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Pliska Stud. Math. Bulgar. **23** (2014), 57–66

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NUMERICAL METHODS FOR DELAYED DIFFERENTIAL EQUATIONS WITH DISCONTINUITES

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Abstract. The numerical approximation of higher order are used to solve differential equations with discontinuous solutions and fixed time delay. The accuracy of these methods is investigated.

1. Preliminaries

Many real problems are described by impulsive differential equations, e.g. population dynamics, process in economy and nanoelectronic or electronic devises and so on.

The basic theory of the systems of impulsive differential equations can be found in [2, 16, 22], where the main qualitative properties of these systems are considered. The optimal control of discontinuous systems is studied in [25]. In the recent books [19, 20] single and multivalued discontinuous systems are studied, and the attention is paid also to averaging techniques for such systems. The papers [5, 13, 14] are devoted to monotone iterative technique for discontinuous systems.

The numerical approximation of discontinuous systems with fixed impulsive times is studied in [12, 18, 21]. The numerical approximation of time-varying discontinuous systems is more difficult. The first successful attempt is [4], where the first order approximation of multivalued discontinuous systems is investigated. In [3] Runge-Kutta methods for such systems are studied.

2000 *Mathematics Subject Classification:* 34A37
Key words: Impulsive differential equations, Runge-Kutta methods, delay.

The systems studied in this paper have the form:

(1) $\dot{x}(t) = f(t, x(t), x(t-\sigma))$ for $t \in I = [0, T],\ t \neq \tau_i,\ x(s) = \psi(s),\ s \in [-\sigma, 0]$,

(2) $\Delta x|_{t=\tau_i(x(t))} = S_i(x))\quad (i = 1, \dots, r)$.

Here $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function, $\tau_i : \mathbb{R}^n \to \mathbb{R}$ and $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous *switching surfaces* and *jump functions*, respectively. Assume for convenience that $\tau_0 = 0$.

The presence of delay makes the system more complicated. In general it is difficult to study numerical approximations of delay systems even in the case of absence of jumps, [6, 10, 15].

Recall that the piecewise continuously differentiable function $x(\cdot)$ is said to be a solution of (1) if:

a) $x(\cdot)$ is right continuously differentiable function satisfying (1), and for all $t \in I$ for which $\tau_i((t)) \neq t$, $i = 1, \dots, r$.

b) It possesses points of discontinuity (jumps) for all $t \in I$, such that $\tau_i(x) = t$, and defined in (2).

Note that in the case considered in [6], i.e., if the system with delay is continuous of classical type without being impulsive perturbed, then the solution is continuous. Thus it turns out that in the case of discontinuous derivatives the situation makes worse. There exist software like Maple, Mathematica, MatLab and so on, established to solve delay DEs of classical type, but the system with delay and jumps is different because the initial function may be discontinuous.

In the present paper we consider a method by which the problem under consideration can be solved effectively. To illustrate this we restrict our consideration only on discontinuous systems with fixed delay. Let assume that there exists continuous extension of the solution obtained with such a precision just as it is in the Runge-Kutta methods. We discuss the existence of such a continuous extension in Lemma 3 and Theorem 2. Our solving approach is characterized by Hermite polynomials being applied here although it is possible to use other known methods (cf. [6]).

Standing hypotheses (SH)

We suppose that $f(\cdot, \cdot)$ is sufficiently smooth (hence locally Lipschitz) with a growth condition, i.e. there exists a continuous function $v : I \times \mathbb{R}^+ \to \mathbb{R}^+$ with $|f(t, x)| \leq v(t, |x|)$ such that the maximal solution of $\dot{r} = v(t, r)$ exists on I for any initial condition $r(0) \geq 0$.

A1. $\tau_i(\cdot)$ are M-Lipschitz and $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are μ-Lipschitz.

A2. $\tau_i(x + S_i(x)) \neq \tau_j(x)$, $\forall\ j \neq i$ and $\forall\ x \in \mathbb{R}^n$.

Assume further that either **A3** or **A4** hold.

A3. $\tau_i(x) < \tau_{i+1}(x)$ for every $x \in \mathbb{R}^n$, and the following two conditions are satisfied:

1) There exists a constant $\alpha < 1$ such that $\langle \partial\tau_i(x), f(t,x,y)\rangle \le \alpha$ for $i = 1, \dots, r$, and for every $(t,x,y) \in I \times \mathbb{R}^n$, where the derivatives exist;

2) $\tau_i(x) \ge \tau_i(x + S_i(x))$.

