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GLOBAL BEHAVIOR OF THE SOLUTIONS TO SIXTH ORDER BOUSSINESQ EQUATION WITH LINEAR RESTORING FORCE

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Abstract. Potential well method is established to sixth order Boussinesq equation with linear restoring force and subcritical initial energy. For supercritical initial energy finite time blow up of the solutions is proved under general structural conditions on the initial data. Numerical experiments, illustrating the theoretical results, are presented.

1. Introduction

We study the Cauchy problem for a sixth order Boussinesq equation with linear restoring force

$$(1)\qquad \beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} + \beta_3 u_{ttxxxx} + mu + \alpha(|u|^{p-1}u)_{xx} = 0,$$
$$x \in \mathbb{R}, \quad t \in [0,T),\ T \le \infty,$$

$$(2)\qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R},$$

where

$$(3)\qquad \begin{array}{l} \beta_1 \ge 0,\ \beta_2 > 0,\ \beta_3 > 0,\ m > 0,\ p \ge 2,\ \alpha > 0, \\ u_0 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}),\ u_1 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}), \end{array}$$

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and $(-\Delta)^{-s}u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$ for $s > 0$, $\mathcal{F}(u)$, $\mathcal{F}^{-1}(u)$ are the Fourier transformation and the inverse Fourier transformation, respectively.

Equation (1) describes the transverse deflections of an elastic rod on elastic foundation, see [1, 7, 8, 9]. The term $-u_{xx}$ is responsible for the tension acting along the axis of the rod which is subject to axial compression, while $\beta_1 u_{ttxx}$ represents the so-called rotational inertia. Equation (1) also occurs in the water wave problems with nonzero surface tension, see [10].

The propagation of solitary wave solutions of (1), (2) with $\beta_3 = 0$ is investigated numerically for the first time in [1, 2]. Theoretically problem (1), (2) with $\beta_3 = 0$ is studied in [6] by the potential well method for subcritical initial energy, $0 < E(0) < d$.

For a sixth order Boussinesq equation without linear restoring force, i.e. $\beta_3 > 0$ and $m = 0$, the long-time behaviour of the solutions to (1),(2) for small data is considered in [12, 13, 15]. In these papers the authors use the contraction mapping theorem. In [5, 11, 14], for $\beta_3 > 0$ and $m = 0$, global existence or finite time blow up of the weak solutions with subcritical or critical initial energy $E(0) \leq d$ is proved by means of the potential well method.

The aim of this paper is to fill up the lack of theoretical investigations of problem (1), (2). For this purpose we apply the potential well method and completely investigate the global behaviour of the solutions to Boussinesq equation (1), (2) for subcritical initial energy, $0 < E(0) < d$. In the case of supercritical energy, $E(0) > d$, we prove finite time blow up of the weak solutions under general structural conditions for the initial data.

The paper is organized in the following way. In Section 2 some preliminary definitions and results are given. In Section 3 we establish the potential well method for (1), (2). By means of this method global existence or finite time blow up of the solutions is proved when the initial energy is subcritical. Finite time blow up of the solutions to (1), (2) with supercritical initial energy is proved in Theorem 6 from Section 4. In Section 5 explicit choice of initial data with arbitrary high positive energy satisfying the conditions of Theorem 6 is given. The performed numerical experiment, supporting the theoretical results, are presented in Section 6.

2. Preliminaries

Throughout the paper we use the following short notations:

$$(u, v) = (u(\cdot, t), v(\cdot, t)) = \int_{\mathbb{R}} u(x, t)v(x, t)\, dx,$$

$$\|u\| = \|u(\cdot,t)\|_{\mathrm{L}^2(\mathbb{R})}, \quad \|u\|_1 = \|u(\cdot,t)\|_{\mathrm{H}^1(\mathbb{R})}.$$

In the space $\{u \in \mathrm{H}^1(\mathbb{R}) : (-\Delta)^{-1/2}u \in \mathrm{L}^2(\mathbb{R})\}$ we define the scalar product

$$\langle u, v\rangle = \langle u(\cdot,t), v(\cdot,t)\rangle = \beta_1(u,v) + \beta_2\left((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v\right) + \beta_3(u_x, v_x).$$

For simplicity we suppose that $m = 1$ in (1).

