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EXPLICIT FORMULAS TO THE SOLUTIONS OF SEVERAL EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract. Explicit formulas to the solutions of several equations of mathematical physics including semilinear multidimensional Klein-Gordon equation, the wave equation, Kadomtsev-Petviashvili equation and cubic first order hyperbolic pseudodifferential equation are proposed.

1. Introduction

This paper deals with the above mentioned PDE of Mathematical Physics having interesting applications in different areas (see [3], [6]). Concerning Klein-Gordon (K-G), respectively multidimensional wave equations we can construct special solutions via appropriate change of the unknown function u and by solving some overdetermined systems of linear and nonlinear PDE. We remind that in the case of 1D sin-Gordon equation solutions containing elliptic and hyperbolic functions appear [10]. The wave solutions of Kadomtsev-Petviashvili (K-P) equation are constructed via Hirota method [12,13,11] but their interaction can give rise to the so called X and Y waves. In studying the linear first order multidimensional wave equation (periodic case in (t, x)) the machinery of small denominators works [2], while in the cubic case we are able to construct classes of Szegö type solutions [14,15].

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2. Special solutions of semilinear K-G type equations in the multidimensional case

Consider the K-G PDE

$$Lu + (\nabla_{t,x}u, \vec{B}) = f(u) \quad \text{in} \quad \mathbf{R}^1_t \times \mathbf{R}^n_x, n \geq 2, \tag{1}$$

where $L = \partial_t^2 - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the wave operator, the constant vector $\vec{B}=(b_0, b_1, \dots, b_n) \in \mathbf{R}^1_t \times \mathbf{R}^n_x$, the real-valued function $f \in C^1(\mathbf{R}^1)$.

According to the classical approach (see [8]), we look for a solution of (1) of the form $u = \varphi(G)$, $\varphi \in C^2(\mathbf{R}^1)$, $G = G(t,x) \in C^2$.

Then (1) takes the form

$$\varphi' LG + \varphi''(G_t^2 - \sum_{j=1}^n G_{x_j}^2) + \varphi'(\nabla_{t,x}G, \vec{B}) = f(\varphi(G)). \tag{2}$$

Further on we shall assume that either

$$\left| \begin{array}{l} LG + (\nabla_{t,x}G, \vec{B}) = -G \\ \sum_{j=1}^n G_{x_j}^2 - G_t^2 = G^2, \end{array} \right. \tag{3}$$

i.e.

$$G\varphi'(G) + G^2\varphi''(G) = -f(\varphi(G)) \tag{4}$$

or

$$\left| \begin{array}{l} LG + (\nabla_{t,x}G, \vec{B}) = 0 \\ \sum_{j=1}^n G_{x_j}^2 - G_t^2 = 1, \end{array} \right. \tag{5}$$

i.e.

$$\varphi''(G) = -f(\varphi(G)) \quad \text{(pendulum equation)}. \tag{6}$$

The change $G = e^t$ in the Euler ODE (4) leads to $\tilde{\varphi}''(t) = -f(\tilde{\varphi}(t))$. Solving the pendulum equation we get $\tilde{\varphi} = \tilde{\varphi}(t) \Rightarrow \varphi = \tilde{\varphi}(ln\ G)$, $G > 0$.

Example 1. We shall illustrate (1) with the following examples: a). $f(u) = e^{-2u}$, b). $f(u) = -u(ln\ u + ln^2\ u)$, c). $f(u) = -sh\ u$, d). $f(u) = \pm sin\ u$, e). $f(u) = -\frac{1}{2}e^u$, f). $f(u) = 3u^2 - 3\beta^2$, $\beta < 0$.

In case a). we take $u = \varphi(G) = ln\ G$, > 0 which satisfies (6); in case b). we take $u = \varphi(G) = e^G$, which satisfies (4); in case c). (6) takes the form

$\varphi^{''} = sh\ \varphi \Rightarrow (\varphi^{'})^2 = 2(ch\ \varphi + A)$ and therefore we take $\varphi = 2ln|tg\frac{G}{2}|$ for $A = 1$, $\varphi = 2ln|cth\frac{G}{2}|$ for $A = -1$; in case e). the equation (6) is written as $\varphi^{''} = \frac{1}{2}e^{\varphi} \Rightarrow \varphi^{'} = \pm e^{\frac{\varphi}{2}}$ and we put $u = \varphi(G) = -2ln|\frac{G}{2}|$, $g \neq 0$; in case d). $f(u) = -sin\ u$ the equation (4) possesses the solution $\varphi = 4arctg\ G$, while if $f(u) = sin\ u$ we take $\varphi = 4(arctgG - \frac{\pi}{4})$. In case f). and after the change $G = e^t$ the equation (4) can be written as: $\tilde{\varphi}^{''} = -(3\tilde{\varphi}^2 - 3\beta^2) \Rightarrow$

$$(\tilde{\varphi}^{'})^2 = -2(\tilde{\varphi}^3 - 3\beta^2\tilde{\varphi} + 2\beta^3), \tag{7}$$

the constant $2\beta^2$ being appropriate chosen after the integration. As $\tilde{\varphi} = \beta$ is a double root of $\tilde{\varphi}^3 - 3\beta^2\tilde{\varphi} + 2\beta^3 = 0$ and $\tilde{\varphi} = -2\beta > 0$ is a simple root we can integrate (7) obtaining $\varphi = \beta - 3\beta sech^2(-\sqrt{\frac{-3\beta}{2}}lnG)$, $G > 0$, $\beta = const < 0$.

