Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **40** (2014), 129–160

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

NEW OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Communicated by I. D. Iliev

Abstract. This paper will study the oscillatory behavior of third order nonlinear difference equation with distributed deviating arguments of the form

$$\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p(n)x\tau\left(\tau(n)\right)\right)\right)\right)+\sum_{\xi=m_0}^{m}q(n,\xi)f\left(x\left(g(n,\xi)\right)\right)=0,$$

where m_0, m $(> m_0)$ be integers. We establish some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results improve and extend some known results in the literature. Examples are given to illustrate the importance of the results.

2010 *Mathematics Subject Classification:* 39A10, 39A12.
Key words: oscillatory solutions, third order, neutral, deviating arguments, difference equation.

1. Introduction. By a Riccati transformation technique, we present some new oscillation criteria for the nonlinear difference equation with distributed deviating arguments of the form

$$\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p(n)x(\tau(n))\right)\right)\right) + \sum_{\xi=m_0}^{m} q(n,\xi)f(x(g(n,\xi))) = 0, \quad n \geq n_0, \tag{1.1}$$

where $n_0 \in N$ is a fixed integer,Δ denotes the forward difference operator defined by $\Delta x(n) = x(n+1)-x(n)$ and $\Delta^i x(n) = \Delta\left(\Delta^{i-1}x(n)\right)$. Throughout this paper, we will assume the following hypotheses:

- (A_1) $a(n)$, $b(n) > 0$ for $n \in N(n_0)$, where $N(n_0) = \{n_0, n_0+1, \ldots\}$.
- (A_2) $\{p_n\}_{n=n_0}^{\infty}$ is positive, $0 \leq p_n \leq p < 1$ and $\tau : N \to N$ satisfies $n \geq \tau(n)$ and $\lim\limits_{n\to\infty} \tau(n) = \infty$.
- (A_3) $q(n,\xi) > 0$ on $N(n_0) \times N(m_0,m)$ and $g : N(n_0) \times N(m_0,m) \to N$ satisfies $n \geq g(n,\xi)$ for $\xi \in N(m_0,m)$ and $\lim\limits_{n\to\infty} \min g(n,\xi) = \infty$, where $N(m_0,m) = \{m_0, m_0+1, \ldots, m\}$ and $m > m_0$.
- (A_4) $f \in C(R,R)$ such that $xf(x) > 0$ for all $x \neq 0$ and f is nondecreasing.

In addition, we will make use of the following conditions:

- (S_1) $f(u)/u \geq K > 0$, K is a real constant, $u > 0$.
- (S_2) there exists a real valued function B such that $f(u(n)) - f(v(n)) = B(u(n), v(n))\ (u(n)+p(n)u(\tau(n))) - (v(n)+p(n)v(\tau(n)))$ for all $u(n)$, $v(n) \neq 0$, $p_n \geq 0$, $n > \tau(n) > 0$ and $B(u(n), v(n)) \geq \mu > 0 \in R$.

By a solution of equation (1.1) we mean a nontrivial sequence $x(n)$ defined on $N(n_0)$, which satisfies equation (1.1) for all $n \geq n_0$. A solution $x(n)$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory. In recent years, there has been an increasing interest in the study of the problem of determining the oscillation and non-oscillation of solutions of difference equations of the form (1.1) and its special cases. For further results concerning the oscillatory and asymptotic behavior of third order difference equation we refer to the books [1, 4, 8–10] and the papers [2, 3, 5–7, 11–19]. The main aim of this paper is to establish some sufficient conditions which guarantee that the equation (1.1) has oscillatory solutions or the solutions tend to zero as $n \to \infty$. In this paper, the details of the proofs of results for nonoscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions. Our results improve and expand some known results, for example, the results obtained by Graef et al. [6], Schmeidel [19], Grace et al. [5], Selvaraj et al. [16, 18], Saker et al. [14] and Thandapani et al. [13] and the references cited therein. See Section 4 below for details. The paper is organized as follows: In Section 2, we state and prove some useful lemmas that will be used in the proofs of the main results. In Section 3, we consider the oscillation of equation (1.1) subject to the conditions (S_1) or (S_2) and (3.1) or (3.2) or (3.3) hold. In this section, we consider the delay cases when $n \geq g(n,\xi) \geq \tau(n) \geq G(n)$ and when $n \geq g(n,\xi) \geq G(n) \geq \tau(n)$. In Section 5, we provide some examples to illustrate the main results.

2. Some preliminary lemmas. In this section, we state and prove some useful lemmas, which will be used in the proofs of the main results. We set $z(n) = x(n) + p(n)x(\tau(n))$.

Lemma 2.1. *Let* $x(n)$ *be an eventually positive solution of* (1.1) *and suppose that* $z(n)$ *satisfies*

$$\Delta z(n) > 0,\ \Delta(b(n)\Delta z(n)) > 0,\ \Delta(a(n)\Delta(b(n)\Delta z(n))) \leq 0, \quad \textit{for all} \ \ n \geq n_1.$$

Then there exists $n_2 \geq n_1$ *such that*

$$\Delta z(n) \geq b^{-1}(n)\left(a(n)\Delta(b(n)\Delta z(n))\right)\sum_{s=n_2}^{n-1} a^{-1}(s), \quad \textit{for} \quad n \geq n_2. \tag{2.1}$$

Proof. Since $\Delta(a(n)\Delta(b(n)\Delta z(n))) \leq 0$, we have $a(n)\Delta(b(n)\Delta z(n))$ is non-increasing. Then we obtain,

$$\begin{aligned} b(n)\Delta z(n) &= b(n_2)\Delta z(n_2) + \sum_{s=n_2}^{n-1} a^{-1}(s)a(s)\Delta\left(b(s)\Delta z(s)\right) \\ &\geq a(n)\Delta(b(n)\Delta z(n))\sum_{s=n_2}^{n-1} a^{-1}(s). \end{aligned}$$

The proof is complete. □

Lemma 2.2. *Assume that*

$$\sum_{n=n_0}^{\infty} a^{-1}(n) = \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty.$$

Let $x(n)$ *be an eventually positive solution of equation* (1.1). *Then for sufficiently large* n, *there are only two possible cases:*

(I): $\Delta z(n) > 0$, $\Delta\left(b(n)\Delta(z(n))\right) > 0$, *or*

(II): $\Delta z(n) < 0$, $\Delta\left(b(n)\Delta(z(n))\right) > 0$.

Proof. The proof can be found in [3, Lemma 2.2]. □

Lemma 2.3. *Assume that* (S_1) *holds. Let* $x(n)$ *be an eventually positive solution of equation* (1.1) *and suppose that* (II) *of Lemma* 2.2 *holds. If*

$$\sum_{v=n_0}^{\infty}\left(b^{-1}(v)\left(\sum_{u=n_0}^{v-1}a^{-1}(u)\left(\sum_{s=n_0}^{u-1}\sum_{\xi=m_0}^{m}q(s,\xi)\right)\right)\right)=\infty, \tag{2.2}$$

then $x(n)\to 0$ *as* $n\to\infty$.

