PLISKA
STUDIA MATHEMATICA

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ASYMPTOTIC BEHAVIOUR OF Y-LINKED GENES THROUGH BISEXUAL BRANCHING PROCESSES FOR GENETIC BALANCED SEX DETERMINATION

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The limiting genotype behaviour of Y-linked genes is studied in a two-sex monogamous population, where the sex designation is balanced. To this end, a multitype bisexual branching process is considered to model the evolution of the numbers of females and males of each genotype. It is assumed perfect fidelity mating with preference of females for males carrying certain allele of the gene. From this model, conditions for having positive probability of coexistence are investigated. Moreover, genotype growth rates on the coexistence event are established. Hence, the dominant genotype is found. Finally, the main results are illustrated by means of a simulated example.

1. Introduction

In last decades, mathematical population genetics is of increasing interest. Branching processes have a long history in population genetics theory (see, for example, [10]–[13]), mainly, motivated by a serious limitation in some cases of the classical Wright-Fisher model, which is the assumption that population size is constant. In recent years, two new stochastic models ranging over the field of branching

processes have been introduced to analyze the evolution of characters associated to genes linked to Y chromosome (see [7] and [9]). Both models describe the evolution of the number of carriers of two alleles (one implying the absence of the other) of a Y-linked gene in a two-sex monogamic population. The so-called Y-linked bisexual branching process with preference was introduced in [7]. In this model it is assumed that characters controlled by a gene may have some influence on the mating process of the species, in the sense that females have preference for males carrying one of the alleles of the gene. On the other hand, in [9], females were considered to choose their mates without caring about their genotypes since some Y-linked characters are not decisive at the time of mating. It is done by introducing the Y-linked bisexual branching process with blind choice. A more general and complex stochastic process, in which the mutation of the gene is considered, has been recently introduced in [5].

For these models, it has been developed the probabilistic theory to determine conditions for the extinction/survival of Y-linked genes as well as for their asymptotic rate of growth in the population (see [1] and [7]-[9]). Moreover, different approaches to inferential problems arising from these models can be found in [3], [4] and [6].

It has been found in those previous studies that the probability of being born female has, among other model parameters, the greatest influence on asymptotic behaviour. In particular, the balanced sex case, i.e. when the probability of being female is equal to the probability of being male, raises special difficulties. This situation has been solved by applying complex mathematical tools from the theory of branching processes (see [2], Chapter XI), for the Y-linked bisexual branching process with blind choice (see [1]). However, this problem has not been considered yet for the Y-linked bisexual branching process with preference. Hence, it is the aim of this paper to deal with it.

Apart from this introduction, the paper is organized as follows. The definition of the Y-linked bisexual branching process with preference is provided in Section 2. In the case of balanced sex designation, sufficient conditions to have a positive probability of coexistence together with the asymptotic rates on this event, are tackled in Section 3. Finally, in Section 4 the proofs of the results are provided.

2. The probability model

The model considered here was introduced in [7] to analyze the evolution generation-by-generation of the number of carriers of two alleles, labelled by R and r (one can mean the absence of the other), of a Y-linked gene in a two-sex monogamous population where the gene has some influence on the mating process

of the species, with females having a preference for males carrying the R alleles. Next, we give the mathematical definition.

Let $\{(FR_{ni}, MR_{ni}) : i = 1, 2, \ldots; n = 0, 1, \ldots\}$ and $\{(Fr_{nj}, Mr_{nj}) : j = 1, 2, \ldots; n = 0, 1, \ldots\}$ be two independent sequences of independent, identically distributed, non-negative and integer-valued bivariate random vectors on the same probability triple $(\Omega, \mathcal{F}, P)$. The sequences

$$\{(ZR_n, Zr_n)\}_{n \geq 0} \text{ and } \{(F_{n+1}, MR_{n+1}, Mr_{n+1})\}_{n \geq 0}$$

are defined recursively, for each $n \geq 0$, as follows:

$$(ZR_0, Zr_0) = (a, b) \in \mathbb{N}_0^2,$$

$$F_{n+1} = \sum_{i=1}^{ZR_n} FR_{ni} + \sum_{j=1}^{Zr_n} Fr_{nj}, \ MR_{n+1} = \sum_{i=1}^{ZR_n} MR_{ni} \text{ and } Mr_{n+1} = \sum_{j=1}^{Zr_n} Mr_{nj},$$

assuming that $\sum_1^0 = 0$; and

$$\begin{aligned} ZR_{n+1} &= \min\{F_{n+1}, MR_{n+1}\} \text{ and} \\ Zr_{n+1} &= \min\{\max\{0, F_{n+1} - MR_{n+1}\}, Mr_{n+1}\}. \end{aligned}$$

The two-dimensional process $\{(ZR_n, Zr_n)\}_{n \geq 0}$ is called Y-linked bisexual branching process with preference.

The process $\{(ZR_n, Zr_n)\}_{n \geq 0}$ is a homogeneous two-type Markov chain. Intuitively, for n fixed, the random vector (ZR_n, Zr_n) represents the total number of couples of type R and r, respectively, at generation n, where the type of a couple is determined by the type of its male. To describe the evolution of the population from this generation on, two phases are considered: reproduction and mating.

In the reproduction phase, each couple, independently of the others, generates females and males of its type (mutation of the gene is not considered) according to some probability distribution depending on its type. So, (FR_{ni}, MR_{ni}) and (Fr_{nj}, Mr_{nj}) denote the total number of females and males given by the ith R-couple and the jth r-couple, respectively, at the generation n. Then, F_{n+1}, MR_{n+1} and Mr_{n+1} denote the total number of females, R- and r-males at generation $n + 1$.

In the mating phase, the total number of individuals in generation $n + 1$ is known, i.e. the random vector $(F_{n+1}, MR_{n+1}, Mr_{n+1})$, and the number of couples of each genotype formed in generation $n + 1$ is obtained taking into account that generations do not overlap, the assumption of perfect fidelity (monogamous

population) and that females prefer R-males as mates. Hence, since R-males are chosen first as mates and each individual mates with only one individual of the opposite sex provided that some of them are still available, the number of R-couples is the minimum between the total number of females and the total number of R-males. Then, females which do not mate with R-type males (if any) mate with r-males.

Since in nature, R- and r-couples may have differences in their reproductive abilities, in general, we allow for different reproduction laws for each genotype and also assume that these reproduction laws have finite means and variances. Let m_R and m_r denote the average number of offspring produced by the R- and r-couple, respectively. Furthermore, an offspring will be female with probability α, $0 < \alpha < 1$, and male with probability $1 - \alpha$. These sex designations are made independently following a binomial scheme among the offspring of any couple, and it is assumed that the genotype has no influence on the sex determination, so that α is the same for both genotypes. As a consequence of this reproduction scheme, we derive that the average number of females and males generated by an R-couple are αm_R and $(1 - \alpha)m_R$, respectively, while the respective values for an r-couple are αm_r and $(1 - \alpha)m_r$, respectively.

It is proved in [7] that each genotype shows the dual behaviour typical for branching processes and known as the *extinction-explosion* dichotomy. This means that the number of couples of any type is bound to undergo either extinction or indefinite growth. The events of extinction of the population and the fixation of a genotype events have been studied for the balanced sex case (see [7] and [8]). However, the event $A_{\infty,\infty} = \{ZR_n \to \infty, Zr_n \to \infty\}$, termed simultaneous survival of both genotypes or coexistence, has not been considered yet. The following section is devoted to the study of asymptotic genotype growth rates on the coexistence event for the balanced sex case.

3.   Growth rates on coexistence event

From now on, we write $P_{(i,j)}(\cdot)$ for $P(\cdot|(ZR_0, Zr_0) = (i,j))$, and we consider that the following assumption holds:

B. $\alpha = 0.5$ (balanced sex designation) and $1 < \alpha m_R < \alpha m_r$.

First, we give conditions for survival of both genotypes.