We need the following local existence result:

Theorem 1. (**Local existence**) *If* (3) *holds then problem* (1), (2) *admits a unique local solution* $u \in \mathrm{C}^1([0,T_m);\mathrm{H}^1(\mathbb{R}))$, *where* T_m *is the maximal existence time. Moreover, the conservation law* $E(t) = E(0)$ *holds for every* $t \in [0, T_m)$, *where*

$$E(t) := E(u(\cdot,t)) = \frac{1}{2}\left(\langle u_t, u_t\rangle + \|u\|_1^2 + \left\|(-\Delta)^{-1/2}u\right\|^2\right) - \frac{\alpha}{p+1}\int_{\mathbb{R}} |u|^{p+1}(x,t)\,dx. \tag{4}$$

The proof of Theorem 1 is similar to the proofs of Theorem 2.3 and Theorem 2.4 in [14] and we omit it.

Let us introduce some important definitions related to problem (1), (2):

- Nehari functional I: $\quad I(u) = \|u\|_1^2 + \left\|(-\Delta)^{-1/2}u\right\|^2 - \alpha\int_{\mathbb{R}} |u|^{p+1}\,dx$;
- Nehari manifold $\mathcal{N}$: $\quad \mathcal{N} = \{u \in \mathrm{H}^1(\mathbb{R}) :\ \|u\|_1 \neq 0,\ I(u) = 0\}$;
- critical energy constant d (mountain pass level) and functional J:

$$d = \inf_{u\in\mathcal{N}} J(u), \qquad J(u) = \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\left\|(-\Delta)^{-1/2}u\right\|^2 - \frac{\alpha}{p+1}\int_{\mathbb{R}} |u|^{p+1}\,dx.$$

When u depends on x and t, we use the short notations $I(u(t)) = I(u(\cdot,t))$ and $J(u(t)) = J(u(\cdot,t))$.

We give two auxiliary lemmas which deal with the properties of the functionals $I(u)$, $J(u)$ and the critical energy d.

Lemma 1. *Let* $u \in \mathrm{H}^1(\mathbb{R})$, $(-\Delta)^{-1/2}u \in \mathrm{L}^2(\mathbb{R})$ *and* $\|u\|_1 \neq 0$. *Then:*

(i) there exists a unique $\lambda^* = \lambda^*(u) \in (0,\infty)$ *such that for* $\lambda \in [0,\lambda^*]$ *function* $J(\lambda u)$ *is an increasing function of* λ; *for* $\lambda \in [\lambda^*,\infty)$ $J(\lambda u)$ *is a decreasing function of* λ *and* $J(\lambda u)$ *takes its maximum at* λ^*.

(ii) $I(\lambda u) > 0$ *for* $\lambda \in (0, \lambda^*)$*;* $I(\lambda u) < 0$ *for* $\lambda \in (\lambda^*, \infty)$ *and* $I(\lambda^* u) = 0$.

Lemma 2. *Let* $u \in \mathrm{H}^1(\mathbb{R})$, $(-\Delta)^{-1/2}u \in \mathrm{L}^2(\mathbb{R})$ *and* r_0 *be the constant defined as*

$$r_0 = \alpha^{-\frac{1}{p-1}}(C_p)^{-\frac{p+1}{p-1}} > 0 \quad \textit{with}$$

$$C_p = \sup_{v \in \mathrm{H}^1, \|v\|_1 \neq 0} \frac{\|v\|_{\mathrm{L}^{p+1}}}{\|v\|_1} = \frac{1}{\sqrt{2(p+1)}} \left((p-1)(p+3) \frac{\Gamma(\frac{4}{p-1})}{\Gamma^2(\frac{2}{p-1})} \right)^{\frac{p-1}{2(p+1)}}.$$

