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ESTIMATING THE EFFECT OF ECONOMIC CRISIS WITH CRUMP-MODE-JAGERS BRANCHING PROCESS*

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Dedicated to the memory of B. A. Sevastyanov and his inspiration to many generations of mathematicians working in the theory of branching processes

The Crump-Mode-Jagers branching process is used to model the populations of Bulgaria, Greece, Ireland, Italy, Portugal and Spain, which were influenced greatly by the economic crisis, started in 2008. The social effects were felt shortly after 2008 with different delay for each country. The unemployment rate in some of these countries was very high and constituted mainly of young people around 25 years old. This paper reviews the demographic changes that were experienced during the crisis and its effect on population growth. It also reviews the current retirement policy in these countries and extrapolates the needed changes in order to keep the current percentage of working people constant.

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1. Introduction

In this paper we study the demographic effects of economic crisis on the most affected European countries. The unemployment rate in Greece, Italy, Portugal and Spain is the highest in EU and constitutes mainly of young people in their twenties. This typically results in postponing of having children, which decreases the net fertility rate and affects directly the demographic condition of the nation. It is a common tendency for most European countries that the population count is declining for many years and the fewer births during this crisis may even deepen the problem. Having insufficient number of children and at the same time a constant increase of the life length leads to ageing population. In turn this leads to smaller percentage of working people and often requires the countries to do pension reforms. We study the differences and the common trends in the countries specified above and forecast their population structure. In addition we provide a possible retirement strategy that will keep the percentage of working people at current level (around 63%) for the next 10 years.

The results in this paper are derived using the Crump-Mode-Jagers Branching Process to model the population. We use this model because of its generality and flexibility (see Jagers [6]). It provides a model for population where each woman gives random number of births in random intervals of time and has a random life length according to specified stochastic laws. This generality allows us to use real data (see [3]) and produce a realistic model of human population (see [14]).

Section 2 reviews the theory on which the model is based on. In Section 3 we discuss the empirical results of the model. The software used for calculations is R [9] with some additional packages – demography [5] and gam [4].

2. A brief theory review and preliminary results

The Crump-Mode-Jagers branching process is also called General Branching Process and we will refer to it as GBP. The notation used in this paper follows Jagers [6]. We denote ξ to be the point process defined on R^+ and $\mu(A) = \mathbb{E}(\xi(A))$, where A is a Borel set in R^+. Let I be the set of all n-tuples of nonnegative integers for all n. This is an index set for individuals in the population. In the GBP model presented in Jagers [6] it is assumed the process starts from a single individual denoted by (0). The woman (0) is assumed to have age 0 and her birth was at time $t = 0$. The n-th child of $x \in I$ is denoted by (x, n). The individual $(x, n) \in I$ exists if $\xi_x(\infty) \geq n$. Let λ_x be a random variable that models the life length of the individual x and ξ_x be a point process of a woman x. For each

$x \in I$ is defined a couple (λ_x, ξ_x) and these couples are assumed to be independent and identically distributed. This means the distributions of ξ and λ are time invariant. Let $\tau_x(k) = \inf\{t : \xi_x(t) \geq k\}$ is the age of birth of child (x, k). Let $\sigma_x = \tau_0(j_1) + \tau_{j_1}(j_2) + \cdots + \tau_{(j_1,\ldots,j_{n-1})}(j_n)$, where $x = (j_1, \ldots, j_n)$. We have that $\sigma_0 = 0$.

An indicator variable $z_t^a(x)$ is defined for the individual x to be alive and younger than $a > 0$ at time $t > 0$ as follows

$$z_t^a(x) = \begin{cases} 1, & \text{when } t - a < \sigma_x \leq t < \sigma_x + \lambda_x \\ 0, & \text{otherwise.} \end{cases}$$

The GBP is defined as

$$z_t^a = \sum_{x \in I} z_t^a(x).$$

Let $z_t = z_t^a$, for all $a > t$. At time t the oldest individual is of age less than t, so the definition is correct. So z_t is the total number of individuals in the population. Let $\mu(t) = \mu([0, t])$ for $t > 0$.