To find a special solution into explicit form of the overdetermined system (3) we put $G = e^{\psi}V$, where the unknown linear function $\psi = \sum_{j=1}^{n} a_j x_j - \sigma t$, $\sigma \neq 0$, has real-valued coefficients and therefore $V(t, x)$ should satisfy

$$\begin{gathered} LV - 2((\vec{a}, \nabla_x V) + \sigma V_t) + (\sigma^2 + 1 - |a|^2)V + \\ +(< V_t - \sigma V, (\nabla_x + \vec{a})V >, \vec{B}), \vec{a} = (a_1, \dots, a_n) \\ \sum_{j=1}^{n} V_{x_j}^2 - V_t^2 + 2V((\vec{a}, \nabla_x V) + \sigma V_t) + V^2(|a|^2 - \sigma^2 - 1) = 0. \end{gathered} \tag{8}$$

We shall assume further on that the following overdetermined system of 4 PDE holds:

$$\left| \begin{array}{l} LV = 0 \\ \sum_{j=1}^{n} V_{x_j}^2 - V_t^2 = 0 \quad \text{(eikonal equations)} \\ \sum_{j=1}^{n} a_j V_{x_j} + \sigma V_t = 0 \\ \sum_{j=1}^{n} b_j V_{x_j} + b_0 V_t = 0 \end{array} \right. \tag{9}$$

under the additional assumptions: $\sum_{j=1}^{n} a_j^2 = \sigma^2 + 1$, $\sum_{j=1}^{n} a_j b_j = b_0\sigma$.

Evidently, (9) $\Rightarrow$ (8).

Consider now (5). We are looking for a solution having the form $G = \psi + W(t, x)$ and the linear function ψ is defined as above. Then (5) is rewritten as:

$$\left| \begin{array}{l} (\nabla_{t,x}\psi, \vec{B}) + (\nabla_{t,x}W, \vec{B}) + LW = 0 \\ \sum_{j=1}^{n}(a_j + W_{x_j})^2 - (W_t - \sigma)^2 = 1. \end{array} \right. \tag{10}$$

Suppose now that W satisfies

$$(11)\qquad \left|\begin{array}{l} LW = 0 \\ \sum_{j=1}^n W_{x_j}^2 - W_t^2 = 0 \\ \sum_{j=1}^n a_j W_{x_j} + \sigma W_t = 0 \\ \sum_{j=1}^n b_j W_{x_j} + b_0 W_t = 0 \end{array}\right.$$

under the additional conditions $\sum_{j=1}^n a_j^2 = \sigma^2 + 1$, $\sum_{j=1}^n a_j b_j = \sigma b_0$. Certainly, (11) $\Rightarrow$ (10) and the systems (9), (11) coincide.

Remark. Let $F \in C^2(\mathbf{R}^1)$ be arbitrary and $V = F(\alpha(t,x))$ verifies (9) for some $\alpha \in C^2$. Evidently, then $(F')^2(\sum_{j=1}^n \alpha_{x_j}^2 - \alpha_t^2) = 0$, $F' L\alpha + F''(\alpha_t^2 - \sum_{j=1}^n \alpha_{x_j}^2) = 0$, $F'(\sum_{j=1}^n a_j \alpha_{x_j} + \sigma\alpha_t) = 0$ etc. This way we conclude that if α verifies the overdetermined system

$$(12)\qquad \left|\begin{array}{l} L\alpha = 0 \\ \sum_{j=1}^n \alpha_{x_j}^2 - \alpha_t^2 = 0 \\ \sum_{j=1}^n a_j \alpha_{x_j} + \sigma\alpha_t = 0 \\ \sum_{j=1}^n b_j \alpha_{x_j} + b_0 \alpha_t = 0, \end{array}\right.$$

then for every $F \in C^2$ the system (9) with $V = F(\alpha)$ holds.