Then the following assertions hold:

(i) If $0 < \|u\|_1 < r_0$ *then* $I(u) > 0$*;*

(ii) If $I(u) < 0$ *then* $I(u) < (p+1)(J(u) - d)$*;*

(iii) If $I(u) = 0$ *and* $\|u\|_1 \neq 0$ *then* $\|u\|_1 \geq r_0$*;*

(iv) The following lower bound for the critical energy constant d *holds:*

$$d \geq d_0 = \frac{p-1}{2(p+1)} r_0^2.$$

The proofs of Lemma 1 and Lemma 2 are analogous to the proofs of Lemma 1 and Lemma 2 in [6] respectively and we omit them.

3. Potential well method

In the framework of the potential well method we introduce two important subsets of $\mathrm{H}^1(\mathbb{R})$ which are invariant under the flow of (1), (2):

$$W = \{u \in \mathrm{H}^1(\mathbb{R}) : \ I(u) > 0\} \cup \{0\}, \qquad V = \{u \in \mathrm{H}^1(\mathbb{R}) : \ I(u) < 0\}.$$

Theorem 2. (**Sign preserving property of $\mathbf{I(u(t))}$**) *Suppose* (3) *holds,* $E(0) < d$ *and* $u(x,t)$ *is the weak solution of* (1), (2)*. Then the following assertions hold:*

(i) If $u_0 \in W$ *then* $u(x,t) \in W$ *for every* $t \in [0, T_m)$*;*

(ii) If $u_0 \in V$ *then* $u(x,t) \in V$ *for every* $t \in [0, T_m)$.

As a consequence of Theorem 2 we have the following global existence and finite time blow up results.

Theorem 3. (Global existence) *Suppose* (3) *holds and* $E(0) < d$. *If* $I(u_0) > 0$ *or* $\|u_0\|_1 = 0$ *then problem* (1), (2) *has a unique global weak solution* $u(x,t)$ *defined for every* $t \in [0,\infty)$.

Theorem 4. (Finite time blow up) *Suppose* (3) *holds and* $E(0) < d$. *If* $I(u_0) < 0$ *then the weak solution* $u(x,t)$ *to problem* (1), (2) *blows up for a finite time.*

The proofs of Theorem 2, Theorem 3 and Theorem 4 follow the ideas of the proofs of the corresponding results in [6, 11, 14] and we omit them. In fact, global existence and finite time blow up of the solutions with subcritical initial energy for a fourth order Boussinesq equation with linear restoring force ($\beta_3 = 0$, $m > 0$) by the potential well method are proved in [6]. For a sixth order Boussinesq equation without linear restoring force ($\beta_3 > 0$, $m = 0$) results analogous to Theorem 2, Theorem 3 and Theorem 4 are obtained in [11, 14].

4. Finite time blow up for arbitrary high positive initial energy

As in Theorem 4 the finite time blow up of the solutions to (1), (2) with arbitrary high positive initial energy is based on the sign preserving properties of the Nehari functional $I(u(t))$. First we prove the following auxiliary lemma:

Lemma 3. *Suppose* (3) *holds and*

$$\langle u_0, u_1\rangle \geq 0. \tag{5}$$

Let $u(x,t)$ *be a weak solution of problem* (1), (2). *If* $I(u(t)) < 0$ *for every* $t \in [0,T]$, $T < T_m$, *then the functions* $\phi(t) = \langle u(t), u(t)\rangle$, $\phi'(t) = 2\langle u(t), u_t(t)\rangle$ *and* $h(t) = \frac{(\phi'(t))^2}{\phi(t)}$ *are strictly increasing ones in* $(0,T]$. *Moreover, the function* $\phi(t)$ *is strictly convex in* $(0,T]$ *and the inequality*

$$\langle u(t), u(t)\rangle \geq \langle u_0, u_0\rangle + 2t\langle u_0, u_1\rangle \tag{6}$$

holds for every $t \in [0,T]$.

