \boldsymbol{t_0}$. If there exists a value $\bar{t}$ for which $i\left(\bar{t};\bar{\eta}\right) = 0$ then $\bar{t} < \boldsymbol{t_0}$. But then $\bar{t} < \boldsymbol{l}$. From the equation (70) with $\boldsymbol{\lambda}(t) \equiv 0$ and from the properties of the function $\boldsymbol{\Lambda}(t)$ it follows that $i\left(t;\eta\right)$ is strictly increasing with respect to t on $t < \boldsymbol{l}$. The conditions of the implicit function theorem are satisfied (see in [11, p. 502, Theorem 1]). According to this theorem there exists a continuous function $t = t(\eta)$ determined in a neighborhood (η_1, η_2) of $\bar{\eta}$ for which $i\left(t(\eta);\eta\right) = 0$ and $\bar{t} = t\left(\bar{\eta}\right)$. We can assume, eventually by decreasing the neighborhood, that $t(\eta) < \boldsymbol{t_0}$ for each $\eta \in (\eta_1, \eta_2)$. Thus we see that $t_p(\eta)$ is continuous at $\eta = \bar{\eta}$ since $t_p(\eta) \equiv \boldsymbol{t_0}$ for each $\eta \in (\eta_1, \eta_2)$.

Let us now consider the third case of (72) when $i\left(t_p\left(\bar{\eta}\right);\bar{\eta}\right) = 0$ and $t_p\left(\bar{\eta}\right) < \boldsymbol{l}$. The implicit function theorem again holds and according to this theorem there exists a continuous function $t = t(\eta)$ which is uniquely determined in a neighborhood of the point $\left(t_p\left(\bar{\eta}\right), \bar{\eta}\right)$ and for which $i\left(t(\eta);\eta\right) = 0$. Since the maximum of two continuous functions is a continuous function then $t_p(\eta)$ is continuous at $\eta = \bar{\eta}$.

It remains to prove the continuity of $t_p(\eta)$ for the values of η for which $t_p(\eta) = \boldsymbol{l} > \boldsymbol{t_0}$ and $v_p(\eta) = i\left(t_p(\eta);\eta\right) = 0$. Let these two conditions be satisfied for $\eta = \bar{\eta}$. Since in this case the inequality $t_p(\eta) \le \boldsymbol{l}$ holds for each η (see the right side of the figure 2), the function $t_p(\eta)$ is upper semicontinuous with respect to η at $\eta = \bar{\eta}$. Suppose that $t_p(\eta)$ is not lower semicontinuous with respect to η at $\eta = \bar{\eta}$. Then there exists $\varepsilon > 0$ and we can find a sequence $\{\eta_n\}_{n=1}^{\infty}$ for which $|\eta_n - \bar{\eta}| < 1/n$ and $t_p\left(\eta_n\right) < t_p\left(\bar{\eta}\right) - \varepsilon$. Since $t_p(\eta) \ge \boldsymbol{t_0}$ for each η and $\boldsymbol{t_0} = \max\{x, 0\} \ge 0$ the members of the sequence $\{t_p\left(\eta_n\right)\}_{n=1}^{\infty}$ are bounded within the interval $[0,\, t_p\left(\bar{\eta}\right) - \varepsilon]$. Therefore we can choose a convergent subsequence. Without changing the notations we have obtained that there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ for which $|\eta_n - \bar{\eta}| < 1/n$ and $\lim_{n\to\infty} t_p\left(\eta_n\right) = s \le t_p\left(\bar{\eta}\right) - \varepsilon$. The inequality $v_p\left(\eta_n\right) = i\left(t_p\left(\eta_n\right);\eta_n\right) \ge 0$ holds true for each member of this sequence. From the continuous dependence of the solution to the equation (70) with $\boldsymbol{\lambda}(t) \equiv 0$ on the terminal condition and from the continuity of $t_p(\eta)$ when $t_p(\eta) < \boldsymbol{l}$ it follows that the inequality $i\left(s;\bar{\eta}\right) \ge 0$ also holds true. But in the case considered now, we have $i\left(t_p\left(\bar{\eta}\right);\bar{\eta}\right) = 0$, therefore $i\left(s;\bar{\eta}\right) \ge i\left(t_p\left(\bar{\eta}\right);\bar{\eta}\right)$. The last is impossible because $i\,(t;\bar{\eta})$ is strictly increasing for $t < \boldsymbol{l} = t_p\,(\bar{\eta})$. The contradiction obtained proves the continuity of the function $t_p(\eta)$.