To solve (12) we look for $\alpha(t,x)$ of linear form, i.e. $\alpha(t,x) = -\sum_{j=1}^n x_j^0 x_j + t$, $x_j^0 = const.$

One gets immediately that

$$(13)\qquad \sum_{j=1}^n x_j^0 = 1, \sum_{j=1}^n a_j x_j^0 = \sigma, \sum_{j=1}^n b_j x_j^0 = b_0$$

and moreover, $|a|^2 = \sum_{j=1}^n a_j^2 = \sigma^2 + 1$, $\sum_{j=1}^n a_j b_j = \sigma b_0$ ($\sigma \neq 0$). If S_1^{n-1} is the unit sphere in $\mathbf{R}_x^n$ and B_1^n is the unit ball in $\mathbf{R}_x^n$ then the point $X^0 = (x_1^0, \dots, x_n^0) \in S_1^{n-1}$, $a = (a_1, \dots, a_n) \notin B_1^n$ and $X^0 \in \gamma_{1\sigma} \cap \gamma_2$, $\gamma_{1\sigma}$ and γ_2 being the hyperplanes $\sum_1^n a_j y_j = \sigma$, $\sum_1^n b_j y_j = b_0$ respectively. Put $b = (b_1, \dots, b_n)$ and assume that $|b_0| < |b|$. Therefore, $|cos(\vec{a}, \vec{b})| < 1$, i.e. $\vec{a}, \vec{b}$ are not colinear and parts of $\gamma_{1\sigma}, \gamma_2$ are contained inside B_1^n.

Proposition 1. *Consider the system (12) and suppose that: $|b_0| < |b|$, there exist a constant $\sigma \neq 0$, a vector $a \in \mathbf{R}^n$ such that: $|a|^2 = \sigma^2 + 1$, $(\vec{a}, \vec{b}) = b_0\sigma$ and $\gamma_{1\sigma} \cap \gamma_2 \cap S_1^{n-1} \neq \{\emptyset\}$. Then (12) possesses infinitely many solutions depending on an arbitrary smooth function. It follows that (1) possesses infinitely many solutions written into explicit form: $u = \varphi(e^\psi F(\alpha(t,x))$.*

Remark. In many cases points $X^0 \in S_1^{n-1} \cap \gamma_{1\sigma} \cap \gamma_2$ do not exist for some $\sigma \neq 0$. Let $n \geq 3$, $\vec{b} \neq 0$ and $b_0 = 0 \Rightarrow \vec{a} \perp \vec{b}$, the point $P^0 = \frac{\sigma}{\sigma^2+1} a \in int B_1^n$ and $P^0 \in \gamma_{1\sigma} \cap \sigma_2$. The plane of codimmension 2 $\gamma_{1\sigma} \cap \gamma_2$, $\sigma \neq 0$ will cross S_1^{n-1}, certainly. If $n = 3$ $\gamma_{1\sigma} \cap \gamma_2$ is a straight line crossing S_1^{n-1} at two points only. Otherwise, it is a smooth set of codimension 2 at S_1^{n-1}.

We shall not discuss the case (5), respectively then $u = \varphi(\psi + F(\alpha(t,x)))$.

3. Interaction of 2 soliton type solutions of the K-P equation. Resonances, non-resonances and X, Y shalow water waves in the oceans

The K-P equation is given by the formula:

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0, \tag{14}$$

$u = u(t,x,y)$, $\alpha^2 = 1$.

Later on we shall deal with $\alpha = 1$. By using Hirota's method [13,16] Satsuma proved in [12] the existence of N-soliton solution of (14) having the form:

$$u = 2(log\ f)_{xx}, f = \sum_{\mu=0,1} exp[\sum_{1 \leq i < j}^{N} \mu_i \mu_j A_{ij} + \sum_{i=1}^{N} \mu_i \eta_i], \tag{15}$$

where $\eta_i = k_i(x + p_i y - C_i t)$, $C_i = k_i^2 + p_i^2$, $e^{A_{ij}} = \frac{3(k_i-k_j)^2-(p_i-p_j)^2}{3(k_i+k_j)^2-(p_i-p_j)^2}$.

We do not have resonances if $e^{A_{ij}} \neq 0$. Resonances appear if for some (i,j): $e^{A_{ij}} = 0$. Thus, $N = 1 \Rightarrow u = \frac{k_1^2}{2} sech^2 \frac{\eta_1}{2}$, $sech\ x = \frac{2}{e^x + e^{-x}}$.

Let $N = 2$. Resonance exists iff $\sqrt{3}(k_1 - k_2) = \pm(p_1 - p_2)$. Further on we shall take $sign" + "$ in front of $p_1 - p_2$, assuming $p_1 > p_2 > 0 \Rightarrow k_1 > k_2 > 0$.

The case of triple resonance $N = 3$

$$\sqrt{3}(k_1 - k_2) = \pm(p_1 - p_2)$$

$$\sqrt{3}(k_1 - k_3) = \pm(p_1 - p_2)$$

can be investigated in a similar way as the case of resonance for $N = 2$. Due to the lack of space we omit it.