P r o o f. Since $\phi''(t) = 2\langle u_t, u_t\rangle - 2I(u(t)) > 0$ for $t \in [0,T]$ it follows that $\phi(t)$ is a strictly convex function and $\phi'(t)$ is a strictly increasing one for $t \in [0,T]$. From (5) we get $\phi'(t) > 0$ for $t \in (0,T]$, i.e. $\phi(t)$ is a strictly increasing function of t. Inequality (6) is a consequence of the convexity of $\phi(t)$. For $h'(t)$ we get the inequality

$$\begin{aligned} h'(t) &= \frac{2\phi''(t)\phi'(t)\phi(t) - (\phi'(t))^3}{\phi^2(t)} = \frac{\phi'(t)}{\phi^2(t)} \left(2\phi''(t)\phi(t) - (\phi'(t))^2\right) \\ &= \frac{8\langle u, u_t\rangle}{\langle u, u\rangle^2} \left(\langle u, u\rangle^2 \langle u_t, u_t\rangle^2 - \langle u, u\rangle^2 I(u) - \langle u, u_t\rangle^2\right) > 0, \end{aligned}$$

i.e. $h(t)$ is a strictly increasing function for $t \in [0, T]$. $\square$

Theorem 5. (**Sign preserving property of $\mathbf{I(u(t))}$**) *Suppose* (3) *and* (5) *hold and* $m_0 = \left(\min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\beta_3}\right)\right)^{\frac{1}{2}}$. *If*

$$\frac{(m_0)^2}{2} \frac{(p-1)}{(p+1)} \langle u_0, u_0\rangle + \frac{1}{2} \frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} > E(0) > 0 \tag{7}$$

then $I(u(t)) < 0$ *for every* $t \in [0, T_m)$. *Moreover, if* $t_b = \frac{1}{2(m_0)^2} \frac{(p+1)}{(p-1)} \frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle} < T_m$ *then the following inequality*

$$I(u(t)) \leq -\frac{p+1}{2} \langle u_t, u_t\rangle \tag{8}$$

holds for every $t \in [t_b, T_m)$.

P r o o f. From the conservation law (4) we have

$$\frac{1}{p+1} I(u(t)) = E(0) - \frac{1}{2}\langle u_t, u_t\rangle - \frac{p-1}{2(p+1)} \left(\|u\|_1^2 + \left\|(-\Delta)^{-1/2} u\right\|^2\right). \tag{9}$$

By means of (7) and (9) we get the following chain of inequalities for $t = 0$

$$\begin{aligned} \frac{I(u(0))}{(p+1)} <& \frac{(m_0)^2}{2} \frac{(p-1)}{(p+1)} \langle u_0, u_0\rangle + \frac{1}{2} \frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{1}{2}\langle u_1, u_1\rangle \\ & - \frac{1}{2} \frac{(p-1)}{(p+1)} \left(\frac{\beta_1}{\beta_1} \|u_0\|^2 + \frac{\beta_3}{\beta_3} \|u_0'\|^2 + \frac{\beta_2}{\beta_2} \left\|(-\Delta)^{-1/2} u_0\right\|^2\right) \\ \leq& \frac{(m_0)^2}{2} \frac{(p-1)}{(p+1)} \langle u_0, u_0\rangle + \frac{1}{2} \frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{1}{2}\langle u_1, u_1\rangle - \frac{(m_0)^2}{2} \frac{(p-1)}{(p+1)} \langle u_0, u_0\rangle \\ =& -\frac{1}{2} \left\langle u_1 - \frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle} u_0,\ u_1 - \frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle} u_0 \right\rangle \leq 0 \end{aligned}$$

i.e. $I(u(0)) < 0$.