A4. $\tau_i(x) > \tau_{i+1}(x)$ for every $x \in \mathbb{R}^n$ and the following two conditions are satisfied:

1) There exists a constant $\beta > 1$ such that $\langle \partial\tau_i(x), f(t,x,y)\rangle \ge \beta$ for $i = 1, \dots, r$ and for every $(t,x,y) \in I \times \mathbb{R}^n$, where the derivatives exist;

2) $\tau_i(x) \le \tau_i(x + S_i(x))$.

The multiple hitting of one switching surface is called **beating phenomena**. Furthermore, we show that if **SH** hold true, then the above stated phenomena should be impossible.

Lemma 1. *Let* **A1, A2** *and* ***A3*** *or* ***A4*** *hold, and let $x(\cdot)$ be a solution of (1)–(2). Then every equation $\tau_i(x(t)) = t$ admits no more than one solution.*

The proof is given in [4].

Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous function. Let $r(\cdot)$ be a real function with $h(\tau) = r(p) = 0$ $(\tau, p \in \mathbb{R})$, and $|h(t) - r(t)| \le \varepsilon$ for some $\varepsilon > 0$.

Proposition 1. *Let $\alpha > 0$, then we conclude that:*

(i) if $(h(t) - h(s))(t - s) \le -\alpha(t - s)^2$, then $|\tau - p| \le \dfrac{\varepsilon}{\alpha}$;

(ii) if $(t - s)(h(t) - h(s)) \ge \alpha(t - s)^2$, then again $|\tau - p| \le \dfrac{\varepsilon}{\alpha}$.

2. Runge-Kutta approximation of the solution

In this section we study discrete approximation of the impulsive system (1) with the Runge-Kutta scheme stated below. We refer the reader to [8] for general theory and to [6] for Runge Kutta method for delay (nonimpulsive) differential equations.

Given a natural number N, set $h = \dfrac{1}{N}$ and let $t_j = jh$ be a uniform grid on $[0, 1]$, where $j = 0, 1, \dots, N$. An *s-stage Runge-Kutta method* (RK method) computes iteratively the solution for the system (1) without jumps using the following relations:

$$\eta_h(t_{j+1}) = \eta_h(t_j) + h\sum_{\nu=1}^{s} b_\nu k_\nu, \tag{3}$$

$$k_\nu = f\left(t_j + c_\nu h, \eta_h(t_j) + h\sum_{l=1}^{\nu} a_{\nu l}k_l, \nu_h(t_j + c_l t - \sigma)\right) \quad (\nu = 1, \dots, s). \tag{4}$$

The RK method is *accurate up to order* p, if it provides the exact approximation of a polynomial solution $x(\cdot)$ up to degree p. It is known that the grid function $\eta_h(\cdot)$ of the RK approximation satisfies

$$\max_{j=0,\dots,N} \|\eta_h(t_j) - x(t_j)\| \le Ch^p, \tag{5}$$

under appropriate smoothness conditions on the right-hand side of the DDE and with suitable choice of the coefficients b_ν, c_ν, $a_{\nu,l}$ in (3)–(4). Here $x(\cdot)$ is the solution of (1) without jumps.

Notice however, that due to the delay terms we have discontinuity of the derivatives of the solution $x(\cdot)$ (see eg. [6]). These points must be included in the grid points. We first include in the grid the points $\sigma, 2\sigma, \dots, k\sigma$ where either $k = p$ or $k\sigma > T$. If we find an impulsive point τ, then we include in the grid $\tau, \tau + \sigma, \tau + 2\sigma$, etc.

Furthermore, since we have delay term, then we must know the value of the solution not only on the grid point, but for every t. There are different methods to solve delay differential equations and we refer the reader to [6] for the theory. In this paper we extend the solutions with their Hermite polynomials of degree 3 to the case when the method is of the second order or of degree 5 when it is of fourth order.

The third order Hermite polynomial is defined for every coordinate x^k of the solution x on (t_j, t_{J+1}) w.r.t. the values $H^k(t_j) = x^k(t_j)$. $\dot{H}^k(t_j) = f_k(t_j, x(t_j))$, $H^k(t_{j+1} = x^k(t_{j+1})$ and $\dot{H}^k(t_{j+1}) = f_k(t_{j+1}, x(t_{j+1}))$.