Suppose now that $N = 2$ and $e_{A_{12}} \neq 0$ (no resonance). Then the corresponding solution of (14) given by formula (15) becomes

$$u = 2\frac{k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} + e^{\eta_1+\eta_2}[(k_1-k_2)^2 + e^{A_{12}}(k_1+k_2)^2 + k_2^2 a^{A_{12}+\eta_1} + k_1^2 e^{A_{12}+\eta_2}]}{(1 + e^{\eta_1} + e^{\eta_2} + e^{A_{12}+\eta_1+\eta_2})^2} \tag{16}$$

and it is called X wave ($u(0) > 0$). Fix η_1, y. Then $u \sim \frac{k_1^2}{2} sech^2 \frac{\eta_1 + A_{12}}{2}$ for $t \to \infty$, $u \sim \frac{k_1^2}{2} sech^2 \eta_1$ for $t \to -\infty$.

As it concerns the resonance case for $N = 2$, the solution is written as:

$$u = 2\frac{k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} + (k_1 - k_2)^2 e^{\eta_1+\eta_2}}{(1 + e^{\eta_1} + e^{\eta_2})^2}, u(0) > 0 \tag{17}$$

and is called Y wave.

Exercise 1. Consider the function

$$f(x, y) = \frac{x + k^2 y + (1 - k)^2 xy}{(1 + x + y)^2}, x, y \geq 0, 0 < k < 1$$

and k is parameter. Then $f(x, y) \leq \frac{1}{4}$ and $f(x, y) = \frac{1}{4} \iff x = 1,\ y = 0$; $lim\ sup_{(x,y)\to(\infty,\infty)} f(x, y)$ can be studied easily.

Hint. Consider two cases a) $y = 0$ and b) $y > 0$. In case a) $f(x, 0) < \frac{1}{4}$ for $x \neq 1$, $x \geq 0$ and $f(1, 0) = \frac{1}{4}$. In case b) fix $y > 0$, $x \geq 0$ and consider the quadratic polynomial in $k \in [0, 1]$ $f_{xy}(k) = \frac{x+k^2y+(1-k)^2xy}{(1+x+y)^2}$. The coefficient in front of k^2 is $\frac{y(x+1)}{(1+x+y)^2} > 0$, i.e. $f_{xy}(k)$ is strictly convex and therefore $f_{xy}(k) < max(f_{xy}(0), f_{xy}(1))$. As $f_{xy}(1) \leq 1/4$ according to a), one must prove only that $f_{xy}(0) \leq 1/4$. Show that $f_{xy}(0) \leq \frac{1}{4} \iff 0 \leq (1 + y - x)^2$. For this nice proof I am undebted to N.Nikolov and A.Ivanov.

As usually, we shall study the profiles of the waves for $t = 0$, $t = \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1$ etc. We shall concentrate at $t = 0$ as the other cases are treated in a similar way.

Thus, $N = 2$, no resonance case, and denote by $l_1 : \eta_1 = 0$, $l_2 : \eta_2 = 0$ the straight lines passing through the origin in $0xy$. Put for l an arbitrary line through 0. Then $u|_{l_1}$ and $u|_{l_2}$ are kinks-antikinks, while $u|_l$ is a soliton if $l \neq l_1, l_2$. Monotonically increasing (decreasing) bdd function $v(s)$ on $\mathbf{R}^1$ is called kink (antikink) - see [5,10] if it possesses two horizontal asymptotes $v = \alpha$, $v = \beta$, $\alpha < (>) \beta$. In our considerations here we assume that kinks possess two horizontal asymptotes $v = \alpha$, $v = \beta$ at $\pm\infty$ but $v(s)$ is not obliged to be strictly monotone everywhere. Those are generalized kinks. In the case $l_1, l_2 : \alpha > 0$. The definition of soliton is standard.

One can easily see that

$$lim_{y\to-\infty} u|_{l_1} = \frac{2k_1^2 e^{A_{12}}}{(1 + e^{A_{12}})^2} = \alpha_1$$

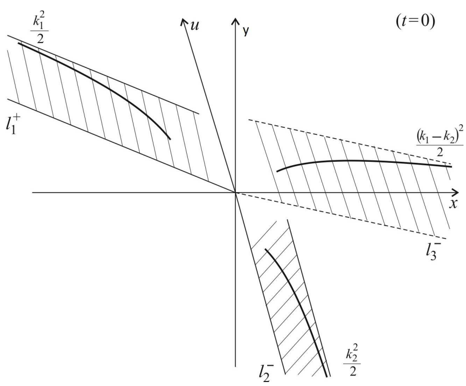
$$lim_{y\to+\infty} u|_{l_1} = \frac{k_1^2}{2} = \beta_1$$

$$lim_{y\to+\infty} u|_{l_2} = 2\frac{k_2^2 e^{A_{12}}}{(1+e^{A_{12}})^2} = \beta$$

$$lim_{y\to-\infty} u|_{l_2} = \frac{k_2^2}{2} = \alpha$$

X waves are formed by $u|_{l_1}$ and $u|_{l_2}$.