Let us suppose that there exists some $t_0 \in (0, T_m)$ such that $I(u(t)) < 0$ for every $t \in [0, t_0)$ and $I(u(t_0)) = 0$. From Lemma 3, (5), (7) and (9) we have the following impossible chain of inequalities:

$$
\begin{aligned}
0 = \frac{I(u(t_0))}{(p+1)} <& \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u(t_0), u(t_0)\rangle \\
&- \frac{1}{2}\langle u_t(t_0), u_t(t_0)\rangle \\
=& \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u(t_0), u(t_0)\rangle \\
&- \frac{1}{2}\left\langle u_t(t_0) - \frac{\langle u(t_0), u_t(t_0)\rangle}{\langle u(t_0), u(t_0)\rangle}u(t_0),\ u_t(t_0) - \frac{\langle u(t_0), u_t(t_0)\rangle}{\langle u(t_0), u(t_0)\rangle}u(t_0)\right\rangle \\
&- \frac{\langle u(t_0), u_t(t_0)\rangle^2}{\langle u(t_0), u(t_0)\rangle} + \frac{1}{2}\frac{\langle u(t_0), u_t(t_0)\rangle^2}{\langle u(t_0), u(t_0)\rangle} \\
\le& - \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\left(\langle u(t_0), u(t_0)\rangle - \langle u_0, u_0\rangle\right) \\
&- \frac{1}{2}\left(\frac{\langle u(t_0), u_t(t_0)\rangle^2}{\langle u(t_0), u(t_0)\rangle} - \frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\right) \le 0.
\end{aligned}
$$

Hence $I(u(t_0)) < 0$ which contradicts our assumption, i.e. $I(u(t)) < 0$ for every $t \in [0, T_m)$.

If $t_b < T_m$ then repeating the above calculations we obtain from Lemma 3, (6) and (9) that the inequalities

$$
\begin{aligned}
\frac{I(u(t))}{(p+1)} \le& \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{1}{2}\langle u_t(t), u_t(t)\rangle \\
&- \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u(t), u(t)\rangle \\
\le& \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{1}{2}\langle u_t(t), u_t(t)\rangle - (m_0)^2 t\frac{(p-1)}{(p+1)}\langle u_0, u_1\rangle \\
=& -\frac{1}{2}\langle u_0, u_1\rangle\left(2(m_0)^2\frac{(p-1)}{p+1}t - \frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle}\right) - \frac{1}{2}\langle u_t(t), u_t(t)\rangle \\
\le& -\frac{1}{2}\langle u_t(t), u_t(t)\rangle
\end{aligned}
$$

are valid for every $t \in [t_b, T_m)$. Thus Theorem 5 is proved. $\square$

As a consequence of the sign preserving properties of the Nehari functional $I(u(t))$ we have the following finite time blow up result.

Theorem 6. (Finite time blow up) *Suppose* (3) *holds,* $\langle u_0, u_1\rangle \geq 0$ *and* $m_0 = \left(\min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\beta_3}\right)\right)^{\frac{1}{2}}$. *If*

$$\frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} > E(0) > 0$$

then every weak solution of problem (1), (2) *blows up for a finite time* t_*. *If* $\langle u_0, u_1\rangle > 0$ *then*

$$\textit{either}\quad t_* \leq t_b = \frac{1}{2(m_0)^2}\frac{(p+1)}{(p-1)}\frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle} \quad \textit{or} \quad t_* \leq T_b = \frac{2\langle u(t_b), u(t_b)\rangle}{(p-1)\langle u(t_b), u_t(t_b)\rangle}.$$

Proof. Let us assume the contrary, i.e. $T_m = \infty$. From Theorem 5 and (8) it follows that for every $t \geq t_b$ the function $\phi(t) = \langle u(t), u(t)\rangle$ satisfies the inequalities

$$\phi''(t) = 2\langle u_t(t), u_t(t)\rangle - 2I(u(t)) \geq (p+3)\langle u_t(t), u_t(t)\rangle$$
$$\phi(t)\phi''(t) - \frac{p+3}{4}(\phi'(t))^2 \geq (p+3)\left(\langle u_t(t), u_t(t)\rangle\langle u(t), u(t)\rangle - \langle u(t), u_t(t)\rangle^2\right) \geq 0.$$