Theorem 1. (see Jagers [6]) *If $\mu(0) < 1$ and $\mu(t) < \infty$ for some t, then $\mathbb{P}(z_t < \infty, \forall t) = 1$.*

Let us denote $f(s) = \mathbb{E}(s^{\xi(\infty)})$, $|s| \leq 1$, $L(t) = \mathbb{P}(\lambda_x \leq t)$, $\hat{\mu}$ is the Laplace-Stieltjes transformation of μ and $S(t) = 1 - L(t)$.

Theorem 2. (see Jagers [6]) *If $f(s) < \infty, |s| \leq 1$, then $m_t = \mathbb{E}(z_t) < \infty, \forall t$ and $m_t^a = \mathbb{E}(z_t^a)$ satisfies*

$$m_t^a = 1_{[0,a)}(t)\{1 - L(t)\} + \int_0^t m_{t-u}^a \mu(du). \tag{1}$$

If $m = \mu(\infty) < 1$, then $\lim_{t\to\infty} m_t = 0$. If $m = 1$ and μ is non-lattice, then

$$m_t^a \to \frac{\int_0^a \{1 - L(u)\} du}{\int_0^\infty u\mu(du)}.$$

If further $\int_0^\infty tL(dt) < \infty$*, then*

$$m_t^a \to \frac{\int_0^\infty uL(du)}{\int_0^\infty u\mu(du)}.$$

If $m > 1$*,* μ *is non-lattice and* $\alpha > 0$ *is the Malthusian parameter defined by* $\hat{\mu}(\alpha) = 1$*, then for* $0 \le a \le \infty$

$$m_t^a \sim e^{\alpha t} \frac{\int_0^a e^{-\alpha u}\{1 - L(u)\}du}{\int_0^\infty u e^{-\alpha u}\mu(du)}.$$

In the lattice cases corresponding assertions hold.

If $a = \infty$ then we skip it in the notation m_t. An important fact is that the expected future population count behaves as an exponent when the forecasting horizon increases. This result is also true for Galton-Watson and Bellman-Harris branching processes (see M. Slavtchova-Bojkova and N. Yanev [12]).

The Theorem 2 shows us one important fact – the expected population m_t is a solution of renewal equation (1). This means that we can solve it using the renewal processes theory (see Mitov and Omey [8]). Another important fact that also follows by Theorem 2 is that the expected population increases/decreases exponentially when $t \to \infty$. This also holds for subcritical branching processes.

The function $S(t) = 1 - L(t)$ is called survivability function of a live birth and we can assume $S(0) = 1$ and $S(\omega) = 0$, where ω is the oldest age in a life table. We know from renewal theory that if the survivability function $1-L(t)$ and the point process density $\mu(t)$ are smooth functions for $t \ge 0$, then the expected population m_t (as solution to equation (1)) is a smooth function too.

To model human population consisting of many individuals on different ages we use the results presented in Trayanov ([14], [15]). In order to use the population data (see Eurostat Database [3]) we must make additional assumption that each woman can give birth only to one child during a period of one year. It becomes obvious from the data that the assumptions for smoothness of $S(t)$

and $\mu(t)$ are appropriate as the birth and death rates change smoothly from one age to the next. We can also see that the fertility interval for each woman is $[12, 50]$ (women very rarely give birth outside of it). This assumption can be written in terms of GBP as: $\mathbb{P}(\xi\,[a, b) = 0)) = 1$, when $[a, b) \cap [12, 50] = \varnothing$ and $\mathbb{P}(\xi[\lambda, \infty) = 0) = 1$.