In the resonance case a new wave appears, namely $l_3 : x = -\alpha_0 y$, $\alpha_0 = p_1 + k_2\sqrt{3}$, while $l_1 : x + p_1 y = 0$, $l_2 : x + p_2 y = 0$. More precisely, the linear functions $\eta_1 = \eta_2 \iff x + \alpha_0 y = 0$, i.e. $\eta_1|_{l_3} = \eta_2|_{l_3}$. We put $l_1^+ = l_1 \cap \{y \geq 0\}$, $l_{2,3}^- = l_{2,3} \cap \{y \leq 0\}$. Then $u > 0$, $u|_{l_1}$, $u|_{l_2}$, $u|_{l_3}$ are kinks with a horizontal asymptote at $u = \alpha = 0$ and the second one at β_j, $j = 1, 2, 3$. Moreover, $lim_{y\to\infty} u|_{l_1} = \frac{k_1^2}{2} = \beta_1$, $lim_{y\to-\infty} u|_{l_2} = \frac{k_2^2}{2} = \beta_2$, while $lim_{y\to-\infty} u|_{l_3} = \frac{(k_1-k_2)^2}{2} = \beta_3$. In other words, the resonance gives rise of a new born wave kink with a maximal amplitude $\frac{(k_1-k_2)^2}{2} = \beta_3$, $\beta_3 < \beta_1$ but $\beta_3 < \beta_2 \iff k_1 < 2k_2$. If $0 \in l \neq l_1, l_2, l_3$ is a straight line in $0xy$ then $u|_l$ is a soliton. l_1^+, l_2^- and l_3^- form the configuration Y wave. Both X, Y waves can be observed in the oceans (even in the Mediterranean see) during lowtides. We propose below a geometrical interpretation of the Y wave and pictures of X, Y waves taken from Mediterranean see on May 25, 2014.



4. Solutions of first order linear and cubic nonlinear first order hyperbolic pseudodifferential equations in $\mathbf{R}^1_t \times \mathbf{R}^n_x, n \geq 2$

This section is devoted to the equations (18), (19), where:

$$(D_t - c|D_x|)u = f(t,x) \in D'(\mathbf{T}^1_t \times \mathbf{T}^n_x) \tag{18}$$

with a solution $u \in D'(\mathbf{T}^1_t \times \mathbf{T}^n_x)$. As usual $\mathbf{T}^n_x$ stands for the n-dimensional 2π torus, $c \in \mathbf{R}^1$.

The cubic nonlinear first order hyperbolic equation (19) is given by the formula:

$$(-D_t + |D_x|)u = u|u|^2 \tag{19}$$

with $x \in \mathbf{T}^n$, $t \in [0,T]$, $0 < T$ being possibly sufficiently small, $D_t = \frac{1}{i}\partial_t$. As we know each $L^2(\mathbf{T}^n_x)$ function $f(x)$ can be developed in Fourier series: $f \to \sum_{\alpha \in \mathbf{Z}^n} f_\alpha e^{i<\alpha,x>}$.

We define the Ψdo $|D_x|$ as follows:

$$|D_x|f = \sum_{\alpha \in \mathbf{Z}^n} a_\alpha |\alpha| e^{i<\alpha,x>},$$

the series being convergent in distribution sense in $D^{'}(\mathbf{T}^n)$. We introduce now the function

$$f_{\geq 0} = P_{\geq 0}(f) = \sum_{\alpha \in \mathbf{Z}_+^n} a\alpha e^{i<\alpha,x>}, \tag{20}$$

$\mathbf{Z}_+ = N \cup \{0\}$. A function u satisfying the equation

$$(-D_t + |D_x|)u_{\geq 0} = P_{\geq 0}(|u_{\geq 0}|^2 u_{\geq 0}) \tag{21}$$

is called Szegö solution of (19).

Having in mind that $f(x,t) = \sum_{(\tau,\alpha)\in \mathbf{Z}^{n+1}} a_{\tau,\alpha} e^{i(t\tau + <\alpha,x>)}$ if $f \in D^{'}(\mathbf{T}^{n+1})$ we look for a solution of (18) of the form $u = \sum_{(\tau,\alpha)\in \mathbf{Z}^{n+1}} u_{\tau,\alpha} e^{i(t\tau+<\alpha,x>)}$, i.e.

$$(\tau - c|\alpha|)u_{\tau,\alpha} = f_{\tau,\alpha}, \forall(\tau,\alpha) \in \mathbf{Z}^{n+1}. \tag{22}$$

Thus, $\tau_0 = c|\alpha_0|$ for some $(\tau_0, \alpha_0) \in \mathbf{Z}^{n+1} \Rightarrow f_{\tau_0,\alpha_0} = 0$, while $\tau \neq c|\alpha|$, $\forall(\tau,\alpha) \in \mathbf{Z}^{n+1} \Rightarrow u_{\tau,\alpha} = \frac{f_{\tau,\alpha}}{\tau - c|\alpha|}$. (18) is nonlovable in $D^{'}(\mathbf{T}^{n+1})$ if $\tau_0 = c|\alpha_0|$ but $f_{\tau_0,\alpha_0} \neq 0$; $c = \frac{\tau_0}{|\alpha_0|} \in Q \setminus 0 \iff |\alpha_0| \in \mathbf{N}$, $|\alpha_0| \notin \mathbf{N} \iff |\alpha_0| \notin Q$.