Theorem 1. *Let the initial states (i,j) be such that $i, j > 0$. If B holds, then* $P_{(i,j)}(A_{\infty,\infty}) > 0.$

Hence, the event of the simultaneous survival of both alleles has positive probability if the mean number of females per r-couple is greater than that of females per R-couple and both are greater than unity. Since the sex determination is balanced, then the mean number of females per couple is the same than that of males and therefore this statement is in keeping with the one obtained in [7] (see Theorem 4.6 therein) and in [8] (see Theorem 4.3 therein), when the probability for an offspring to be female is different from 0.5. Notice that since R-males are preferred by females and the sex designation is balanced, then the condition $m_R < m_r$ guarantees that there are enough females to mate with r-males on $A_{\infty,\infty}$.

Next, we consider the study of the limiting growth rates on the set of coexistence of both genotypes. The preference of R-males makes that we distinguish this study by genotypes.

3.1. R genotype

It has been shown in [8] (see proof of Theorem 5.1) that, on the set $A_{\infty,\infty}$, the total number of R-couples is almost surely (a.s.) the total number of R-males, which behaves as a standard Bienaymé-Galton-Watson process defined by the reproduction law of the R-males. Hence, applying the theory about the asymptotic evolution of the Bienaymé-Galton-Watson process (see [2], Proposition 1.3 and Theorem 2.1), the following result is derived.

Theorem 2. *If* B *holds, then there exists a nonnegative and finite random variable W_R, which is positive on $A_{\infty,\infty}$, such that a.s. on this event*

$$\lim_{n\to\infty} \frac{ZR_n}{(\alpha m_R)^n} = \lim_{n\to\infty} \frac{MR_n}{(\alpha m_R)^n} = W_R.$$

Intuitively, the total numbers of R-couples and R-males grow geometrically at the same rate, defined by the mean number of females generated by an R-couple (see Figure 1, left and middle). In fact, both sequences are the same.

3.2. r genotype

First we establish some preliminary results. The following one shows that, on a subset of the event of survival of both genotypes, the asymptotic growth rate of the number of R- and r-couples over one generation is determined by the average number of females generated by each type of couples, respectively. Moreover, the asymptotic behavior of the ratio between the total number of R-couples and the total number of r-couples is zero.

Proposition 1. *Let the initial states (i, j) be such that $i, j > 0$. If* B *holds, then there exists a set $A \subseteq A_{\infty, \infty}$, with $P_{(i,j)}(A) > 0$, such that, a.s. on A*

$$\lim_{n \to \infty} \frac{ZR_{n+1}}{ZR_n} = \alpha m_R \quad and \quad \lim_{n \to \infty} \frac{Zr_{n+1}}{Zr_n} = \alpha m_r,$$

and

$$\frac{ZR_n}{Zr_n} = O(a^n), \ as \ n \to \infty, \ with \ 0 < a < 1.$$

The asymptotic behavior of the ratio between the number of males, respectively females, and the number of couples in the previous generation on the same subset of the coexistence event, is given in the following result.

Proposition 2. *If* B *holds, then, for each $0 < \rho < 1/2$, a.s. on A, as $n \to \infty$,*

$$\frac{MR_{n+1}}{ZR_n} = \alpha m_R + O(ZR_n^{-\rho}), \quad \frac{Mr_{n+1}}{Zr_n} = \alpha m_r + O(Zr_n^{-\rho}),$$

and

$$\frac{F_{n+1}}{m_R ZR_n + m_r Zr_n} = \alpha + O(Z_n^{-\rho}),$$

with $Z_n = ZR_n + Zr_n$, $n \geq 1$, and A as in Proposition 1.

Then, we derive the asymptotic growth rate for the total number of r-couples and r-males, as well as for the total number of females.