Proof. Pick $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$, for $n \geq n_1$. Since $\{x(n)\}$ is a positive decreasing solution of equation (1.1). Then $\lim_{n\to\infty} x(n) = b \geq 0$. Now we claim that $b = 0$. If $b > 0$ then $x(g(n,\xi)) \geq b$ for $n \geq n_2 \geq n_1$. Therefore from (S_1) and (1.1), we have

$$\Delta\left(a(n)\Delta(b(n)\Delta z(n))\right) + Kb\sum_{\xi=m_0}^{m} q(n,\xi) \leq 0,\ n \geq n_2.$$

Define the sequence $u(n) = a(n)\Delta\left(b(n)\left(\Delta z(n)\right)\right)$ for $n \geq n_2$. Then $\Delta u(n) \leq -A\sum_{\xi=m_0}^{m} q(n,\xi)$, where $A = Kb > 0$. Summing the above inequality from n_2 to $n-1$, we obtain

$$u(n) \leq u(n_2) - A\sum_{s=n_2}^{n-1}\sum_{\xi=m_0}^{m} q(s,\xi).$$

From equation (2.2), it is possible to choose an integer n_3 sufficiently large such that

$$u(n) \leq -\frac{A}{2}\sum_{s=n_2}^{n-1}\sum_{\xi=m_0}^{m} q(s,\xi),$$

for all $n \geq n_3$. Hence

$$\Delta(b(n)\Delta z(n)) \leq -\frac{A}{2a(n)}\sum_{s=n_2}^{n-1}\sum_{\xi=m_0}^{m} q(s,\xi).$$

Summing the above inequality from n_3 to $n-1$, we find

$$b(n)\Delta z(n) \leq b(n_3)\, z(n_3) - \frac{A}{2}\left(\sum_{u=n_3}^{n-1} a^{-1}(u)\left(\sum_{s=n_2}^{u-1}\sum_{\xi=m_0}^{m} q(s,\xi)\right)\right).$$

Since $\Delta z(n) < 0$ for $n \geq n_0$, the last inequality implies that

$$\Delta z(n) \leq -\frac{A}{2b(n)}\left(\sum_{u=n_3}^{n-1} a^{-1}(u)\left(\sum_{s=n_2}^{u-1}\sum_{\xi=m_0}^{m} q(s,\xi)\right)\right).$$

Summing from n_4 to $n-1$, we find

$$z(n) \leq z(n_4) - \frac{A}{2}\sum_{l=n_4}^{n-1} b^{-1}(l)\left(\sum_{u=n_3}^{l-1} a^{-1}(u)\left(\sum_{s=n_2}^{u-1}\sum_{\xi=m_0}^{m} q(s,\xi)\right)\right).$$

Condition (2.2) implies that $z(n) \to -\infty$ as $n \to \infty$ which is contradiction with the fact that $z(n) > 0$. Then $b = 0$, i.e. $\lim_{n\to\infty} z(n) = 0$. Since $0 < x(n) \leq z(n)$ then $\lim_{n\to\infty} x(n) = 0$. The proof is complete. $\square$

3. Main results. In this section, we establish some new oscillation criteria for the equation (1.1) under the following conditions:

$$\sum_{n=n_0}^{\infty} a^{-1}(n) = \infty, \quad \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty. \tag{3.1}$$

$$\sum_{n=n_0}^{\infty} a^{-1}(n) < \infty, \quad \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty. \tag{3.2}$$

$$\sum_{n=n_0}^{\infty} a^{-1}(n) < \infty, \quad \sum_{n=n_0}^{\infty} b^{-1}(n) < \infty. \tag{3.3}$$

In the following results, we shall use the following notations:

$$Q(n,\xi) := \min\left\{q(n,\xi), q\left((n,\xi)-\tau\right)\right\},$$

$$\varphi(n) := \frac{\rho(n)}{\rho^2(n+1)b(G(n))}\sum_{s=n_2}^{G(n)-1} a^{-1}(s), \quad \delta(n) := \sum_{v=n}^{\infty}\frac{1}{a(v)},$$

$$\Theta(n) := \frac{\rho(n)}{\rho^2(n+1)b(\tau(n))}\sum_{s=n_2}^{\tau(n)-1} a^{-1}(s), \quad \vartheta(m,n) := \left(\frac{\Delta\rho(n)}{\rho(n+1)} - \frac{h(m,n)}{\sqrt{H(m,n)}}\right).$$

We assume that there exists a double sequence $\{H(m,n) \mid m \geq n \geq 0\}$ and $h(m,n)$ such that

(*i*) $H(m,m) = 0$ for $m \geq 0$,

(*ii*) $H(m,n) > 0$ for $m > n > 0$,

(*iii*) $\Delta_2 H(m,n) = H(m,n+1) - H(m,n) \leq 0$ for $m > n \geq 0$,

(*iv*) $h(m,n) = -\frac{\Delta_2 H(m,n)}{\sqrt{H(m,n)}}$.

Next, we state and prove the main theorems.

First, we establish an oscillation criterion for (1.1) when $n \geq g(n,\xi) \geq \tau(n) \geq G(n)$ and (S_1) holds.

Theorem 3.1. *Assume that* (2.2) *and* (3.1) *hold. Further, assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi) - \frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\varphi(s)}\right) = \infty. \tag{3.4}$$

Then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From equation (1.1), we see that $z(n) > x(n) > 0$, and

$$\Delta(a(n)\Delta(b(n)\Delta z(n))) = -\sum_{\xi=m_0}^{m} q(n,\xi)f(x(g(n,\xi))) \leq 0. \tag{3.5}$$

Then, $a(n)\Delta(b(n)\Delta z(n))$ is non-increasing sequence and thus $\Delta z(n)$ and $\Delta(b(n)\Delta z(n))$ are eventually of one sign. By Lemma 2.2, there exist two possible cases (I) and (II). Assume that (I) holds. From equation (1.1), (S_1) and the definition of $z(n)$, we have

$$\begin{aligned}(3.6)\quad &[\Delta(a(n)\Delta(b(n)\Delta z(n))) + p\left(\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)\right)]\\ &\qquad + K\sum_{\xi=m_0}^{m} Q(n,\xi)z(g(n,\xi)) \le 0.\end{aligned}$$

Further, it is clear from (A_3)

$$(3.7)\qquad g(n,\xi) \ge \min\left\{g\left(n,m_0\right), g(n,m)\right\} \equiv G(n), \quad \xi \in N(m_0,m).$$

Thus

$$\begin{aligned}(3.8)\quad &[\Delta(a(n)\Delta(b(n)\Delta z(n))) + p\left(\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)\right)]\\ &\qquad + Kz(G(n))\sum_{\xi=m_0}^{m} Q(n,\xi) \le 0.\end{aligned}$$

Define a Riccati substitution

$$(3.9)\qquad \omega(n) := \rho(n)\frac{a(n)\Delta(b(n)\Delta z(n))}{z(G(n))}.$$

Then $\omega(n) > 0$. From (3.9), we have

$$\begin{aligned}(3.10)\quad \Delta\omega(n) = {}&\Delta\rho(n)\frac{a(n+1)\left(\Delta\left(b(n+1)\Delta z(n+1)\right)\right)}{z(G(n+1))}\\ &+ \rho(n)\frac{\Delta\left(a(n)\left(\Delta(b(n)\Delta z(n))\right)\right)}{z(G(n))}\\ &- \rho(n)\frac{a(n+1)\left(\Delta\left(b(n+1)\Delta z(n+1)\right)\right)\Delta(z(G(n)))}{z(G(n+1))z(G(n))}.\end{aligned}$$