From Lemma 3 we have $\phi(t_b) = \langle u(t_b), u(t_b)\rangle > 0$, $\phi'(t_b) = 2\langle u(t_b), u_t(t_b)\rangle > 0$ when $t_b > 0$, i.e. $\langle u_0, u_1\rangle > 0$. Since $\frac{p+3}{4} > 1$ it follows from Lemma 1.1 in [3] that

$$\langle u(t), u(t)\rangle \to \infty \quad \text{for} \quad t \to t_*, \quad t_* \leq \frac{4\phi(t_b)}{(p-1)\phi'(t_b)} = \frac{2\langle u(t_b), u(t_b)\rangle}{(p-1)\langle u(t_b), u_t(t_b)\rangle}.$$

If $t_b = 0$, i.e. $\langle u_0, u_1\rangle = 0$, then the same argument holds if t_b is replaced by some $t_0 > 0$. This contradicts our assumption and Theorem 6 is proved. $\square$

Remark. The comparison between Theorem 4 and Theorem 6 shows that for positive subcritical initial data, i.e. $0 < E(0) < d$, Theorem 4 gives a better result than the one of Theorem 6. Indeed, if the conditions of Theorem 6 hold then $I(u_0) < 0$ and hence the conditions of Theorem 4 are also satisfied. The main advantage of Theorem 6 is the validity of the blow up result not only for subcritical initial energy but also for arbitrary high positive initial energy.

5. Choice of initial data

In this section we choose explicitly initial data with arbitrary high positive energy and satisfying all conditions of Theorem 6.

Let w, v be arbitrary $\mathrm{H}^2(\mathbb{R})$ functions such that

$$\|w\|_{\mathrm{H}^2(\mathbb{R})} \neq 0, \quad \|v\|_{\mathrm{H}^2(\mathbb{R})} \neq 0, \quad (w,v) = 0, \quad (w',v') = 0, \quad (w'',v'') = 0.$$

For example, a possible choice of w, v is when w is an even function and v is an odd one.

We define the initial data in the following way:

$$u_0(x) = \frac{r}{\sigma}(w(\sigma x))'_x, \quad u_1(x) = \frac{r}{\sigma}(qw(\sigma x) + \mu v(\sigma x))'_x, \tag{10}$$

where the constants $r > 0$, $\sigma > 0$, $q \geq 0$ and $\mu > 0$ will be chosen below.

Straightforward computations give us the following formula for the energy $E(0)$ in terms of norms of w and v:

$$E(0) = \frac{r^2}{2\sigma}\left(\mu^2\left(\beta_1\|v'\|^2 + \frac{\beta_2}{\sigma^2}\|v\|^2 + \beta_3\sigma^2\|v''\|^2\right) - R(\sigma,r,q)\right), \quad \text{where}$$

$$\begin{aligned} R(\sigma,r,q) = \frac{2\alpha r^{p-1}}{p+1}\int_{\mathbb{R}} |w'|^{p+1}\,dx - q^2\left(\beta_1\|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3\sigma^2\|w''\|^2\right)& \\ - \|w'\|^2 - \frac{1}{\sigma^2}\|w\|^2 - \sigma^2\|w''\|^2.& \end{aligned}$$

It is clear that initial data (10) satisfy conditions (3) and $\langle u_0, u_1\rangle \geq 0$. One has to choose constants r, σ, q and μ so that the inequalities

$$K \leq E(0) < \frac{(m_0)^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} \tag{11}$$

hold for arbitrary positive fixed constant K and $m_0 = \left(\min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\beta_3}\right)\right)^{\frac{1}{2}}$. Inequality (11) is equivalent to

$$\begin{aligned} \frac{2\sigma K}{r^2} &\leq \mu^2\left(\beta_1\|v'\|^2 + \frac{\beta_2}{\sigma^2}\|v\|^2 + \beta_3\sigma^2\|v''\|^2\right) - R(\sigma,r,q) \\ &< \left((m_0)^2\frac{p-1}{p+1} + q^2\right)\left(\beta_1\|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3\sigma^2\|w''\|^2\right). \end{aligned} \tag{12}$$

We propose the following algorithm for determining the constants r, σ, q and μ:

Step 1: Choose arbitrary constants $\sigma > 0$, $q \geq 0$ and $K > 0$.