Theorem 3. *If* B *holds, then there exists a nonnegative random variable W_r, such that*

$$\lim_{n \to \infty} \frac{Zr_n}{(\alpha m_r)^n} = \lim_{n \to \infty} \frac{Mr_n}{(\alpha m_r)^n} = \lim_{n \to \infty} \frac{F_n}{(\alpha m_r)^n} = W_r \ a.s. \ on \ A,$$

and $A \subseteq \{0 < W_r < \infty\}$, with A as in Proposition 1.

Intuitively speaking, the total numbers of r-couples, r-males and females grow geometrically at the same rate, defined by the mean number of females generated by an r-couple (see Figure 1, left and middle). Notice that a difference with R allele case, is that now the sequences of total number of r-couples and r-males could be different. Since $m_R < m_r$, then we derive that, in this situation, the r allele is the dominant one, in spite of the preference of R allele (see Figure 1, left and middle).

Finally, from Propositions 1 and 2, and Theorem 3, we derive the following result about the asymptotic behavior of the ratio between the total number of females and males, which shows the balanced sex designation (see Figure 1, right).
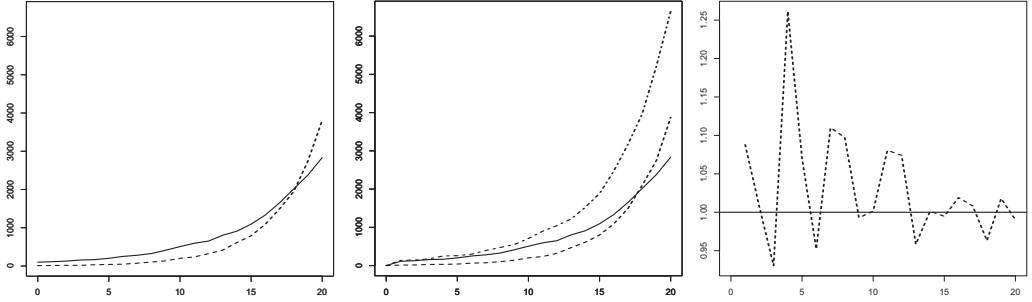
Figure 1: A simulated path from a Y-linked bisexual branching process with preference, where $ZR_0 = 100$ and $Zr_0 = 10$, $\alpha = 0.5$, and reproduction laws following Poisson distributions with parameters $m_R = 2.4$ and $m_r = 2.8$. Left: ZR_n (solid line) and Zr_n (dash line). Middle: MR_n (solid line), Mr_n (dash line) and F_n (dotdash line). Right: F_n/M_n (dash line).

Corollary 1. *If* B *holds, then*

$$\lim_{n \to \infty} \frac{F_n}{M_n} = 1 \quad a.s. \ \ on \ A,$$

with A as in Proposition 1 and $M_n = MR_n + Mr_n$, $n \geq 1$.

4. Proofs

In this section we provide the proofs of the results in the paper.

Proof of Theorem 1

First notice that, since every pair of states of the set $\{(i,j) : i,j > 0\}$ are communicating (see Proposition 3.1 in [7]), we can assume without loss of generality that the starting state (i,j), is such that $i \leq j$.

Let $\varepsilon > 0$ and $\eta_1 = \alpha(m_R - \varepsilon)$, $\eta_2 = \alpha(m_R + \varepsilon)$, $\eta_3 = \alpha(m_r - 3\varepsilon)$, $\eta_4 = \alpha(m_r + 3\varepsilon)$. One can fix ε small enough in order that $\eta_4 > \eta_3 > \eta_2 > \eta_1 > 1$.