From the continuous dependence of the solution to a differential equation on parameters and on initial conditions it follows that $k(t;\eta)$ is also continuous with respect to η. Therefore the terminal value $k\,(\boldsymbol{t_1};\eta)$ is also continuous with respect to η. So the continuity of the function $\Sigma(\eta)$ is proven and all assertions in this proposition are proven too. $\square$

Algorithm for solving the boundary value problem. *We find a solution to the boundary value problem* (59)–(64) *for these characteristics for which* $\boldsymbol{t_1} \geq \boldsymbol{l}$, *by finding a solution* $(i(t), k(t))$ *to the problem* (69)–(71). *The steps are as follows:*

1. *We shoot with the minimal value of* η, *that is with* $\eta{=}\eta_{\min}\overset{\text{def}}{=}-\dfrac{\boldsymbol{b_0}}{c}e^{-(r+\delta+\boldsymbol{\Delta})\boldsymbol{t_1}}$. *If* $\Sigma\,(\eta_{\min}) \geq 0$ *then we go to the final step 3. Otherwise, we have to look for a solution of the problem* (69)–(71) *with* $\eta > \eta_{\min}$. *According to the condition in the line after* (71), *the value of the shot* $\Sigma(\eta) = k\,(\boldsymbol{t_1})$ *must vanish. Therefore we go to the step 2.*

2. *We have received* $\Sigma\,(\eta_{\min}) < 0$ *in the previous step. So we shoot with the maximal value of* η, *that is with* $\eta_{\max} \overset{\text{def}}{=} 0$. *In this shot* $i(t) \geq 0$ *for each* $t \in [t_p, \boldsymbol{t_1}]$ *and therefore* $\Sigma\,(\eta_{\max}) \geq 0$. *If* $\Sigma\,(\eta_{\max}) = 0$ *we go to the final step 3.*

   Let $\Sigma\,(\eta_{\max}) > 0$. *According to the Bolzano's intermediate value theorem there exists* $\eta \in (\eta_{\min}, \eta_{\max})$ *for which* $\Sigma(\eta) = 0$. *Since* $\Sigma(\eta)$ *is a strictly increasing and continuous function this value is unique. It can be found by the bisection method: We divide the current interval* $[\eta_1, \eta_2]$ *for which* $\Sigma\,(\eta_1) < 0$ *and* $\Sigma\,(\eta_2) > 0$, *and shoot with the midpoint* $(\eta_1 + \eta_2)\,/2$. *Having the value of* $\Sigma\Big(\,(\eta_1 + \eta_2)\,/2\Big)$ *we determine the next interval and so on until we approximate the value of* η, *for which* $\Sigma(\eta) = 0$.

3. *We extend the control and state variables* ($i(t)$ *and* $k(t)$) *found in the last shot determining them as* $i(t) \equiv k(t) \equiv 0$ *for* $t \in [\boldsymbol{t_0}, t_p]$. *Then we determine the Lagrange multiplier as* $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\Lambda}(t)$ *for* $t \in [\boldsymbol{t_0}, t_p]$ *and as* $\boldsymbol{\lambda}(t) \equiv 0$ *for* $t \in (t_p, \boldsymbol{t_1}]$. *From the definitions* (68) *we find the control variable* $\boldsymbol{I}(t)$ *and the state variable* $\boldsymbol{K}(t)$. *Finally from* (61) *we find the adjoint variable* $\boldsymbol{\xi}(t)$.

From the above algorithm it becomes clear that the boundary value problem (59)–(64) has a solution for these characteristics for which $\boldsymbol{t_1} \geq \boldsymbol{l}$. Remember that according to the Proposition 2, the boundary value problem has also a solution for the remaining characteristics that are the characteristics for which $\boldsymbol{t_1} \leq \boldsymbol{l}$. The solution in these cases is $\boldsymbol{K}(t) \equiv \boldsymbol{I}(t) \equiv 0$. In order to prove the applicability of Theorem 2 to the solution of the problem (59)–(64) it remains to prove that the following conditions are satisfied for the functions found: $\boldsymbol{\gamma} = \gamma(x) \in L^1([-\omega, T])$, $\boldsymbol{\lambda}(t) = \lambda(t, t-x) \in L^1(Q)$ and $\boldsymbol{\xi}(t) = \xi(t, t-x) \in L^\infty(Q)$. We will prove these conditions in the next proposition.