Analogously Hermite polynomial of degree 5 is defined by using two successive intervals (t_{2k}, t_{2k+1}) and (t_{2k+1}, t_{2k+2}). The reader can consult [15] for details.

We will now apply the Runge-Kutta method to discontinuous systems and set then

$$\begin{aligned} \varphi_i(t) &\equiv \tau_i(x(t)) - t, \quad (t \in [0, T]), \\ \varphi_{i,h}(t_j) &\equiv \tau_i(\eta_h(t_j)) - t_j, \quad (j = 0, 1, \dots, N). \end{aligned}$$

Calculate for this purpose some approximations by RK method to the differential system in (1) and for subsequent grid points t_j, $j = 1, \dots, N$. On each interval

$[t_j, t_{j+1}]$ we check, whether one of the functions $\varphi_{i,h}(\cdot)$ changes its sign. If it does (for some i), then the discrete trajectory $\eta_h(\cdot)$ needs to jump within the interval (t_j, t_{j+1}) which is close to the i-th jump of the exact solution $x(\cdot)$.

Although Runge–Kutta method provides the values of approximate solutions only on the grid points, we will consider the approximate solutions as it is defined on the whole interval $[0, T]$ with unknown values outside the grid. The following theorem hold:

Theorem 1. *Under (SH) the system (1)–(2) admits a unique solution defined on* $[0, T]$*. Furthermore, there exists a constant* K *such that the exact solution along with the approximate solutions are* K *Lipschitzean on the intervals of continuity. Moreover, there exists a constant* $\lambda > 0$ *such that* $\tau_{i+1}(x(t)) - \tau_i(x(t)) \geq \lambda$ *(*$i = 1, 2, \dots, r-1$*) for every approximate solution* $x(\cdot)$*,* $t \in [0, T]$*.*

Existence and uniqueness of the solution is proved in [7], and the existence of λ is proved in [4].

Let us discuss some strategies to find the approximate jump time $\tau^\star_{i,h} \in [t_j, t_{j+1}]$ of $\eta_h(\cdot)$, i. e. $\tau_i(\eta_h(\tau^\star_{i,h})) = \tau^\star_{i,h}$, where $\tau_i(\eta_h(\cdot))$ is the i-th jump point of $\eta_h(\cdot)$.

This problem is studied in the literature in the case of numerical approximation of nonsmooth systems (events location). We refer to [1], where a review of these studies was presented. Such a problem arises in case of delayed systems (see [6] for instance). Notice also [11, 24].

One interesting approach used first in [9] and developed in [23] is to associate to (1)–(2) also the equation

$$\dot{\varphi}(t) = \langle \nabla \tau_i(x(t)), f(t, x(t), x(t-\sigma)) \rangle - 1.$$

Next we start with τ_i and verify the function $\varphi_i(0)$. In the case **A3** it should be the smallest i for which $\varphi_i(0) > 0$, if $\varphi_i(0) < 0$ then $\varphi_i(t) < 0$ on I. In the case of **A4** we start with the smallest i for which $\varphi_i(t) < 0$. Then find j with $\varphi_i(t_j)\varphi_i(t_{j+1}) < 0$, and $\tau_i(x) \in (t_j, t_{j+1})$. Further, we use the polynomial extension of the approximate solution (in fact $\varphi_i(\cdot)$), and then solve $\varphi_i(t) = 0$ in order to find approximate τ_i.

We may use also successively *Piecewise linear interpolation:*, *Newton method interpolation:* or *Hermite approximation:* depending on the smoothness of $\tau_i(\cdot)$.

Remark. In practice we use *combined strategies*, depending on the smoothness of $\tau(\cdot)$. For instance use strategy A and with t_j, approximated τ_i, and t_{j+1} define Hermite polynomial of degree 5. It is possible to use also C and then starting from the approximated τ_i make iteratively use of B.

Notice that the approximate jump times τ_i (founded with some strategy) and also $\tau_i + \sigma$, $\tau_i + 2\sigma, \ldots,$ $\tau_i + p\sigma$ are included in the grid points, because the k-th derivative of the solution at $\tau_i + k\sigma$ is discontinuous.

Furthermore, the strategies B, C are applicable in the case when $\tau_j(\cdot)$ is sufficiently smooth.