Step 2: Fix constant r so that

$$r \geq \max\{r_0, r_1\}, \qquad \text{where}$$

$$r_0 = (2\sigma K)^{\frac{1}{2}} \left((m_0)^2 \frac{p-1}{p+1} + q^2\right)^{-\frac{1}{2}} \left(\beta_1 \|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3 \sigma^2 \|w''\|^2\right)^{-\frac{1}{2}},$$

$$r_1 = \left(q^2 \left(\beta_1 \|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3 \sigma^2 \|w''\|^2\right) + \|w'\|^2 + \frac{1}{\sigma^2}\|w\|^2 \right.$$
$$\left. + \sigma^2 \|w''\|^2\right)^{\frac{1}{p-1}} \times \left(\frac{2\alpha}{p+1}\int_{\mathbb{R}} |w'|^{p+1}\, dx\right)^{-\frac{1}{p-1}}.$$

This choice of r guarantees that

$$\frac{2\sigma K}{r^2} < \left((m_0)^2 \frac{p-1}{p+1} + q^2\right) \left(\beta_1 \|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3 \sigma^2 \|w''\|^2\right)$$

and that $R(\sigma, q, r) > 0$.

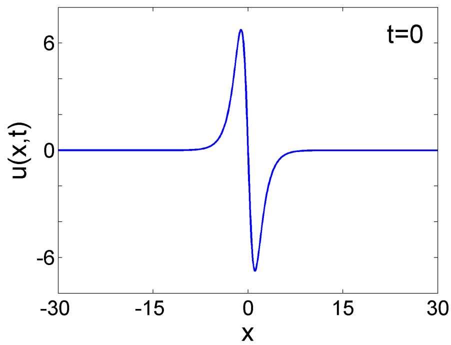
Step 3: Choose constant μ such that $\mu \in [\mu_0, \mu_1)$, where

$$\mu_0 = \left(\frac{2\sigma K}{r^2} + R(\sigma, q, r)\right)^{\frac{1}{2}} \left(\beta_1 \|v'\|^2 + \frac{\beta_2}{\sigma^2}\|v\|^2 + \beta_3 \sigma^2 \|v''\|^2\right)^{-\frac{1}{2}},$$

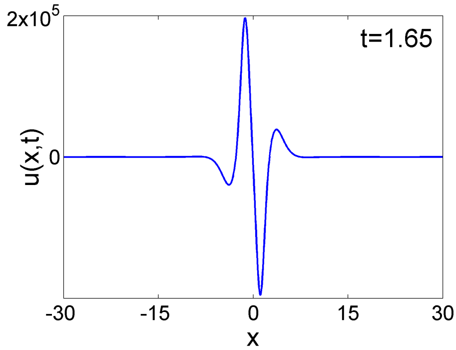
$$\mu_1 = \left(\beta_1 \|v'\|^2 + \frac{\beta_2}{\sigma^2}\|v\|^2 + \beta_3 \sigma^2 \|v''\|^2\right)^{-\frac{1}{2}} \times$$
$$\left(\left((m_0)^2 \frac{p-1}{p+1} + q^2\right) \left(\beta_1 \|w'\|^2 + \frac{\beta_2}{\sigma^2}\|w\|^2 + \beta_3 \sigma^2 \|w''\|^2\right) + R(\sigma, q, r)\right)^{\frac{1}{2}}.$$

Let us note that if we set $\mu = \mu_0$ in (12) then the lhs of this condition becomes an equality, while for $\mu = \mu_1$ the rhs of (12) becomes an equality. Hence for $\mu \in [\mu_0, \mu_1)$ condition (12) is satisfied.