Proposition 4. *The function $\xi(t, t-x)$ is continuous with respect to x, therefore this function is continuous on the compact domain Q and the function $\gamma(x)$ is integrable on the interval $[-\omega, T]$. The function $\lambda(t, t-x)$ is integrable on the domain Q.*

Proof. Let us remind that the functions and the constants, denoted by bold font in the boundary value problem (69)–(71) with $\boldsymbol{\lambda}(t) \equiv 0$, depend continuously on x. For convenience, here we will represent this dependence and the dependence on the terminal condition $\eta = i(\boldsymbol{t_1})$ by the denotations $t_0(x)$ and $t_1(x)$ for the ends of the time interval, $i(t; \eta, x)$ and $k(t; \eta, x)$ for the control and state variables, $t_p(\eta, x)$ and $v_p(\eta, x)$ for the coordinates of the point P determined by the shot, and $\Sigma(\eta; x)$ for the function associated with the shot. Of course, the terminal condition $\eta = i(\boldsymbol{t_1})$ of the equation (70) depends on x, too. We will denote by $l(x)$ the zero $\boldsymbol{l}$ represented by (66) of the function $\boldsymbol{\Lambda}(t)$ defined by (65) (clearly $l(\cdot)$ is continuous in x). In order to prove the assertions, first we will prove that $i(t; \eta, x)$, $t_p(\eta, x)$, $v_p(\eta, x)$, $k(t; \eta, x)$ and $\Sigma(\eta; x)$ are continuous in x and that the terminal condition η in the solution of (69)–(71) depends continuously on x.

Let us first begin with the proof of the continuity of the functions considered with respect to x. For this purpose let us fix the value of η. Here we will repeat the same arguments which we have used in the proof of the continuity in Proposition 3. From the continuous dependence on parameters and on initial conditions it follows that $i(t; \eta, x)$ is continuous with respect to x, therefore the ordinate $v_p(\eta, x)$ of the point P is continuous with respect to x, too. As in the proof of Proposition 3 we see that the abscissa $t_p(\eta, x)$ is determined implicitly by the formula

$$t_p(\eta, x) = \begin{cases} t_0(x), & \text{if} \quad l(x) \leq t_0(x), \\ l(x), & \text{if} \quad t_0(x) < l(x) \ \text{ and } \ v_p(\eta, x) < 0, \\ \max\{t_0(x), \{t : i(t; \eta, x) = 0, t \leq l(x)\}\} & \text{otherwise.} \end{cases} \tag{73}$$

In the first two cases of (73) the continuity of $t_p(\eta, x)$ on x is an obvious consequence from the continuity of the functions $t_0(x)$, $l(x)$ and $v_p(\eta, x)$. Let us begin with the consideration of the third case of (73) when $i(t_p(\eta, \bar{x}); \eta, \bar{x}) > 0$. In this case $t_p(\eta, \bar{x}) = t_0(\bar{x})$. If there exists a value $\bar{t}$ for which $i(\bar{t}; \eta, \bar{x}) = 0$ then $\bar{t} < t_0(\bar{x})$. But then $\bar{t} < l(\bar{x})$, therefore the conditions of the implicit function theorem are satisfied for the function $(t, x) \to i(t; \eta, x)$. According to this theorem there exists a continuous function $t = t(x)$ defined on a neighborhood of $\bar{x}$ and for this function $i(t(x); \eta, x) = 0$. Decreasing the neighborhood we can assume that $t(x) < t_0(\bar{x})$ for each x of the neighborhood which we consider. Therefore in this neighborhood $t_p(\eta, x) = t_0(x)$ and from here in turn it follows that $x \to t_p(\eta, x)$ is continuous at $x = \bar{x}$.

Let us continue with the consideration of the third case of (73) when $i(t_p(\eta, \bar{x}); \eta, \bar{x}) = 0$ and $t_p(\eta, \bar{x}) < l(\bar{x})$. The conditions of the implicit function theorem again hold for the function $(t, x) \to i(t; \eta, x)$. According to this theorem the equality $i(t; \eta, x) = 0$ determines uniquely a continuous function $t = t(x)$ in a neighborhood of $\bar{x}$ and $i(t(x); \eta, x) = 0$. Since the maximum of continuous functions is a continuous function then the function $x \to t_p(\eta, x)$ determined through the third case of (73) is continuous with respect to x at $x = \bar{x}$.