On every strategy the approximate jump point is the zero of the approximate function. That motivates the stopping criterion resulting in the common estimate $\mathcal{O}(h^q)$ for all strategies, where $q \geq p + 1$.

Studying problem (1) we say that two solutions $x(\cdot)$ and $y(\cdot)$ are *in distance* $\rho(x(\cdot), y(\cdot)) \leq \varepsilon$ (see [4, Definition 1]), if they intersect successively the impulsive surfaces, i. e. $\tau_i(x) < \tau_{i+1}(y)$ or vice versa. Moreover, $\sum\limits_{i=1}^{r} \left(\tau_i^+ - \tau_i^-\right) < \varepsilon$ and $|x(t) - y(t)| < \varepsilon$ for every $t \in I \setminus \left(\bigcup\limits_{i=1}^{r} [\tau_i^-, \tau_i^+] \right)$. Here $\tau_i^- = \min\{\tau_i(x), \tau_i(y)\}$ and τ_i^+ is the maximal one.

Lemma 2. *Denote by $\tau_1^\star$ the first jump time of the solution $x(\cdot)$ and by $\tau_{1,h}^\star$ the jump time of $\eta_h(\cdot)$, and assume that $x(\cdot)$ is Lipschitz continuous on the interval $[a, \tau_1^\star] \subset I$. Under assumption **SH** with **A3** or **A4** we have for sufficiently small step sizes h that*

$$|\tau_1^\star - \tau_{1,h}^\star| \leq \frac{NC}{1-\alpha} \cdot h^p, \quad \textit{and} \quad |\tau_1^\star - \tau_{1,h}^\star| \leq \frac{NC}{\kappa - 1} \cdot h^p\,,$$

respectively.

Theorem 2. *Under assumption **SH** the measure of distance between the exact solution $y(\cdot)$ and the approximate solution $\eta_h(\cdot)$ is $O(h^p)$ for N being big enough.*

3. Numerical example

In numerical calculations we use the implicit Runge–Kutta methods. Our calculations are realized by Maple 17.

$$
\begin{aligned}
\dot{x}(t) = x + \frac{\cos^2(x-0.5)+\sin^2(y-0.5)}{2} - 0.1,\ t \in [0,1] \\
\dot{y}(t) = \frac{\sin^2(x-0.5)+\cos^2(y-0.5)}{2} + y + 0.1,\ t \neq \tau_i(x,y), \\
x(t) \equiv y(t) \equiv 0,\ t \in [-0.5,0], \\
\tau_1 :\ t = x+y-0.08,\ \Delta_1 = \left(\frac{\sin^2(x)}{10}; \frac{\cos^2(x)}{10}\right) \\
\tau_2 :\ t = x+y-1,\ \Delta_2 = \left(\frac{\cos^2(y)}{5}; \frac{\sin^2(y)}{5}\right)
\end{aligned}
$$

In this case we are not able to find the exact solutions, however, if we set $z = x+y$ then the system for z becomes: $\dot{z} = z+1$, $z(0) = 0$.

We use implicit Runge–Kutta method of order 4. Exact impulsive times are $\tau_1 = 0.3750188688$ and $\tau_2 = 0.9537208885$. The approximate jump points are $\tau_{1ap} = 0.3750188689$ and $\tau_{2ap} = 0.9537208886$. In the table we put the exact values of $z(t)$, and approximate values of $x(\cdot)$ and $y(\cdot)$. The error is $r(t) = |x_{ap}(t) + y_{ap}(t) - z(t)|$.

t	z_{ex}	x_{ap}	y_{ap}	$r(t)$
0.10	0.105170918	0.031551338	0.073619790	0.00000021
0.20	0.221402758	0.066421558	0.154983635	0.00000244
0.30	0.349858808	0.104957779	0.244901484	0.00000045
0.40	0.594354274	0.178306460	0.416048408	0.00000059
0.50	0.762033976	0.228610421	0.533424317	0.00000076
0.60	0.947348707	0.284204896	0.663144758	0.00000095
0.70	1.152153158	0.345646293	0.806508017	0.00000115
0.80	1.378497081	0.413549166	0.964948053	0.00000014
0.90	1.628645804	0.488594230	1.140053203	0.00000163
1.00	1.905102899	0.571531441	1.333573363	0.00000191

Table 1 – Exact and approximate values with error terms

Acknowledgement.

The research is supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0154.

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