From Lemma 2.1, $\Delta\left(a(n)\left(\Delta(b(n)\Delta z(n))\right)\right) \le 0$ and $G(n) < n$, we get

$$\begin{aligned}(3.11)\quad &\Delta z(G(n))\\ &\ge b^{-1}(G(n))\left(a(n+1)\left(\Delta\left(b(n+1)\Delta z(n+1)\right)\right)\right)\sum_{s=n_2}^{G(n)-1} a^{-1}(s).\end{aligned}$$

From (3.9) and (3.11), we obtain

$$\Delta\omega(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) + \rho(n)\frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{z(G(n))} - \varphi(n)\omega^2(n+1). \tag{3.12}$$

Similarly, define another sequence $v(n)$ by

$$v(n) := \rho(n)\frac{a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)}{z(G(n))}. \tag{3.13}$$

Then $v(n) > 0$. From (3.13), we have

$$\begin{aligned}\Delta v(n) = {}& \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) + \rho(n)\frac{\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)}{z(G(n))} \\ & - \rho(n)\frac{a(\tau(n+1))\left(\Delta\left(b(\tau(n+1))\Delta z(\tau(n+1))\right)\right)\Delta z(G(n))}{z(G(n+1))z(G(n))}.\end{aligned} \tag{3.14}$$

From Lemma 2.1, and $G(n) < \tau(n)$, we get

$$\begin{aligned}&\Delta z(G(n)) \\ &\quad \geq \left(a(\tau(n+1))\left(\Delta\left(b(\tau(n+1))\Delta z(\tau(n+1))\right)\right)\right)b^{-1}(G(n))\sum_{s=n_2}^{G(n)-1} a^{-1}(s).\end{aligned}$$

Then from (3.13) and (3.14) and the above inequality, we have

$$\Delta v(n) \leq \rho(n)\frac{\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)}{z(G(n))} + \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) - \varphi(n)v^2(n+1) \tag{3.15}$$

From (3.12) and (3.15), we obtain

$$\begin{aligned}&\Delta\omega(n) + p\Delta v(n) \\ &\quad \leq \rho(n)\frac{\Delta\left[(a(n)\Delta(b(n)\Delta z(n))) + p\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)\right]}{z(G(n))} \\ &\qquad + \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) - \varphi(n)\omega^2(n+1) \\ &\qquad + p\left[\frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) - \varphi(n)v^2(n+1)\right].\end{aligned}$$

From (3.8), we have

$$\begin{aligned}(3.16)\quad &\Delta\omega(n)+p\Delta v(n)\\ &\leq -\rho(n)K\sum_{\xi=m_0}^{m}Q(n,\xi)+\frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)-\varphi(n)\omega^2(n+1)\\ &\qquad +p\left[\frac{\Delta\rho(n)}{\rho(n+1)}v(n+1)-\varphi(n)v^2(n+1)\right].\end{aligned}$$

Using (3.16) and the inequality

$$Bu-Au^2\leq\frac{B^2}{4A},A>0, \tag{3.17}$$

we have

$$\begin{aligned}&\Delta\omega(n)+p\Delta v(n)\\ &\leq -\rho(n)K\sum_{\xi=m_0}^{m}Q(n,\xi)+\frac{1}{4}\frac{(\Delta\rho(n))^2}{(\rho(n+1))^2\varphi(n)}+\frac{p}{4}\frac{(\Delta\rho(n))^2}{(\rho(n+1))^2\varphi(n)}.\end{aligned}$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi)-\frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\varphi(s)}\right)\leq\omega(n_2)+pv(n_2),$$

which yields

$$\sum_{s=n_2}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi)-\frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\varphi(s)}\right)\leq c_1,$$

where $c_1>0$ is a finite constant. But, this contradicts (3.4). Next we assume that (II) holds. We are then back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty}x(n)=0$. The proof is complete. □

Theorem 3.2. *Assume that* (2.2), (3.2) *and* (3.4) *hold. Further, assume that there exists a positive nondecreasing sequence* $\rho(n)$. *If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left(K\sum_{\xi=m_0}^{m}Q(s,\xi)\frac{\sum\limits_{u=n_2}^{G(s)-1}\frac{\sum\limits_{v=n_1}^{u-1}\frac{1}{a(v)}}{b(u)}}{\sum\limits_{v=n_1}^{s}\frac{1}{a(v)}}\delta(s+1)-\frac{1+p}{4}\frac{1}{a(s)\delta(s+1)}\right)=\infty, \tag{3.18}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\geq n_0$. Based on condition (3.2), there exist three possible cases (I), (II) (as those of Theorem 3.1), and

(III): $\Delta z(n)>0$, $\Delta(b(n)\Delta z(n))<0$ for all large n.

Assume that (I) holds. Then we are back to the proof of Theorem 3.1 to get contradiction by (3.4). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty}x(n)=0$. Assume that (III) holds. Define the sequence $\omega(n)$ by

$$\omega(n):=\frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)}. \tag{3.19}$$

Then $\omega(n)<0$ for $n\geq n_1$. Noting that $a(n)\Delta(b(n)\Delta z(n))$ is non-increasing sequence. Thus, we get

$$a(s)\Delta\left(b(s)\Delta z(s)\right)\leq a(n)\Delta(b(n)\Delta z(n)),\quad s\geq n\geq n_1. \tag{3.20}$$

Dividing the last inequality by $a(s)$ and summing it from n to $l-1$, we find

$$b\left(l\right)\Delta z\left(l\right)\leq b(n)\Delta z(n)+a(n)\Delta(b(n)\Delta z(n))\sum_{u=n}^{l-1}a^{-1}(u).$$

Letting $l\to\infty$, we have

$$0\leq b(n)\Delta z(n)+a(n)\Delta(b(n)\Delta z(n))\delta(n). \tag{3.21}$$

Which yields

$$-\frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)}\delta(n) \leq 1. \tag{3.22}$$

Therefore, form (3.19), we have

$$-1 \leq \omega(n)\delta(n) \leq 0, \quad n \geq n_2. \tag{3.23}$$

Similarly, we define the sequence $v(n)$ by

$$v(n) := \frac{a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)}{b(n)\Delta z(n)}, \quad n \geq n_2. \tag{3.24}$$

Clearly, $v(n) < 0$ for $n \geq n_2$. Noting that $a(n)\Delta(b(n)\Delta z(n))$ is non-increasing sequence and $\tau(n) \leq n$, we get

$$a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right) \geq a(n)\Delta(b(n)\Delta z(n)).$$

Then $v(n) \geq \omega(n)$. Thus, by (3.23), we have

$$-1 \leq v(n)\delta(n) \leq 0, \quad n \geq n_2. \tag{3.25}$$

From (3.19), we obtain

$$\begin{aligned}\Delta\omega(n) = {} & \frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n+1)\Delta z(n+1)} \\ & - \frac{a(n)\Delta(b(n)\Delta z(n))a(n)\Delta(b(n)\Delta z(n))}{a(n)b(n)\Delta z(n)\left(b(n+1)\Delta z(n+1)\right)}.\end{aligned} \tag{3.26}$$

Since $\Delta(b(n)\Delta z(n)) \leq 0$, we get

$$b(n+1)\Delta z(n+1) \leq b(n)\Delta z(n).$$

From (3.26) and the above inequality, we obtain

$$\Delta\omega(n) \leq \frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n+1)\Delta z(n+1)} - \frac{\omega^2(n)}{a(n)}. \tag{3.27}$$