To complete the proof of the continuity of $t_p(\eta, x)$ with respect to x in the case $\boldsymbol{t_1} \geq \boldsymbol{l}$ which we consider so far, it remains to prove the continuity for the values of x for which $t_p(\eta, x) = l(x)$ and $v_p(\eta, x) = i(t_p(\eta, x); \eta, x) = 0$. Let these equations hold for $x = \bar{x}$, that is $t_p(\eta, \bar{x}) = l(\bar{x})$ and $v_p(\eta, \bar{x}) = i(t_p(\eta, \bar{x}); \eta, \bar{x}) = 0$. Since $t_p(\eta, x) \leq l(x)$ for each x and the function $l(x)$ is continuous, then $x \to t_p(\eta, x)$ is upper semicontinuous at $x = \bar{x}$. Suppose that $x \to t_p(\eta, x)$ is not lower semicontinuous at $x = \bar{x}$. Then there exists $\varepsilon > 0$ and we can find a sequence $\{x_n\}_{n=1}^{\infty}$ for which $|x_n - \bar{x}| < 1/n$ and $t_p(\eta, x_n) < t_p(\eta, \bar{x}) - \varepsilon$. Since $t_0(x) = \max\{x, 0\} \geq 0$ the members of the sequence $\{t_p(\eta, x_n)\}_{n=1}^{\infty}$ are bounded within the interval $[0, t_p(\eta, \bar{x}) - \varepsilon]$. Therefore we can choose a convergent subsequence. Without changing the notations we have obtained that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ for which $|x_n - \bar{x}| < 1/n$ and $\lim_{n\to\infty} t_p(\eta, x_n) = s \leq t_p(\eta, \bar{x}) - \varepsilon = l(\bar{x}) - \varepsilon$. From the continuity of the function $l(x)$ it follows that $l(\bar{x}) - \varepsilon < l(x_n) - \varepsilon/2$ for all sufficiently large numbers n, therefore $t_p(\eta, x_n) < l(x_n) - \varepsilon/2$ for these sufficiently large numbers. But then the inequality $v_p(\eta, x_n) = i(t_p(\eta, x_n); \eta, x_n) \geq 0$ holds true for these sufficiently large numbers. From the continuity of the functions $t \to i(t; \eta, \bar{x})$ and $x \to t_p(\eta, x)$ when $t_p(\eta, x) < l(x)$ it follows that the inequality $i(s; \eta, \bar{x}) \geq 0$ also holds true. But in the case considered now, we have $i(t_p(\eta, \bar{x}); \eta, \bar{x}) = 0$, therefore $i(s; \eta, \bar{x}) \geq i(t_p(\eta, \bar{x}); \eta, \bar{x})$. The last is impossible because $i(t; \eta, \bar{x})$ is strictly increasing for $t < t_p(\eta, \bar{x}) = l(\bar{x})$. This contradiction proves the continuity of $x \to t_p(\eta, x)$.

From the continuous dependence on parameters and initial conditions it follows that $k(t;\eta,x)$ is continuous with respect to x and therefore $\Sigma(\eta;x)$ is also continuous with respect to x.

Now let us continue with the proof that the terminal condition η in the solution of (69)–(71) depends continuously on x. First let's consider the case when for a fixed $x=\bar{x}$ the solution is obtained in the first shot of the algorithm and $\Sigma\left(\eta_{\min}\left(\bar{x}\right);\bar{x}\right)>0$. Here by $\eta_{\min}(x)$ we have denoted the continuous function of the variable x which is $\eta_{\min}(x)\stackrel{\text{def}}{=}-\frac{1}{c}b_0(x)e^{-(r+\delta+\Delta(x))t_1(x)}$. From the continuity of the function $x\to\Sigma\left(\eta_{\min}(x);x\right)$ it follows that $\bar{x}$ has a neighborhood in which $\Sigma\left(\eta_{\min}(x);x\right)>0$. It is clear that in this neighborhood the choice of η depends continuously on x, besides $\eta(x)=\eta_{\min}(x)$. Let us continue with the cases in which for the fixed value $\bar{x}$ either $\Sigma\left(\eta_{\min}\left(\bar{x}\right);\bar{x}\right)=0$ or the parameter η is determined in the second step of the algorithm. In these cases the value of $\eta\left(\bar{x}\right)$ which corresponds to $\bar{x}$ is determined implicitly by the formula

$$\eta\left(\bar{x}\right)=\max\left\{\eta_{\min}\left(\bar{x}\right),\left\{\eta:\Sigma\left(\eta;\bar{x}\right)=0,\eta\leq 0\right\}\right\} \tag{74}$$