Denote by $A_n = \{\eta_1 ZR_n < ZR_{n+1} < \eta_2 ZR_n, \eta_3 Zr_n < Zr_{n+1} < \eta_4 Zr_n\}$, for

all $n \geq 0$. Then one has that

$$
\begin{aligned}
P_{(i,j)}(A_{\infty,\infty}) &\geq P_{(i,j)}\left(\bigcap_{n=0}^{\infty}\{\eta_1 ZR_n < ZR_{n+1} < \eta_2 ZR_n, \eta_3 Zr_n < Zr_{n+1} < \eta_4 Zr_n\}\right) \\
&= \lim_{n\to\infty} P_{(i,j)}\left(\bigcap_{l=0}^{n} A_l\right) \\
&= \lim_{n\to\infty} P_{(i,j)}(A_0)\prod_{l=1}^{n} P_{(i,j)}\left(A_l \,\Bigg|\, \bigcap_{k=0}^{l-1} A_k\right).
\end{aligned}
$$
(1)

Since $\{(ZR_n, Zr_n)\}_{n\geq 0}$ satisfies the Markov property, one further infers that, for any $n \geq 1$,

$$
\begin{aligned}
P_{(i,j)}\left(A_n \,\Bigg|\, \bigcap_{k=0}^{n-1} A_k\right) &= P_{(i,j)}\left(A_n \,\Bigg|\, \bigcup_{i',j'>0}\{(ZR_n, Zr_n)=(i',j')\}\cap\bigcap_{k=0}^{n-1} A_k\right) \\
&\geq \inf_{\substack{\eta_1^n i < i' < \eta_2^n i \\ \eta_3^n j < j' < \eta_4^n j}} P_{(i,j)}\left(A_n \,\Bigg|\,\{(ZR_n, Zr_n)=(i',j')\}\cap\bigcap_{k=0}^{n-1} A_k\right) \\
&= \inf_{\substack{\eta_1^n i < i' < \eta_2^n i \\ \eta_3^n j < j' < \eta_4^n j}} P_{(i',j')}(A_0).
\end{aligned}
$$
(2)

Since $\eta_2 < \eta_3$ and $i \leq j$, we deduce that $\eta_2^n i < \eta_3^n j$ for any $n \geq 1$ and consequently $i' < j'$ for all i', j' such that $\eta_1^n i < i' < \eta_2^n i$ and $\eta_3^n j < j' < \eta_4^n j$. Therefore, a suitable lower positive bound for $P_{(i',j')}(A_0)$, with $i' < j'$, needs to be found in order to conclude that $P_{(i,j)}(A_{\infty,\infty}) > 0$. Towards this end, one first notes that

$$
\begin{aligned}
A_0^c &= \{ZR_1 \leq \eta_1 ZR_0\} \cup \{ZR_1 \geq \eta_2 ZR_0\} \cup \{Zr_1 \leq \eta_3 Zr_0\} \cup \{Zr_1 \geq \eta_4 Zr_0\} \\
&\subseteq \{ZR_1 \leq \eta_1 ZR_0, MR_1 > \eta_1 ZR_0, FR_1 > \eta_1 ZR_0\} \\
&\quad \cup \{ZR_1 \geq \eta_2 ZR_0, MR_1 < \eta_2 ZR_0\} \\
&\quad \cup \{Zr_1 \leq \eta_3 Zr_0, FR_1 > \eta_1 ZR_0, MR_1 < \eta_2 ZR_0, Fr_1 > \eta_5 Zr_0, Mr_1 > \eta_5 Zr_0\} \\
&\quad \cup \{Zr_1 \geq \eta_4 Zr_0, Mr_1 < \eta_4 Zr_0\} \\
&\quad \cup \{|MR_1 - \alpha m_R ZR_0| \geq \alpha\varepsilon ZR_0\} \cup \{FR_1 \leq \eta_1 ZR_0\} \\
&\quad \cup \{|Mr_1 - \alpha m_r Zr_0| \geq \alpha\varepsilon Zr_0\} \cup \{Fr_1 \leq \eta_5 Zr_0\},
\end{aligned}
$$
(3)

where $\eta_5 = \alpha(m_r - \varepsilon)$.