From (3.24), we obtain

$$\Delta v(n) \leq \frac{\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)}{b(n+1)\Delta z(n+1)} - \frac{v^2(n)}{a(n)}. \tag{3.28}$$

Combining (3.27) and (3.28), we have

$$\Delta\omega(n)+p\Delta v(n)\le\frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n+1)\Delta z(n+1)}-\frac{\omega^2(n)}{a(n)}+p\frac{\Delta\left(a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)\right)}{b(n+1)\Delta z(n+1)}-p\frac{v^2(n)}{a(n)}. \tag{3.29}$$

From (3.8) and (3.29), we obtain

$$\Delta\omega(n)+p\Delta v(n)\le-K\frac{z(G(n))}{b(G(n))\Delta z(G(n))}\frac{b(G(n))\Delta z(G(n))}{b(n+1)\Delta z(n+1)}\sum_{\xi=m_0}^{m}Q(n,\xi)-\frac{\omega^2(n)}{a(n)}-p\frac{v^2(n)}{a(n)}. \tag{3.30}$$

Since

$$b(n)\Delta z(n)\ge b(n)\Delta z(n)-b\left(n_1\right)\Delta z\left(n_1\right)=\sum_{s=n_1}^{n-1}\frac{a(s)\Delta\left(b(s)\Delta z(s)\right)}{a(s)}\ge a(n)\Delta(b(n)\Delta z(n))\sum_{s=n_1}^{n-1}\frac{1}{a(s)} \tag{3.31}$$

we have that

$$\Delta\left(\frac{b(n)z(n)}{\sum\limits_{s=n_1}^{n-1}\frac{1}{a(s)}}\right)\le 0.$$

Thus

$$z(n)\;=\;z(n_2)+\sum_{s=n_2}^{n-1}\frac{b(s)\Delta z(s)}{\sum\limits_{u=n_1}^{s-1}\frac{1}{a(u)}}\frac{\sum\limits_{u=n_1}^{s-1}\frac{1}{a(u)}}{b(s)}\;\ge\;\frac{b(n)\Delta z(n)}{\sum\limits_{u=n_1}^{n-1}\frac{1}{a(u)}}\sum_{s=n_2}^{n-1}\frac{\sum\limits_{u=n_1}^{s-1}\frac{1}{a(u)}}{b(s)}. \tag{3.32}$$

From (3.30), (3.31) and (3.32), we obtain

$$\Delta\omega(n)+p\Delta v(n)\le-\frac{\omega^2(n)}{a(n)}-p\frac{v^2(n)}{a(n)}-K\sum_{\xi=m_0}^{m}Q(n,\xi)\frac{\sum\limits_{s=n_2}^{G(n)-1}\frac{\sum\limits_{u=n_1}^{s-1}\frac{1}{a(u)}}{b(s)}}{\sum\limits_{u=n_1}^{n}\frac{1}{a(u)}}. \tag{3.33}$$

Multiplying (3.33) by $\delta(n+1)$ and summing it from n_2 to $n-1$, we find

$$\begin{aligned}(3.34)\quad & \omega(n)\delta(n) - \omega(n_2)\delta(n_2) + \sum_{s=n_2}^{n-1}\frac{\omega(s)}{a(s)}\\ & + \sum_{s=n_2}^{n-1}\delta(s+1)\frac{\omega^2(s)}{a(s)} + pv(n)\delta(n) - pv(n_2)\delta(n_2)\\ & + p\sum_{s=n_2}^{n-1}\frac{v(s)}{a(s)} + p\sum_{s=n_2}^{n-1}\delta(s+1)\frac{v^2(s)}{a(s)}\\ & + \sum_{s=n_2}^{n-1} K \sum_{\xi=m_0}^{m} Q(s,\xi)\frac{\sum\limits_{u=n_2}^{G(s)-1}\frac{\sum_{v=n_1}^{u-1}\frac{1}{a(v)}}{b(u)}}{\sum\limits_{v=n_1}^{s}\frac{1}{a(v)}}\delta(s+1) \le 0.\end{aligned}$$

It follows from (3.17) and (3.34), that

$$\begin{aligned}(3.35)\quad & \omega(n)\delta(n) - \omega(n_2)\delta(n_2) + pv(n)\delta(n)\\ & - pv(n_2)\delta(n_2) - \frac{1+p}{4}\sum_{s=n_2}^{n-1}\frac{1}{a(s)\delta(s+1)}\\ & + \sum_{s=n_2}^{n-1} K \sum_{\xi=m_0}^{m} Q(s,\xi)\frac{\sum\limits_{u=n_2}^{G(s)-1}\frac{\sum_{v=n_1}^{u-1}\frac{1}{a(v)}}{b(u)}}{\sum\limits_{v=n_1}^{s}\frac{1}{a(v)}}\delta(s+1) \le 0.\end{aligned}$$

Therefore,

$$\begin{aligned}(3.36)\quad & \omega(n)\delta(n) + pv(n)\delta(n)\\ & + \sum_{s=n_2}^{n-1}\left(K \sum_{\xi=m_0}^{m} Q(s,\xi)\frac{\sum\limits_{u=n_2}^{G(s)-1}\frac{\sum_{v=n_1}^{u-1}\frac{1}{a(v)}}{b(u)}}{\sum\limits_{v=n_1}^{s}\frac{1}{a(v)}}\delta(s+1) - \frac{1+p}{4}\frac{1}{a(s)\delta(s+1)}\right)\\ & \le \omega(n_2)\delta(n_2) + pv(n_2)\delta(n_2).\end{aligned}$$

From (3.18) and the above inequality, we obtain a contradiction to (3.23) and (3.25). This completes the proof of Theorem 3.2. □

Theorem 3.3. *Assume that* (2.2), (3.3), (3.4) *and* (3.18) *hold. If*

$$\sum_{u=n_0}^{\infty}\left(b^{-1}(u)\left(\sum_{s=n_1}^{u-1}a^{-1}(s)\right)\right)=\infty, \tag{3.37}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\geq n_0$. Based on condition (3.3), there exist four possible cases: (I), (II), (III) (as those of Theorem 3.1, 3.2) and

(IV): $\Delta z(n)<0$, $\Delta(b(n)\Delta z(n))<0$ for all large n.

Assume that (I) holds. Then we are back to the proof of Theorem 3.1 to get contradiction to (3.4). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty}x(n)=0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.2 to get contradiction by (3.18). Assume that (IV) holds. Since $a(n)\Delta(b(n)\Delta z(n))$ is non-increasing sequence there exists a negative constant K_1 and $n_2\geq n_1$ such that

$$a(n)\Delta(b(n)\Delta z(n))\geq K_1 \quad \text{for} \quad n\geq n_2.$$

Dividing by $a(n)$ and summing the last inequality from n_1 to $n-1$, we obtain

$$\Delta z(n)\leq b^{-1}(n)K_1\left(\sum_{s=n_1}^{n-1}a^{-1}(s)\right).$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$z(n)\leq z(n_1)+K_1\sum_{u=n_1}^{n-1}\left(b^{-1}(u)\left(\sum_{s=n_1}^{u-1}a^{-1}(s)\right)\right).$$

Letting $n\to\infty$ then, by (3.37) we deduce that $z(n)\to-\infty$, which is contradiction to the fact that $z(n)>0$. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Assume that* (3.1) *and* (2.2) *hold. Let* $\{\rho(n)\}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence*