We saw that in these cases the equation $\Sigma\left(\eta,\bar{x}\right)=0$ has a unique solution $\bar{\eta}$. In Proposition 3 and in the present proposition we have proven that the function $\Sigma(\eta;x)$ is continuous with respect to η and x, and it is strictly increasing with respect to η. Then according to the implicit function theorem there exists a continuous function $\eta=\eta(x)$ which is uniquely determined in a neighborhood of the point $(\bar{\eta},\bar{x})$, and for which both $\bar{\eta}=\eta\left(\bar{x}\right)$ and $\Sigma\left(\eta(x);x\right)=0$ hold. We saw in the algorithm that $\eta\leq 0$. Since the maximum of continuous functions is a continuous function then the function $\eta(x)$ determined implicitly by (74) is continuous.

We have proven so far that the functions $x\to i\left(t;\eta(x),x\right)$, $t_p\left(\eta(x),x\right)$, $v_p\left(\eta(x),x\right)$, $x\to k\left(t;\eta(x),x\right)$ and $\Sigma\left(\eta(x);x\right)$ are continuous with respect to x on the domain $\{x:l(x)\leq t_1(x)\}$ for each $t\in[t_0(x),t_1(x)]$. On the other hand, according to Proposition 2 the state and the control variables are continuous with respect to x on the domain $\{x:l(x)\geq t_1(x)\}$ for each $t\in[t_0(x),t_1(x)]$ because in this domain $i(t)\equiv k(t)\equiv 0$. Therefore, in order to prove the continuity of the state and the control variables as well as the correctness of their determination, it remains to prove that the values of these variables determined by the algorithm vanish at the values of x for which $l(x)=t_1(x)$. But the last fact is obvious. Let the equality $l(x)=t_1(x)$ hold for $x=\bar{x}$. Then $\Sigma(\eta)\leq 0$ for each $\eta\in[\eta_{\min},\eta_{\max}]$ besides $\Sigma(\eta)=0$ if and only if $\eta=\eta_{\max}$, that is if $\eta=0$. Therefore we will receive $\eta=0$ in the last step of the algorithm. Hence $t_p\left(\eta\left(\bar{x}\right),\bar{x}\right)=l\left(\bar{x}\right)$, $v_p\left(\eta\left(\bar{x}\right),\bar{x}\right)=0$, $i\left(l\left(\bar{x}\right);\eta\left(\bar{x}\right),\bar{x}\right)=0$ and $k\left(l\left(\bar{x}\right);\eta\left(\bar{x}\right),\bar{x}\right)=0$. Thus the continuity with respect to x of the state and the control variables is proven. It follows from (61) that the adjoint variable $\xi(t, t-x)$ is also continuous with respect to x and therefore it is continuous on the compact domain Q. Since the function $\gamma(x) = \boldsymbol{\xi}\,(t_1(x))$, this function is continuous on the bounded interval $[-\omega, T]$.

It remains to prove that $\lambda(t, t-x)$ is integrable. Remember that we have determined the Lagrange multiplier as $\lambda(t, t-x) \equiv \Lambda(t; x)$ on the set $Q \cap \{(t, x) \colon t \le t_p\,(\eta(x), x)\}$ and as $\lambda(t, t-x) \equiv 0$ on the set $Q \cap \{(t, x) \colon\ t > t_p\,(\eta(x), x)\}$. Here $\Lambda(t; x)$ is the continuous function which is defined by the equality (65). We saw that the function $x \to t_p\,(\eta(x), x)$ is continuous, therefore the above two sets are measurable as curvilinear trapezoids. Since the multiplier $\lambda(t, t-x)$ is continuous upon each of the two sets considered, then $\lambda(t, t-x)$ is measurable upon the set Q. Thus, the proposition is completely proven. $\square$

From the proposition proven above it follows that the solution of the boundary value problem (59)–(64), which we find by the algorithm and by Proposition 2, satisfies all conditions of the Theorem 2. Therefore, this solution is optimal for the model (41)–(45).

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Department of Mathematics and Statistics
Tsenov Academy of Economics
2, Em. Chakarov Str.
5250 Svishtov, Bulgaria
e-mail: v_krastev@mail.ru
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