Let ${}_bz_t$ be the branching process started from a woman aged b at time $t = 0$, ${}_b\xi$ to be her point process, ${}_b\mu$ to be the expectation of the point process and ${}_bS$ to be her survivability function. Let $\nu_b = \mathbb{P}(\xi\,[b, b+1) = 1|\lambda \geq b)$ be the probability a woman to give birth at age b if she survived to the beginning of this age interval. Then we have the following corollaries that allow us to use the data and compute the expected population count:

Theorem 3. (see Trayanov [14]) *For $k \geq 1$ the distribution of ${}_b\xi$ satisfies*

$$\mathbb{P}({}_b\xi[b+k-1, b+k) = 1) = 1 - \mathbb{P}({}_b\xi[b+k-1, b+k) = 0) = {}_bS(b+k-1) \cdot \nu_{b+k-1}$$

and the expected number of births in $[b+k-1, b+k)$ of a woman aged b is

$${}_b\mu[b+k-1, b+k) = {}_bS(b+k-1) \cdot \nu_{b+k-1}.$$

Theorem 4. (see Trayanov [14]) *The expected population count at time t started from woman aged b at time zero is given by the following equation*

$$ {}_bm_t = {}_bS(t) + \int_0^t m_{t-u}\ {}_b\mu(b+du), \tag{2}$$

where ${}_bS(t)$ denotes the probability a woman of age b to survive to $b+t$, i.e.

$${}_bS(t) = \frac{S(b+t)}{S(b)}.$$

The GBP model requires two smooth functions $S(t)$ and μ_t in order to solve equation (1) and (2). We model these functions with functional data analysis and GAM package (see [4], [10]). In order to produce a forecast based on our expected future change of $S(t)$ and μ_t, we use the approach described in more details in Trayanov and Slavtchova-Bojkova (see [15]). The population and death count by age and sex and the number of births by age are used to calculate the demographic coefficient and probabilities (see Chiang [2] and Keyfitz [7]). We must first deal with some missing data for which we use the methodology described in Kannisto [13], Human Mortality Database [16] and Shkolnikov [11]. Then we use smoothing splines (see Ramsay [10] and DeBoor [1]) to find the functional form of the birth and death distributions for each year. We find the principal components of the fitted distributions and make a forecast based on random walk with a drift, which is used to find the trends in the age-specific birth and death probabilities. Substituting the forecasts for $S(t)$ and μ_t in the GBP model (equations (1) and (2)) gives us the expected future age structure of the population. The Malthusian parameter calculation methodology is described in Trayanov [14].

3. Results

We review the countries with most prominent debt crisis and unemployment ratio and do a comparison with the demographic conditions in Bulgaria. We analyze data for Bulgaria, Greece, Ireland, Italy, Portugal and Spain and forecast their population structure (excluding the migration). The data we use are published in Eurostat Database [3]. These are data for female population count by age, number of deaths by Lexis triangles and number of births by last birthday and by birth order. The life tables are complete and open-ended. The calculation results for these countries are presented in the Appendix.

The GBP gives us a forecast not only for the total population count of each of the reviewed countries but for the particular age structure too. Thus the GBP generates forecasts for different characteristics of the population like the percentage of people on working age, those younger than 18, those on pension age and other. Fitting the GBP for future years according to our expectations for birth and death rates gives us also the expected Malthusian parameter.

The Malthusian parameters (MP) of the countries are presented in Table 1 and 2. The highlighted cells mark the negative growth in MP. We calculated the historical Malthusian parameter up to year 2012. The presented values in Table 1 and 2 after year 2012 are forecasts based on the tendencies in birth and death probabilities by age. The results show that all of the countries reviewed had an increase of the Malthusian parameter in the year 2008, which is just before the crisis began, and then the MP began decreasing shortly after. We must note that a decrease in MP happens even in some years before the crisis. This can