Since R-males are preferred by females at mate, then we deduce that, a.s. on the event $\{MR_1 > \eta_1 ZR_0, FR_1 > \eta_1 ZR_0\}$, $ZR_1 > \eta_1 ZR_0$, and therefore, we

have that

$$P_{(i',j')}(\{ZR_1 \leq \eta_1 ZR_0, MR_1 > \eta_1 ZR_0, FR_1 > \eta_1 ZR_0\}) = 0.$$

Moreover, $ZR_1 < \eta_2 ZR_0$ a.s. on the event $\{MR_1 < \eta_2 ZR_0\}$, and then

$$P_{(i',j')}(\{ZR_1 \geq \eta_2 ZR_0, MR_1 < \eta_2 ZR_0\}) = 0.$$

Similar arguments give

$$P_{(i',j')}(\{Zr_1 \geq \eta_4 Zr_0, Mr_1 < \eta_4 Zr_0\}) = 0.$$

Also, we deduce that

$$P_{(i',j')}(\{Zr_1 \leq \eta_3 Zr_0, FR_1 > \eta_1 ZR_0, MR_1 < \eta_2 ZR_0, Fr_1 > \eta_5 Zr_0, Mr_1 > \eta_5 Zr_0\}) = 0,$$

since $MR_1 - FR_1 < 2\varepsilon ZR_0$ a.s. on the set $\{FR_1 > \eta_1 ZR_0, MR_1 < \eta_2 ZR_0\}$ and $Fr_1 - (MR_1 - FR_1) > \eta_5 Zr_0 - 2\varepsilon ZR_0 > \eta_3 Zr_0$ a.s. on the set

$$\{FR_1 > \eta_1 ZR_0, MR_1 < \eta_2 ZR_0, Fr_1 > \eta_5 Zr_0, ZR_0 < Zr_0\},$$

and therefore $Zr_1 > \eta_3 Zr_0$ a.s. on the set

$$\{Zr_1 \leq \eta_3 Zr_0, FR_1 > \eta_1 ZR_0, MR_1 < \eta_2 ZR_0, Fr_1 > \eta_5 Zr_0, Mr_1 > \eta_5 Zr_0, ZR_0 < Zr_0\}.$$

Finally, since $\alpha = 1 - \alpha = 0.5$ and the reproduction laws are assumed to have finite variances, it follows with the help of Chebyshev's inequality that

$$P_{(i',j')}(|MR_1 - \alpha m_R ZR_0| \geq \alpha \varepsilon ZR_0) \quad = \quad P_{(i',j')}\left(\left|\sum_{k=1}^{i'}(MR_{k0} - (1-\alpha)m_R)\right| \geq \alpha \varepsilon i'\right)$$

$$(4) \qquad\qquad\qquad\qquad\qquad \leq \quad \frac{C_1}{i'},$$

for some positive constant C_1. Similar arguments give

$$P_{(i',j')}(FR_1 \leq \eta_1 ZR_0) \quad \leq \quad \frac{C_2}{i'}, \quad P_{(i',j')}(Fr_1 \leq \eta_5 Zr_0) \quad \leq \quad \frac{C_3}{j'}$$

and

$$(5) \qquad\qquad P_{(i',j')}(|Mr_1 - \alpha m_r Zr_0| \geq \alpha \varepsilon ZR_0) \quad \leq \quad \frac{C_4}{j'},$$

for suitable positive constants $C_2, C_3,$ and C_4.

By combining (3)–(5), for all $i' < j'$, one finds that

$$P_{(i',j')}(A_0) = 1 - P_{(i',j')}(A_0^c) \geq 1 - \frac{C_5}{i'} - \frac{C_6}{j'},$$

for some positive constants C_5 and C_6.

Since $\eta_1, \eta_3 > 1$, it finally follows from (1) and (2) that

$$
\begin{aligned}
P_{(i,j)}(A_{\infty,\infty}) &\geq P_{(i,j)}(A_0) \lim_{n\to\infty} \prod_{l=1}^{n} \inf_{\substack{\eta_1^l i < i' < \eta_2^l i \\ \eta_3^l j < j' < \eta_4^l j}} P_{(i',j')}(A_0) \\
&\geq P_{(i,j)}(A_0) \lim_{n\to\infty} \prod_{l=1}^{n} \left(1 - \frac{C_5}{\eta_1^l i} - \frac{C_6}{\eta_3^l j}\right) > 0,
\end{aligned}
$$

which completes the proof.