Moreover, it is easy to check that $E(0) = K$ for $\mu = \mu_0$ and $E(0) > K$ for $\mu \in (\mu_0, \mu_1)$. In this way we find a wide class of initial data (10) with arbitrary high positive energy K which satisfy all conditions of Theorem 6.



(a)



(b)

Figure 1 – Profiles of the numerical solution of (1), (10) with $\sigma = 0.8$, $r = 13.5$, $q = 0.3$, $\mu = 0.15$ computed at evolution times: (a) – $t = 0$; (b) – t=1.65; $\widetilde{t}_* = 1.66$.

6. Numerical experiments

The aim of the presented numerical experiments is to illustrate the blow up result from Theorem 6 applying the algorithm for choosing of initial data proposed in Section 5.

We solve equation (1) with initial data (10), where

$$w(x) = \frac{1}{\cosh(x)}, \qquad v(x) = w'(x) = -\frac{\sinh(x)}{\cosh^2(x)}.$$

Let us note that $w(x)$ is an even function, while $v(x)$ is an odd one.

For the numerical solution of problem (1), (10) we use conservative, implicit with respect to the nonlinearity, finite difference schemes. These schemes are modifications of the numerical schemes proposed and studied in [5] (for case $m = 0$) and in [6] (for case $\beta_3 = 0$). Following the ideas and technique from [4] we can prove that these schemes have second order of convergence in space and time. In a similar to [4] way we introduce a discrete energy functional $E_h(v_i^n)$, where v_i^n is a discrete approximation to u on a regular mesh. This discrete functional approximates the energy functional $E(t)$ in (4) and the discrete energy $E_h(v_i^n)$ is conserved in time.

Numerical experiments are performed for $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 1$, $m = 1$, $\alpha = 2$ and $p = 2$. In order to demonstrate the validity of Theorem 6 for supercritical initial energy, i.e. $E(0) > d$, we need an upper bound for d. Since $d = \inf_{u\in\mathcal{N}} J(u)$, we may take for an upper bound of d the value $d^0 = J(z)$ for an arbitrary function

$z \in \mathcal{N}$. Here we find out that $z(x) = \delta u_0(0.8x)$ with $\delta \approx 2.8941$ belongs to $\mathcal{N}$ and $d^0 = \widetilde{J}(z) \approx 6.7332$. That is why we have to choose the constant r, σ, q and μ not only to satisfy conditions of Theorem 6 but also to ensure that $E(0) > d^0 \geq d$.

Following the algorithm proposed in Section 5 we chose:

Step 1: $\sigma = 0.8$, $q = 0.3$ and $K = 7$.

Step 2: $r = 13.5 \geq \max\{r_0, r_1\}, \quad r_0 \approx 2.4552, \quad r_1 \approx 12.4550.$

Step 3: $\mu = 0.15 \in [\mu_0, \mu_1), \quad \mu_0 \approx 0.1416, \quad \mu_1 \approx 0.6969.$

In that way we construct initial data satisfying all conditions of Theorem 6 with discrete initial energy $E_h(0) \approx 8.0803 > K$. Profile of the numerical solution at $t = 0$ is presented on Fig. 1(a) whereas Fig. 1(b) shows the typical blow up profile of the numerical solution at time $t = 1.65$ which is very close to the computed blow up time $\widetilde{t}_* = 1.66$. One can see that the behaviour of the numerical solution is fully consistent with the statements of Theorem 6. The computed blow up time $\widetilde{t}_* = 1.66$ is greater then $t_b = \frac{3}{2}q = 0.45$ but is bounded above by the computed value of T_b, $\widetilde{T}_b = 4.3086$ (see Theorem 6).

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