$\{H(m,n)| \geq n \geq 0\}$. *If*

$$(3.38)\quad \limsup_{m\to\infty} \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left(H(m,n)K\rho(n) \sum_{\xi=m_0}^{m} Q(n,\xi) - (1+p)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)} \right) = \infty,$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.1, there are two possible cases. If (I) holds, from the proof of Theorem 3.1, we find that (3.16) holds for all $n \geq n_2$. From(3.16), we have

$$(3.39)\quad \rho(n)K \sum_{\xi=m_0}^{m} Q(n,\xi) \leq -\Delta\omega(n) - p\Delta v(n) + \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) - \varphi(n)\omega^2(n+1) + p\left[\frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) - \varphi(n)v^2(n+1)\right].$$

Therefore, we have

$$\sum_{n=k}^{m-1} H(m,n)K\rho(n) \sum_{\xi=m_0}^{m} Q(n,\xi) \leq -\sum_{n=k}^{m-1} H(m,n)\Delta\omega(n) - p\sum_{n=k}^{m-1} H(m,n)\Delta v(n) + \sum_{n=k}^{m-1} H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) - \sum_{n=k}^{m-1} H(m,n)\varphi(n)\omega^2(n+1) + p\sum_{n=k}^{m-1}(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) - p\sum_{n=k}^{m-1} H(m,n)\varphi(n)v^2(n+1),$$

which yields after summing by parts

$$\sum_{n=k}^{m-1} H(m,n)K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi) \le H(m,k)\omega(k) + \sum_{n=k}^{m-1}\vartheta(m,n)H(m,n)\omega(n+1)$$
$$- \sum_{n=k}^{m-1} H(m,n)\varphi(n)\omega^2(n+1) + pH(m,k)v(k)$$
$$+ p\sum_{n=k}^{m-1}\vartheta(m,n)H(m,n)v(n+1) - p\sum_{n=k}^{m-1} H(m,n)\varphi(n)v^2(n+1).$$

From (3.17), we have

$$\sum_{n=k}^{m-1} H(m,n)K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi) \le H(m,k)\omega(k) \tag{3.40}$$
$$+ \sum_{n=k}^{m-1}\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)} + pH(m,k)v(k) + p\sum_{n=k}^{m-1}\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)}.$$

Then,

$$\sum_{n=k}^{m-1}\left(H(m,n)K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi) - (1+p)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)}\right)$$
$$\le H(m,k)\omega(k) + pH(m,k)v(k),$$

which implies

$$\sum_{n=k}^{m-1}\left(H(m,n)K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi) - (1+p)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)}\right)$$
$$\le H(m,0)|\omega(k)| + pH(m,0)|v(k)|.$$

Hence,

$$\sum_{n=0}^{m-1}\left(H(m,n)K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi) - (1+p)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)}\right)$$
$$\le H(m,0)\left\{\sum_{n=0}^{k-1}\left|K\rho(n)\sum_{\xi=m_0}^{m} Q(n,\xi)\right| + |\omega(k)| + p|v(k)|\right\}.$$

Hence,

$$\limsup_{m\to\infty}\frac{1}{H(m,0)}\sum_{n=0}^{m-1}\left(H(m,n)K\rho(n)\sum_{\xi=m_0}^{m}Q(n,\xi)-(1+p)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi(n)}\right)$$
$$\leq\left\{\sum_{n=0}^{k-1}\left|K\rho(n)\sum_{\xi=m_0}^{m}Q(n,\xi)\right|+|\omega(k)|+p|v(k)|\right\}<\infty,$$

which is contrary to (3.38). If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty}x(n)=0$. This completes the proof of Theorem 3.4. □

Theorem 3.5. *Assume that* (2.2), (3.2) *and* (3.18) *hold. Let* $\{\rho(n)\}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H(m,n)\mid m\geq n\geq 0\}$. *If* (3.38) *holds, then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\geq n_0$. From the proof of Theorem 3.2 there are three possible cases. If (I) holds, then we are back to the proof of Theorem 3.4 to get contradiction by (3.38). If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty}x(n)=0$. If (III) holds, then we are back to the proof of Theorem 3.2 to get contradiction by (3.18). This completes the proof of Theorem 3.5. □

Theorem 3.6. *Assume that* (2.2), (3.3), (3.18) *and* (3.37) *hold. Let* $\{\rho(n)\}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H(m,n)\mid m\geq n\geq 0\}$. *If* (3.38) *holds, then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1 we assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\geq n_0$. From the proof of Theorem 3.3, there are four possible cases. If (I), (III) and (IV) hold, then we are back to the proof of Theorems 3.4, 3.2 and 3.3 respectively to get contradiction by (3.38), (3.18) and (3.37) respectively. If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty}x(n)=0$. This completes the proof of Theorem 3.6. □

Next, we establish an oscillation criterion for (1.1) when $n\geq g(n,\xi)\geq G(n)\geq\tau(n)$ and (S_1) holds.

Theorem 3.7. *Assume that* (2.2) *and* (3.1) *hold. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi)-\frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\Theta(s)}\right)=\infty, \tag{3.41}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$. From the proof of Theorem 3.1, there are two possible cases. Assume that (I) holds. Define the sequence $\omega(n)$ by

$$\omega(n) := \rho(n)\frac{a(n)\Delta(b(n)\Delta z(n))}{z(\tau(n))}. \tag{3.42}$$

Then $\omega(n) > 0$. From (3.42), we have

$$\begin{aligned}\Delta\omega(n) = {} & \Delta\rho(n)\frac{a(n+1)(\Delta(b(n+1)\Delta z(n+1)))}{z(\tau(n+1))} \\ & + \rho(n)\frac{\Delta(a(n)\Delta((b(n)\Delta z(n))))}{z(\tau(n))} \\ & - \rho(n)\frac{a(n+1)(\Delta(b(n+1)\Delta z(n+1)))\Delta(z(\tau(n)))}{z(\tau(n+1))z(\tau(n))}.\end{aligned} \tag{3.43}$$

From Lemma 2.1, and $\tau(n) \leq n$, we get

$$\Delta z(\tau(n)) \geq b^{-1}(\tau(n))(a(n+1)(\Delta(b(n+1)\Delta z(n+1))))\sum_{s=n_2}^{\tau(n)-1}a^{-1}(s).$$

It follows that from(3.42) , (3.43) and the above inequality, we obtain

$$\Delta\omega(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)+\rho(n)\frac{\Delta\left(a(n)\Delta(b(n)\Delta z(n))\right)}{z(\tau(n))}-\Theta(n)\omega^2(n+1). \tag{3.44}$$

Similarly, define another sequence $v(n)$ by

$$v(n) := \rho(n)\frac{a(\tau(n))\Delta\left(b(\tau(n))\Delta z(\tau(n))\right)}{z\left(\tau(n)\right)}. \tag{3.45}$$

Then $v(n) > 0$. From (3.45), we have

$$\Delta v(n) = \frac{\Delta\rho(n)}{\rho(n+1)} v(n+1) + \rho(n)\frac{\Delta(a(\tau(n))\Delta(b(\tau(n))\Delta z(\tau(n))))}{z(\tau(n))} - \rho(n)\frac{a(\tau(n+1))(\Delta(b(\tau(n+1))\Delta z(\tau(n+1))))\Delta(z(\tau(n)))}{z(\tau(n+1))z(\tau(n))}.$$