The operator $D_t - c|D|$ possesses an infinite dimensional kernel and is not $C^\infty(\mathbf{T}^{n+1})$ hypoelliptic if $\tau = c|\alpha|$ for infinitely many $(\tau,\alpha) \in \mathbf{Z}^{n+1}$. For example, $c = 1 \Rightarrow \tau = |\alpha| \to \tau^2 = |\alpha|^2$ and the Pytagorean numbers are infinitely many.

Assume now that $c^2 > 0$ satisfies the small denominators condition [2]:

$$|c^2 - \frac{p}{q}| \geq \frac{K}{|q|^{2+\sigma}} \tag{23}$$

for each $p, q \in \mathbf{Z} \setminus 0$ and for some $\sigma > 0$, $K = K(c^2, \sigma) > 0$. Let $c > 0$. Then (23) implies that $|\tau - c|\alpha|| \geq \frac{\tilde{K}}{(|\alpha|+|\tau|)^{2\sigma+3}}$, $(\tau,\alpha) \in \mathbf{Z}^{n+1} \setminus 0$, $\tilde{K} = \mathit{const} > 0$.

Proposition 2. *For almost all $c \in \mathbf{R}^1$ in the sense of Lebesgue measure the operator $D_t - c|D_x|$ is C^∞, analytic and Gevrey hypoelliptic on $\mathbf{T}^{n+1}$.*

The Cauchy problem for (18) with initial condition $u_0(x)$ can be easily studied in $D^{'}([0,T) \times \mathbf{T}_x^n)$ as then $f = \sum_\alpha f_\alpha(t) e^{i<\alpha,x>}$, $u = \sum_\alpha u_\alpha(t) e^{i<\alpha,x>}$, $u_0(x) = \sum u_{0\alpha} e^{i<\alpha,x>}$ and therefore

$$\left| \begin{array}{l} u_\alpha^{'}(t) - ic|\alpha|u_\alpha(t) = if_\alpha(t) \\ u_\alpha(0) = u_{0\alpha}. \end{array} \right. \tag{24}$$

Our second step is to investigate (21) (see [14,15]). We look for a solution having the form

$$u = \frac{a(t)}{1 - c(t)e^{i[<\alpha,x>+\beta t]}}, 0 \neq \alpha \in \mathbf{Z}^n, \beta \in \mathbf{R}^1. \tag{25}$$

If $|c(t)| < 1$ the geometric progression formula shows that $u = u_{\geq 0} = P_{\geq 0}(u)$. Put $z = e^{i[<\alpha,x>+\beta t]}$, $w = c(t)z$. From (25) we obtain that

$$i\frac{\partial u}{\partial t} = \frac{ia'}{1-w} + ia\frac{(c' + i\beta c)z}{(1-w)^2}, \tag{26}$$

where $a(t)$, $c(t)$, α and β are unknown, $|c| < 1$.

On the other hand,

$$|D_x|(\frac{1}{1-cz}) = \sum_{k=1}^{\infty} c^k(t)|D_x|(z^k) = \sum_{k=1}^{\infty} c^k(t)|\alpha|kz^k.$$

As $|w| = |c| < 1$ we have

$$\sum_{k=0}^{\infty} w^k = \frac{1}{1-w}, \sum_{k=1}^{\infty} kw^{k-1} = \frac{1}{(1-w)^2} \Rightarrow \sum_{k=1}^{\infty} kw^k = \frac{w}{(1-w)^2}.$$

This way we conclude that

$$|D_x|(\frac{a(t)}{1-w}) = |\alpha|\frac{aw}{(1-w)^2}. \tag{27}$$

Combining (26), (27) we get for $\beta = |\alpha|$:

$$i\frac{\partial u}{\partial t} + |D_x|u = \frac{ia'(t)}{1-w} + \frac{ia(t)c'(t)z}{(1-w)^2} = \tag{28}$$

$$= i(\frac{a}{c})' \frac{c}{1-cz} + i\frac{a}{c}\frac{c'}{(1-cz)^2}.$$

V.Georgiev, N.Tzvetkov and N.Visciglia have shown the following algebraic lemma [15].