Proof of Proposition 1

First notice again that, since every pair of states of the set $\{(i,j) : i,j > 0\}$ are communicating (see Proposition 3.1 in [7]), we can assume without loss of generality that the starting state (i,j), is such that $i \leq j$.

Let $\varepsilon > 0$ and $\eta_1 = \alpha(m_R - \varepsilon)$, $\eta_2 = \alpha(m_R + \varepsilon)$, $\eta_3 = \alpha(m_r - 3\varepsilon)$, $\eta_4 = \alpha(m_r + 3\varepsilon)$ be such that $\eta_4 > \eta_3 > \eta_2 > \eta_1 > 1$. Also, for all $n \geq 0$, let $A_n = \{\eta_1 ZR_n < ZR_{n+1} < \eta_2 ZR_n, \eta_3 Zr_n < Zr_{n+1} < \eta_4 Zr_n\}$, and $A = \cap_{n=0}^{\infty} A_n$. It was shown in the proof of Theorem 1 that $A \subseteq A_{\infty,\infty}$ and $P_{(i,j)}(A) > 0$. By definition of A we derive that

$$\lim_{n\to\infty} \frac{ZR_{n+1}}{ZR_n} = \alpha m_R \quad \text{and} \quad \lim_{n\to\infty} \frac{Zr_{n+1}}{Zr_n} = \alpha m_r \quad \text{a.s. on } A.$$

Moreover, it is verified that, for $n \geq 1$, $\eta_1^n i < ZR_n < \eta_2^n i$ and $\eta_3^n j < Zr_n < \eta_4^n j$ a.s. on A. Consequently, since $\eta_2 < \eta_3$, then, we derive that,

$$0 \leq \liminf_{n\to\infty} \frac{ZR_n}{Zr_n} \leq \limsup_{n\to\infty} \frac{ZR_n}{Zr_n} \leq \lim_{n\to\infty} \frac{\eta_2^n i}{\eta_3^n j} = 0 \quad \text{a.s. on } A,$$

and therefore $ZR_n/Zr_n = O(a^n)$, a.s. on A as $n \to \infty$, with some $0 < a < 1$.

Proof of Proposition 2

Since the first two assertions are obtained in a similar manner, we confine ourselves to the proof of the first one. For $n \geq 0$ and $0 < \rho < 1/2$, define

$$A_n = \{|MR_{n+1} - (1-\alpha)m_R ZR_n| \geq ZR_n^{1-\rho}\}.$$

Let $\mathcal{G}_n = \sigma(ZR_0, Zr_0, FR_k, MR_k, Fr_k, Mr_k, \ k = 1, \ldots, n, ZR_n, Zr_n)$, $n \geq 1$ ($\mathcal{G}_0 = \sigma(ZR_0, Zr_0)$). By an appeal to Chebyshev's inequality and Proposition 1, we infer that

$$\sum_{n=0}^{\infty} P(A_n | \mathcal{G}_n) \leq \sum_{n=0}^{\infty} \frac{Var(MR_{n+1} | \mathcal{G}_n)}{ZR_n^{2(1-\rho)}} \leq C \sum_{n=0}^{\infty} \frac{1}{ZR_n^{1-2\rho}} < \infty \quad \text{a.s. on } A,$$

for some positive constant C. Hence, since $\alpha = 1 - \alpha$, by the conditional Borel-Cantelli lemma,

$$A \subseteq \left\{ \sum_{n=0}^{\infty} P(A_n | \mathcal{G}_n) < \infty \right\} = \liminf_{n \to \infty} \left\{ \left| \frac{MR_{n+1}}{ZR_n} - \alpha m_R \right| < ZR_n^{-\rho} \right\} \quad \text{a.s.,}$$

which is the desired conclusion.