From Lemma 2.1, and $\tau(n) \le n$, we get

$$\Delta z(\tau(n)) \ge b^{-1}(\tau(n))\,(a(\tau(n+1))(\Delta(b(\tau(n+1))\Delta z(\tau(n+1))))) \sum_{s=n_2}^{\tau(n)-1} a^{-1}(s).$$

Thus

$$\Delta v(n) \le \rho(n)\frac{\Delta(a(\tau(n))\Delta(b(\tau(n))\Delta z(\tau(n))))}{z(\tau(n))} + \frac{\Delta\rho(n)}{\rho(n+1)} v(n+1) - \Theta(n)v^2(n+1). \tag{3.46}$$

From (3.44) and (3.46), we obtain

$$\begin{aligned} &\Delta\omega(n) + p\Delta v(n) \\ &\le \rho(n)\frac{[\Delta(a(n)\Delta(b(n)\Delta z(n))) + p\Delta(a(\tau(n))\Delta(b(\tau(n))\Delta z(\tau(n))))]}{z(\tau(n))} \\ &\quad + \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) - \Theta(n)\omega^2(n+1) \\ &\quad + p\left[\frac{\Delta\rho(n)}{\rho(n+1)} v(n+1) - \Theta(n)v^2(n+1)\right]. \end{aligned} \tag{3.47}$$

From (I), (3.8), (3.47) and $G(n) \ge \tau(n)$, we have

$$\begin{aligned} \Delta\omega(n) + p\Delta v(n) &\le -\rho(n)K\sum_{\xi=m_0}^{m} Q(n,\xi) + \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) \\ &\quad - \Theta(n)\omega^2(n+1) + p\left[\frac{\Delta\rho(n)}{\rho(n+1)} v(n+1) - \Theta(n)v^2(n+1)\right]. \end{aligned} \tag{3.48}$$

From (3.48) and (3.17), we have

$$\begin{aligned} &\Delta\omega(n) + p\Delta v(n) \\ &\le -\rho(n)K\sum_{\xi=m_0}^{m} Q(n,\xi) + \frac{(\Delta\rho(n))^2}{4(\rho(n+1))^2\Theta(n)} + p\frac{(\Delta\rho(n))^2}{4(\rho(n+1))^2\Theta(n)}. \end{aligned}$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi)-\frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\Theta(s)}\right)\le \omega(n_2)+pv(n_2),$$

which yields

$$\sum_{s=n_2}^{n-1}\left(\rho(s)K\sum_{\xi=m_0}^{m}Q(s,\xi)-\frac{(1+p)}{4}\frac{(\Delta\rho(s))^2}{(\rho(s+1))^2\Theta(s)}\right)\le c_1,$$

where $c_1>0$ is a finite constant. But, this contradicts (3.41). If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty}x(n)=0$. This completes the proof of Theorem 3.7. □

Theorem 3.8. *Assume that* (2.2), (3.2) *and* (3.41) *hold. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left(K\delta(s+1)\sum_{\xi=m_0}^{m}q(s,\xi)(1-p(g(s,\xi)))\sum_{v=n_3}^{G(s)-1}\frac{1}{b(v)}-\frac{1}{4a(s)\delta(s+1)}\right)=\infty, \tag{3.49}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $n_1\ge n_0$ such that $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$. From the proof of Theorem 3.2, there are three possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.7 to get contradiction by (3.41). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty}x(n)=0$. Assume that (III) holds, since $x(n)\le z(n)$, we see that

$$x(g(n,\xi)-\tau)\le z(g(n,\xi)-\tau)\le z(g(n,\xi)),\quad n\in N(n_2),\quad \xi\in N(m_0,m) \tag{3.50}$$

Form (3.7), we have

$$z(g(n,\xi))\ge z(G(n)),\quad n\in N(n_3),\quad \xi\in N(m_0,m)\ \text{ for some }\ n_3\ge n_2.$$

Using the above inequality together with (3.50) and (S_1) in equation (1.1) for $n \geq n_3$, we get

$$0 \geq \Delta(a(n)\Delta(b(n)\Delta z(n))) + Kz(G(n)) \sum_{\xi=m_0}^{m} q(n,\xi)(1 - p(g(n,\xi))). \tag{3.51}$$

Define $\omega(n)$ by (3.19). Proceeding as in the proof of Theorem 3.1, we obtain (3.23) and (3.26). From (3.26) and (3.51), we have

$$\begin{aligned}\Delta\omega(n) \leq -K\frac{z(G(n))}{b(n+1)\Delta z(n+1)} \sum_{\xi=m_0}^{m} q(n,\xi)(1 - p(g(n,\xi)))\\ - \frac{a(n)\Delta(b(n)\Delta z(n))a(n)\Delta(b(n)\Delta z(n))}{a(n)b(n)\Delta z(n)(b(n+1)\Delta z(n+1))}.\end{aligned} \tag{3.52}$$

Since

$$z(n) \geq z(n) - z(n_3) = \sum_{s=n_3}^{n-1} \frac{b(s)\Delta z(s)}{b(s)} \geq b(n)\Delta z(n) \sum_{s=n_3}^{n-1} \frac{1}{b(s)}, \tag{3.53}$$

we have that

$$\Delta\left(\frac{z(n)}{\sum\limits_{s=n_3}^{n-1} \frac{1}{b(s)}}\right) \leq 0.$$

Which implies that

$$\frac{z(G(n))}{b(n+1)\Delta z(n+1)} \geq \frac{b(G(n))\Delta z(G(n)) \sum\limits_{s=n_3}^{G(n)-1} \frac{1}{b(s)}}{b(n+1)\Delta z(n+1)} \geq \sum_{s=n_3}^{G(n)-1} \frac{1}{b(s)}. \tag{3.54}$$

From (3.52) and (3.54), we get

$$\Delta\omega(n) \leq -K \sum_{\xi=m_0}^{m} q(n,\xi)(1 - p(g(n,\xi))) \sum_{s=n_3}^{G(n)-1} \frac{1}{b(s)} - \frac{\omega^2(n)}{a(n)}. \tag{3.55}$$

Multiplying (3.55) by $\delta(n+1)$ and summing it from n_4 to $n-1$, we find

$$\omega(n)\delta(n)-\omega(n_4)\delta(n_4)+\sum_{s=n_4}^{n-1}\frac{\omega(s)}{a(s)}+\sum_{s=n_4}^{n-1}\delta(s+1)\frac{\omega^2(s)}{a(s)} \\ +\sum_{s=n_4}^{n-1}K\delta(s+1)\sum_{\xi=m_0}^{m}q(s,\xi)(1-p(g(s,\xi)))\sum_{v=n_3}^{G(s)-1}\frac{1}{b(v)}\le 0. \tag{3.56}$$

It follows from (3.17) and (3.56), that

$$\omega(n)\delta(n)-\omega(n_4)\delta(n_4)-\sum_{s=n_4}^{n-1}\frac{1}{4a(s)\delta(s+1)} \\ +\sum_{s=n_4}^{n-1}K\delta(s+1)\sum_{\xi=m_0}^{m}q(s,\xi)(1-p(g(s,\xi)))\sum_{v=n_3}^{G(s)-1}\frac{1}{b(v)}\le 0.$$

From (3.23), we get

$$\sum_{s=n_4}^{n-1}\left(K\delta(s+1)\sum_{\xi=m_0}^{m}q(s,\xi)(1-p(g(s,\xi)))\sum_{v=n_3}^{G(s)-1}\frac{1}{b(v)}-\frac{1}{4a(s)\delta(s+1)}\right) \\ \le 1+\omega(n_4)\delta(n_4).$$

But this contradicts (3.49). This completes the proof of Theorem 3.8. □

Theorem 3.9. *Assume that* (2.2)*,* (3.3)*,* (3.37) *and* (3.49) *hold. Let* $\{\rho(n)\}$ *be a positive sequence, such that* (3.41) *holds. Then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\ge n_0$. From the proof of Theorem 3.3, there are four possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.7 to get contradiction by (3.41). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty}x(n)=0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.8 to get contradiction by (3.49). Assume that (IV) holds. Then we are back to the proof of Theorem 3.3 to get contradiction by (3.37). This completes the proof of Theorem 3.9. □

Next, we establish an oscillation criterion for (1.1) when $n \geq g(n,\xi) \geq G(n)$ and (S_2) holds.