Lemma. *Consider the function* $\frac{1}{1-cz}$, $|c| < 1$, $c \in \mathbf{C}^1$, $z = e^{i\Theta}$, $\Theta \in [0, 2\pi]$. *Then*

$$P_{\geq 0}((\frac{1}{1-cz})^2\frac{1}{1-\bar{c}\bar{z}}) = \frac{1-|c|^2cz}{(1-cz)^2(1-|c|^2)^2} = \frac{a|a|^2}{(1-|c|^2(1-cz)^2} + \frac{a|a|^2|c|^2}{(1-|c|^2)^2(1-cz)} =$$

$$= \frac{h(t)-h(\bar{c})}{z-\bar{c}}, h = \frac{z}{(1-cz)^2}.$$

Due to (28) and the Lemma equation (21 leads to the ODE system

$$(29)\qquad \left| \begin{array}{l} i(\frac{a}{c})' = \frac{a}{c}\frac{|a|^2|c|^2}{(1-|c|^2)^2} \\ ic' = \frac{c|a|^2}{1-|c|^2}. \end{array} \right.$$

Thus, with some $c_0 \in (0,1)$ the function $c(t) = c_0 e^{-ip_1 t}$, $p_1 = \frac{a_0^2}{1-c_0^2} > 0$, $c(0) = c_0$, $a(0) = a_0 \neq 0$ and $a(t) = a_0 e^{-ip_2 t}$, $p_2 = \frac{a_0^2}{(1-c_0^2)^2}$ satisfy (29).

Proposition 3. *For each $\alpha \in \mathbf{Z}^n_+$ and $\beta = |\alpha|$ the equation (21) possesses the solution*

$$u_{\geq 0} = \frac{a_0 e^{-ip_2 t}}{1 - c_0 e^{i[<\alpha,x>+t(|\alpha|-p_1)]}}.$$

5. Possible generalizations of Proposition 3

Consider the same equation (21) and look for a solution having the form

$$u_{\geq 0} = \sum_{m=1}^{N} \frac{a_m(t)}{1 - c_m(t)z}, z = e^{i[<\alpha,x>+\beta t]},$$

$$0 \neq \alpha \in \mathbf{Z}^n_+, \beta = |\alpha|, 0 < |c_1(0)| < |c_2(0)| < \ldots < |c_N(0)| < 1.$$

Then

$$(i\frac{\partial}{\partial t} + |D_x|)u_{\geq 0} = i\sum_{m=1}^{N}(\frac{a_m}{c_m})'\frac{c_m}{1-c_m z} + i\sum_{m=1}^{N}\frac{a_m}{c_m}\frac{c_m'}{(1-c_m z)^2}, |c_m(t)| < 1, 1 \leq m \leq N.$$

Evidently,

$$|u_{\geq 0}|^2 u_{\geq 0} = u_{\geq 0}^2 |\bar{u}_{\geq 0} = \sum_{j,k=1}^{N} \frac{a_j^2 \bar{a}_k}{(1-c_j z)^2(1-\bar{c}_k\bar{z})} + 2\sum_{1\leq j<k\leq N}\sum_{l=1}^{N}\frac{a_j a_k \bar{a}_l}{(1-c_j z)(1-c_k z)(1-\bar{c}_l\bar{z})}$$

Moreover, $c_j(0) \neq c_k(0)$ for $j < k$; for $l \neq j,k$ $c_k(0) \neq \bar{c}_l(0)$, $c_j(0) \neq \bar{c}_l(0)$. Certainly, we must find $P_{\geq 0}(|u_{\geq 0}|^2 u_{\geq 0})$. We observe that $P_{\geq 0}(\frac{1}{(1-c_j z)^2}\frac{1}{1-\bar{c}_k\bar{z}}) = P_{\geq 0}(\frac{z}{(1-c_j z)^2}\frac{1}{z-\bar{c}_k}) = \frac{f_j(z)-f_j(\bar{c}_k)}{z-\bar{c}_k}$, where $f_j(z) = \frac{z}{(1-c_j z)^2}$.

In fact, $\frac{1}{z-\bar{c}_k} = \sum_{p=0}^{\infty}\frac{\bar{c}_k^p}{z^{p+1}}$, on the other hand,

$$P_{\geq 0}(\frac{1}{1-c_j z}\frac{1}{1-c_k z}\frac{1}{1-\bar{c}_l\bar{z}}) =$$

$$P_{\geq 0}\left(\frac{z}{(1-c_jz)(1-c_kz)}\frac{1}{z-\bar{c}_l}\right) = \frac{g_{jk}(z)-g_{jk}(\bar{c}_l)}{z-\bar{c}_l},$$

where $g_{jk}(z) = \frac{z}{(1-c_jz)(1-c_kz)}, c_j \neq c_k$. Certainly, $\frac{z}{(1-c_jz)(1-c_kz)} - \frac{\bar{c}_l}{(1-c_j\bar{c}_l)(1-c_k\bar{c}_l)} = = \frac{z-\bar{c}_l}{c_j-c_k}[\frac{c_j}{(1-c_jz)(1-c_j\bar{c}_l)}) - \frac{c_k}{(1-c_kz)(1-c_k\bar{c}_l)}]$. As in the previous case, we compare the coefficients participating in the left hand side and in the right hand side of (21) and in front of $\frac{1}{1-c_mz}$, $\frac{1}{(1-c_mz)^2}$, $1 \leq m \leq N$.