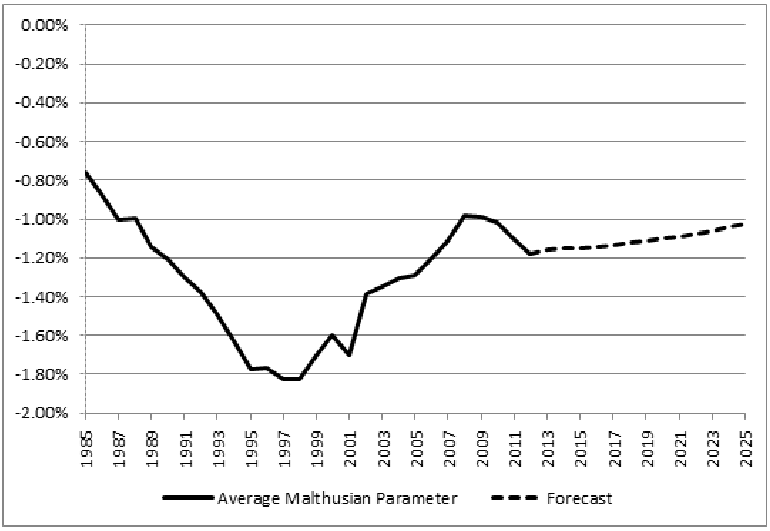

Figure 1: Average Malthusian parameter across the reviewed countries

be explained by some country-specific demographic problems and the stochastic nature of the functions $S(t)$ and μ_t. Also the decrease in Malthusian parameters in some years before 2008 was much smaller and non-persistent as opposed to the trend after the beginning of the crisis. This result is pronounced much more clearly in Figure 1 where the average MP across countries is expressed. We can see the drop after the burst of the "dot-com" bubble in the year 2001 and then the latest trend of decrease which began in year 2008. The average cumulative decrease for the interval 2008–2012 is -0.20%. The forecasts of countries' MPs show an expected average increase of 0.15% by the year 2025. In addition to rate of growth (i.e. the Malthusian parameter) we have calculated the percentage of people on working age for all countries – historical and predicted (see Table 4). This ratio is of particular importance to social policy and economy in each of these countries. Notice the ratio is similar for all countries – around 62%-63%. This is mainly due to the similarities between European countries – they have had similar birth and death distributions for a long time which resulted in similar age structures. Some small differences in this percentage arise from the fact that retirement age is different in each of them. In Bulgaria this is 63 years for women and 65 for men. In Greece, Portugal and Spain it is 65 for both women and men. In Ireland it is 66 for both women and men and in Italy it is 66 for men and 62 for women. The GBP forecasts the working force percentage will decline in all of

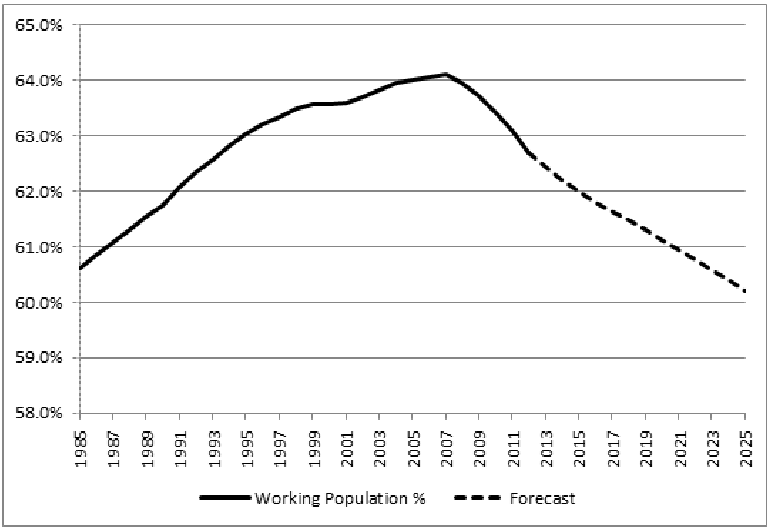

Figure 2: Average percentage of population on working age across the reviewed countries