Theorem 3.10. *Let condition* (2.2) *and* (3.1) *hold. Assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$. *Furthermore, we assume that there exists a double sequence* $\{H(m,n) \mid m \geq n \geq 0\}$ *and* $h(m,n)$ *such that* $(i)-(iv)$ *hold. If*

$$\limsup_{m\to\infty} \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left(H(m,n)\rho(n) \sum_{\xi=m_0}^{m} q(n,\xi) - \frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(G(n))}{4\mu\rho(n) \sum\limits_{s=n_2}^{G(n)-1} a^{-1}(s)} \right) = \infty, \tag{3.57}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.1, there are two possible cases. If (I) holds, from equation (1.1) and (3.7), we have

$$0 \geq \Delta(a(n)\Delta(b(n)\Delta(z(n)))) + f(x(G(n))) \sum_{\xi=m_0}^{m} q(n,\xi). \tag{3.58}$$

Define

$$\omega(n) := \rho(n)\frac{a(n)\Delta(b(n)\Delta z(n))}{f(x(G(n)))}. \tag{3.59}$$

Then $\omega(n) > 0$. From (3.58), (3.59) and (S_2), we have

$$
\begin{aligned}
(3.60)\quad & \Delta\omega(n) \\
&= \Delta\rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))}{f(x(G(n+1)))} + \rho(n)\frac{\Delta(a(n)\Delta(b(n)(\Delta z(n))))}{f(x(G(n)))} \\
&\quad - \rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))}{f(x(G(n+1)))f(x(G(n)))}B(x(G(n+1)),x(G(n))) \\
&\quad \times [x(G(n+1)) + p(G(n+1))x(G(\tau(n+1)))] \\
&\quad - [x(G(n)) + p(G(n))x(G(\tau(n)))] \\
&= \Delta\rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))}{f(x(G(n+1)))} + \rho(n)\frac{\Delta(a(n)\Delta(b(n)(\Delta z(n))))}{f(x(G(n)))} \\
&\quad - \rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))}{f(x(G(n+1)))f(x(G(n)))}B(x(G(n+1)),x(G(n)))\Delta(z(G(n))).
\end{aligned}
$$

From (3.11), (3.60) and (S_2), we obtain

$$
\begin{aligned}
(3.61)\quad & \Delta\omega(n) \\
&\leq \Delta\rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))}{f(x(G(n+1)))} + \rho(n)\frac{\Delta(a(n)\Delta(b(n)(\Delta z(n))))}{f(x(G(n)))} \\
&\quad - \mu\rho(n)\frac{a(n+1)\Delta(b(n+1)(\Delta z(n+1)))(a(n+1)(\Delta(b(n+1)\Delta z(n+1))))}{f(x(G(n+1)))f(x(G(n)))b(G(n))} \\
&\quad \times \sum_{s=n_2}^{G(n)-1} a^{-1}(s).
\end{aligned}
$$

It follows from (3.58) and (3.61) that

$$
\begin{aligned}
(3.62)\quad \Delta\omega(n) \leq {} & \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) - \rho(n)\sum_{\xi=m_0}^{m} q(n,\xi) \\
& - \frac{\mu\rho(n)}{\rho^2(n+1)b(G(n))}\omega^2(n+1)\sum_{s=n_2}^{G(n)-1} a^{-1}(s).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\sum_{n=k}^{m-1} H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi) &\\ \leq -\sum_{n=k}^{m-1} H(m,n)\Delta\omega(n) &+ \sum_{n=k}^{m-1} H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)\\ &- \sum_{n=k}^{m-1} H(m,n)\frac{\mu\rho(n)}{\rho^2(n+1)b(G(n))}\omega^2(n+1)\sum_{s=n_2}^{G(n)-1} a^{-1}(s),\end{aligned}$$

which yields after summing by parts

$$\begin{aligned}\sum_{n=k}^{m-1} H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi) &\\ \leq H(m,k)\omega(k) &+ \sum_{n=k}^{m-1} \vartheta(m,n)H(m,n)\omega(n+1)\\ &- \sum_{n=k}^{m-1} H(m,n)\frac{\mu\rho(n)}{\rho^2(n+1)b(G(n))}\omega^2(n+1)\sum_{s=n_2}^{G(n)-1} a^{-1}(s).\end{aligned}$$

From (3.17), we have

$$\begin{aligned}\sum_{n=k}^{m-1} H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi) &\\ \leq H(m,k)\omega(k) &+ \sum_{n=k}^{m-1} \frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(G(n))}{4\mu\rho(n)\sum_{s=n_2}^{G(n)-1} a^{-1}(s)}.\end{aligned} \tag{3.63}$$

Then,

$$\begin{aligned}\sum_{n=k}^{m-1}\left(H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi) - \frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(G(n))}{4\mu\rho(n)\sum_{s=n_2}^{G(n)-1} a^{-1}(s)}\right)&\\ \leq H(m,k)\omega(k) \leq H(m,0)\left|\omega(k)\right|.&\end{aligned}$$

Hence,

$$\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi)-\frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(G(n))}{4\mu\rho(n)\sum\limits_{s=n_2}^{G(n)-1}a^{-1}(s)}\right)\leq$$

$$H(m,0)\left\{\sum_{n=0}^{k-1}\left|\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi)\right|+|\omega(k)|\right\}.$$

Hence,

$$\limsup_{m\to\infty}\frac{1}{H(m,0)}\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi)\right.$$

$$\left.-\frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(G(n))}{4\mu\rho(n)\sum\limits_{s=n_2}^{G(n)-1}a^{-1}(s)}\right)$$

$$\leq\left\{\sum_{n=0}^{k-1}\left|\rho(n)\sum_{\xi=m_0}^{m} q(n,\xi)\right|+|\omega(k)|\right\}<\infty,$$

which is contrary to (3.57). If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty} x(n)=0$. This completes the proof of Theorem 3.10. □

Theorem 3.11. *Let conditions* (2.2), (3.2) *and* (3.18) *hold. Further, assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that* (3.57) *holds. Then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n)>0$, $x(\tau(n))>0$ and $x(g(n,\xi))>0$ for all $n\geq n_0$. From the proof of Theorem 3.2, there are three possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.10 to get contradiction by (3.57). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty} x(n)=0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.2 to get contradiction by (3.18). This completes the proof of Theorem 3.11. □

Theorem 3.12. *Let conditions* (2.2), (3.3), (3.18) *and* (3.37) *hold. Further, assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that* (3.57) *holds. Then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.3, there are four possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.10 to get contradiction by (3.57). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim\limits_{n\to\infty} x(n) = 0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.2 to get contradiction by (3.18). Assume that (IV) holds. Then we are back to the proof of Theorem 3.3 to get contradiction by (3.37). This completes the proof of Theorem 3.12. □

Finally, we establish an oscillation criterion for (1.1) when $n \geq g(n,\xi) \geq G(n) \geq \tau(n)$ and (S_2) holds.