To prove the last assertion, we define

$$A_n = \{ |F_{n+1} - (\alpha m_R ZR_n + \alpha m_r Zr_n)| \geq Z_n^{-\rho}(m_R ZR_n + m_r Zr_n) \},$$

for $n \geq 0$ and $0 < \rho < 1/2$. Applying Chebyshev's inequality and Propositon 1, it follows that, for some positive constant C,

$$\sum_{n=0}^{\infty} P(A_n | \mathcal{G}_n) \leq \sum_{n=0}^{\infty} \frac{Var(F_{n+1} | \mathcal{G}_n)}{Z_n^{-2\rho}(m_R ZR_n + m_r Zr_n)^2} \leq C \sum_{n=0}^{\infty} \frac{1}{Z_n^{1-2\rho}} < \infty \quad \text{a.s. on } A,$$

where we have also used that $Var(F_{n+1} | \mathcal{G}_n) \leq C(m_R ZR_n + m_r Zr_n)$ a.s. for all $n \geq 0$. Therefore, the conditional Borel-Cantelli lemma yields

$$A \subseteq \left\{ \sum_{n=0}^{\infty} P(A_n | \mathcal{G}_n) < \infty \right\} = \liminf_{n \to \infty} \left\{ \left| \frac{F_{n+1}}{m_R ZR_n + m_r Zr_n} - \alpha \right| < Z_n^{-\rho} \right\} \quad \text{a.s.,}$$

and this gives the desired result.

Proof of Theorem 3

Since $F_n - MR_n > 0$ a.s. on $A \subseteq A_{\infty,\infty}$, then, we obtain, for all $n \geq 1$, that

$$(6) \qquad Zr_n = \begin{cases} Mr_n, & \text{if } F_n > M_n \\ F_n - MR_n, & \text{if } F_n \leq M_n \end{cases} \quad \text{a.s. on } A,$$

with $M_n = MR_n + Mr_n$.

On A, we can write

$$\frac{F_{n+1}}{Zr_n} = \frac{F_{n+1}}{m_R ZR_n + m_r Zr_n} \frac{m_R ZR_n + m_r Zr_n}{Zr_n} \quad \text{and} \quad \frac{MR_{n+1}}{Zr_n} = \frac{MR_{n+1}}{ZR_n} \frac{ZR_n}{Zr_n},$$

for all $n \geq 0$. Since $ZR_n \leq Z_n = ZR_n + Zr_n$, then, we infer, by using Propositions 1 and 2, that, for each $0 < \rho < 1/2$,

$$\frac{F_{n+1} - MR_{n+1}}{Zr_n} = (\alpha + O(Z_n^{-\rho}))(m_r + O(a^n)) - (\alpha m_R + O(ZR_n^{-\rho}))O(a^n)$$

$$(7) \qquad\qquad = \alpha m_r + O(a^n) + O(ZR_n^{-\rho}) \quad \text{a.s. on } A, \text{ as } n \to \infty,$$

with $0 < a < 1$.

Then, from (6), (7) and Proposition 2, we obtain that

$$\frac{Zr_{n+1}}{Zr_n} = \alpha m_r + O(a^n) + O(ZR_n^{-\rho}) + O(Zr_n^{-\rho}) \quad \text{a.s. on } A, \text{ as } n \to \infty.$$

Since, furthermore,

$$\frac{Zr_N}{(\alpha m_r)^N} = Zr_0 \prod_{n=0}^{N-1} \frac{Zr_{n+1}}{\alpha m_r Zr_n}$$

for each $N \geq 0$, a combination of Proposition 1 and Theorem 7.28 in [14] allows us to conclude

$$0 < \prod_{n=0}^{\infty} \frac{Zr_{n+1}}{\alpha m_r Zr_n} < \infty \quad \text{a.s. on } A,$$

and thereby the first assertion of Theorem 3. From this and Propositions 1 and 2, the same result for Mr_n and F_n can be deduced.

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