The corresponding complex system of ODE takes the form:

$$\left| \begin{array}{ll} i(\frac{a_m}{c_m})' c_m = P_m(a,c,\bar{a},\bar{c}), & 1 \leq m \leq N \\ ic_m' \frac{a_m}{c_m} = Q_m(a,c,\bar{a},\bar{c}), & 1 \leq m \leq N, \end{array} \right.$$

Q_m, P_m being algebraic functions of the arguments $a = (a_1,\dots,a_N)$, $c = (c_1,\dots,c_N)$, $\bar{a} = (\bar{a}_1,\dots,\bar{a}_N)$.

Separating the real and imaginary parts of a_j, c_j, P_m, Q_m we obtain a real-valued system of 4N ODE in normal form with 4N unknown functions $Re\ a_j$, $Im\ a_j$, $Re\ c_j$, $Im\ c_j$. Taking the Cauchy data $c_j(0)$, $a_j(0)$ such that $0 < |c_1(0)| < \dots < |c_N(0)| < 1$, $a_j(0) \neq 0$. $1 \leq j \leq N$ we construct a local in t solution, i.e. $|t| \leq T$, $0 < T \ll 1$. Unfortunately, it does not have the elegant form proposed in Proposition 3. Put $\tilde{P}_N = \prod_{m=1}^N (z - \frac{1}{c_m(t)})$, $P_j(z,t) = \frac{\tilde{P}_N(z,t)}{z-\frac{1}{c_j}(t)}$, $\tilde{Q}(z,t) = (-1)^N \sum_{m=1}^N \frac{a_m(t)}{c_m(t)} P_j(z,t)$. Then the local in t solution $u_{\geq 0} = \frac{\tilde{Q}(z,t)}{\tilde{P}_N(z,t)}$, where $\tilde{Q}$, $\tilde{P}_N$ are polynomials in z of degrees $N-1$, N respectively having coefficients depending on $a_m(t)$, $c_m(t)$ or on $\frac{1}{c_m(t)}$, $1 \leq m \leq N$ only. Therefore, we can find rational solution $u_{\geq 0}$ of (21) and $z = e^{i[<\alpha,x>+|\alpha|t]}$, $0 \neq \alpha \in \mathbf{Z}_+^n$.

References

[1] R.F. Dressler, Mathematical solutions of the problem of roll waves in inlined open channels, *Comm. Pure Appl.Math.*,*2*,(1949), 140–190.

[2] V.I.Arnold, Additional chapters on the theory of ordinary differential equatios,Nauka, Moscow, 1978.

[3] G.B. Whitham, Linear and nonlinear waves, A Wiley Interscience Publication, J.Wiley and sons, 1974.

[4] M.R.Abbott, A theory of the propagation of bores in channels and rivers, *Proc. Camb. Phil.Soc.*, *52*, (1956), 344–362.

[5] R.Knobel, An introduction to the mathematical theory of waves, *Student Math.Library*, **vol.3**, 2000, AMS.

[6] B.Rozhdestvenskij, N.Yanenko, Systems of quasilinear equations and their applications to gas dynamics, *Transl. Math.Monographs*, *55*,AMS, Providence,RI, 1983.

[7] A.Goritskii, S.Krujkov, G.Chechkin, Quasilinear first order partial differential equations, Moscow State Univ. Lomonosov ed., Moscow, 1997.

[8] E.L.Aero, A.Buligin, Yu.Pavlov, Solutions of the 3 dimensional sin-Gordon equation, *Theoret. and Math.Physics*, *158:3*, (2009), 370–377.

[9] P.Popivanov, Geometrical methods for solving of fully nonlinear partial differential equations, Union of Bulg.Math. ed., Sofia, 2006.

[10] P.Popivanov, A.Slavova, Nonlinear Waves.An Introduction, *Series on Analysis, Appl. and Computation*,**vol.4**, World Scientific, New Jersey-Singapore, 2011.

[11] M.Ablowitz, H.Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.

[12] J.Satsuma, N-soliton solution of the two-dimensional KdV equation, *J.Phys.Soc.Japan*, *40*, (1976), 286–290.

[13] R.Hirota, Direct methods of finding exact solutions of nonlinear evolution equations in Bäcklund transformations, R.Miura ed., Lecture Notes in Math., *515*, Springer-Verlag, New York,1976.

[14] P.Gérard, S.Grellier, The cubic Szegö equation, *Ann.Sci.de L'ENS*, *43:5*, (2010), 761–810.

[15] V.Georgiev, N.Tzvetkov, N.Visciglia, On the continuity of the solution map for the cubic 1d periodic NLW equation, preprint, Univ. of Pisa, 2013.

[16] Solitons, Edited by R.Bullough and P.Caudrey, Springer-Verlag, Berlin-New York, 1980.

[17] H.Dwight, Tables of integrals and other mathematical data, The McMillan company, New York, 1961.

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