Theorem 3.13. *Let conditions* (2.2) *and* (3.1) *hold. Assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$. *Furthermore, we assume that there exists a double sequence* $\{H(m,n) \mid m \geq n \geq 0\}$ *and* $h(m,n)$ *such that* (i)*–*(iv) *hold. If*

$$\limsup_{m\to\infty} \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left(H(m,n)\rho(n) \sum_{\xi=m_0}^{m} q(n,\xi) - \frac{\vartheta^2(m,n)H(m,n)\rho^2(n+1)b(\tau(n))}{4\mu\rho(n) \sum\limits_{s=n_2}^{\tau(n)-1} a^{-1}(s)} \right) = \infty, \tag{3.64}$$

then every solution of equation (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.1, there are two possible cases. If (I) holds, from equation (1.1) and (3.7), we have

$$\begin{aligned}(3.65)\quad 0 &\geq \Delta(a(n)\Delta(b(n)\Delta(z(n)))) + f(x(G(n))) \sum_{\xi=m_0}^{m} q(n,\xi).\\ &\geq \Delta(a(n)\Delta(b(n)\Delta(z(n)))) + f\left(x(\tau(n))\right) \sum_{\xi=m_0}^{m} q(n,\xi).\end{aligned}$$

Define

$$\omega(n) := \rho(n)\frac{a(n)\Delta(b(n)\Delta z(n))}{f\left(x(\tau(n))\right)}. \tag{3.66}$$

The rest of the proof is similar to that of Theorem 3.10 and hence the details are omitted. □

Theorem 3.14. *Let conditions* (2.2), (3.2) *and* (3.49) *hold. Further, assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that* (3.64) *holds. Then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.2, there are there possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.13 to get contradiction by (3.64). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty} x(n) = 0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.8 to get contradiction by (3.49). This completes the proof of Theorem 3.14. □

Theorem 3.15. *Let condition* (2.2), (3.3), (3.49) *and* (3.37) *hold. Further, assume that there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that* (3.64) *holds. Then every solution of equation* (1.1) *either oscillates or tends to zero.*

Proof. Proceeding as in Theorems 3.1, we assume that equation (1.1) has a non-oscillatory solution, say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n,\xi)) > 0$ for all $n \geq n_0$. From the proof of Theorem 3.3, there are four possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 3.13 to get contradiction by (3.64). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n\to\infty} x(n) = 0$. Assume that (III) holds. Then we are back to the proof of Theorem 3.8 to get contradiction by (3.49). Assume that (IV) holds. Then we are back to the proof of Theorem 3.3 to get contradiction by (3.37). This completes the proof of Theorem 3.15. □

4. Conclusion. In this paper, we established some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results improved and expanded some known results, see e.g. the following results:

Remark 4.1. If $p(n) \equiv 0$, $q(n,\xi) \equiv q(n)$ and $g(n,\xi) \equiv n+l$, then Theorem 3.1 extended and improved Theorem 3 in [19].

Remark 4.2. If $b(n) \equiv 1$, $p(n) \equiv 0$, $q(n,\xi) \equiv q(n)$ and $g(n,\xi) \equiv n-\sigma$, then Theorem 3.1 extended and improved Theorem 1 in [16].

Remark 4.3. If $b(n) \equiv 1$, $q(n,\xi) \equiv q(n)$, $g(n,\xi) \equiv n-\tau$ and $f(x) \equiv x^{\alpha}$, then Theorem 3.1 extended and improved Theorem 2.3 in [13].

Remark 4.4. If $p(n) \equiv 0$, $q(n,\xi) \equiv q(n)$ and $g(n,\xi) \equiv n-m+1$, then Theorem 3.4 extended and improved Theorem 1 in [6].

Remark 4.5. If $p(n) \equiv 0$, $q(n,\xi) \equiv q(n)$, $g(n,\xi) \equiv n-m+1$ and $H(m,n) \equiv 1$, then Theorem 3.10 extended and improved Theorem 2 in [6].

Remark 4.6. If $b(n) \equiv 1$, $p(n) \equiv 0$, $q(n,\xi) \equiv q(n)$ and $g(n,\xi) \equiv n-\sigma$, then we reduced to Theorem 1 in [18].

Remark 4.7. If $b(n) \equiv 1$, $q(n,\xi) \equiv q(n)$ and $g(n,\xi) \equiv n+1$, then we reduced to Theorem 3 in [17].

Remark 4.8. If $a(n) \equiv b(n) \equiv 1$, $p(n) \equiv -1$, $q(n,\xi) \equiv q(n)$, $g(n,\xi) \equiv g(n)$ and $f(x) \equiv x^{\alpha}$, then we reduced to Theorems in [7].

5. Examples. In this section, we will show the applications of our oscillation criteria by three examples. We will see that the equation in the examples is oscillates or tends to zero based on the results in Section 3.

Example 5.1. Consider the third order nonlinear neutral difference equation

$$\delta\left(\frac{1}{n}\left(\Delta^2\left(x_n+\frac{3}{4}x(n-2)\right)\right)\right) + (n^2+2)\sum_{\xi=1}^{2} x^3(n-\xi)(1+x^2(n-\xi)) = 0, \quad n \geq 1. \tag{5.1}$$

All the conditions of Theorem 3.1are satisfied (with $\rho(n) = n$). Hence every solution of (5.1) either oscillates or tends to zero. We should note that the oscillation criteria given in [13], [16] and [19] fail to apply for this difference equation.

Example 5.2. Consider the linear delay difference equation

$$\Delta^3\left(x(n) + \frac{1}{3}x(n-\lambda_1)\right) + \left(\frac{27}{32}\right)\sum_{\xi=0}^{1} x(n-\lambda_2\xi) = 0, \quad n \geq 1. \tag{5.2}$$

All the conditions of Theorem 3.4 are satisfied (with $K = 1$, $\rho(n) = 1$ $\lambda_2 \geq \lambda_1$, $H(m,n) = m - n$). Hence every solution of (5.2) either oscillates or tends to zero. We should note that the oscillation criteria given in [6], fail to apply for this difference equation.

Example 5.3. Consider the linear delay difference equation

$$\Delta^3\left(x(n) + \frac{1}{3}x(n-2)\right) + \frac{\lambda}{n^2}\sum_{\xi=0}^{1} x(n-\xi) = 0, \quad n \geq 1. \tag{5.3}$$

All the conditions of Theorem 3.7 are satisfied (with $K = 1$, $\rho(n) = 1$). Hence every solution of (5.3) either oscillates or tends to zero.

Acknowledgements. The authors would like to thank the anonymous referees very much for valuable suggestions, corrections and comments, which results in a great improvement in the original manuscript.

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Received September 12, 2013
Revised March 7, 2014