these countries if the retirement age stays the same. The average decline across countries (see Figure 2) is predicted to be 2.5% by the end of year 2025. This will affect the economy and might even deepen the crisis if the retirement age does not get adjusted. Calculations show the retirement age must be incrementally raised by 3 years in Bulgaria, Greece and Ireland, 2 years in Portugal, 4 years in Italy and 2 years in Spain in order to keep the working force at 63% by the year 2025. The total population count is presented in Table 3. Although the age structure forecasts for each country are also very interesting they require much larger tables to be presented so they are not reviewed in details in this paper. The results show that by the end of year 2025 the population of Bulgaria is expected to decrease by 10%, Greece by 5%, Italy by 4%, Portugal by 6% and Spain by 3%. An exception is Ireland where we have expected increase by 10%. The main reason behind such a big decrease (like 10% for Bulgaria) is the age structure of the population – fewer young people and more people of old age. When the Malthusian parameter is below 0 for a long time (i.e. the GBP is subcritical) the population structure changes in a particular way – the number of young people starts decreasing and the number of old people increases. The effect is even more clearly expressed if the process was supercritical in the distant past – it generated a large number of newborns (in the years before 1960) who have been getting older and by the year

2025 constitute a large number of pensioners.

The results support the well known fact that there are some common tendencies in most European countries. An exception is Ireland for which the Malthusian parameter is close to 0 and the age structure consists of more young people than other EU countries. The GBP is subcritical for Ireland but the population is expected to increase in the next 10 years due to the favourable age structure. On a longer horizon it will eventually decline if the GBP continues to be subcritical.

Appendix

Table 1: Malthusian Parameter 1960–2000

Value	1960	1970	1980	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
BG	0.38%	0.21%	-0.04%	-0.21%	-0.42%	-0.58%	-0.95%	-1.27%	-1.46%	-1.73%	-2.16%	-2.14%	-2.63%	-2.54%	-2.12%	-1.99%
GR				-1.17%	-1.41%	-1.43%	-1.44%	-1.43%	-1.53%	-1.50%	-1.61%	-1.67%	-1.65%	-1.72%	-1.75%	-1.66%
IE																
IT				-1.43%	-1.51%	-1.50%	-1.55%	-1.58%	-1.68%	-1.81%	-1.85%	-1.82%	-1.76%	-1.76%	-1.72%	-1.65%
PT			0.32%	-0.94%	-1.04%	-1.08%	-1.04%	-1.08%	-1.11%	-1.29%	-1.37%	-1.28%	-1.21%	-1.20%	-1.13%	-1.01%
ES			0.22%	-1.23%	-1.35%	-1.44%	-1.51%	-1.53%	-1.66%	-1.81%	-1.89%	-1.91%	-1.86%	-1.90%	-1.78%	-1.66%
Changes	**1960**	**1970**	**1980**	**1988**	**1989**	**1990**	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
BG		-0.17%	-0.25%	0.02%	-0.21%	-0.16%	-0.37%	-0.31%	-0.20%	-0.27%	-0.42%	0.01%	-0.48%	0.09%	0.42%	0.12%
GR				0.03%	-0.25%	-0.02%	-0.01%	0.01%	-0.10%	0.04%	-0.11%	-0.06%	0.02%	-0.07%	-0.03%	0.09%
IE																
IT				0.09%	-0.08%	0.01%	-0.05%	-0.03%	-0.10%	-0.13%	-0.04%	0.02%	0.06%	0.00%	0.05%	0.06%
PT				-0.01%	-0.10%	-0.04%	0.04%	-0.04%	-0.03%	-0.17%	-0.09%	0.09%	0.07%	0.01%	0.08%	0.11%
ES				-0.11%	-0.12%	-0.09%	-0.07%	-0.02%	-0.12%	-0.16%	-0.08%	-0.02%	0.05%	-0.04%	0.12%	0.12%

Table 2: Malthusian Parameter 2001–2012. Forecast for 2015–2025

Value	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2015	2020	2025
BG	-2.27%	-2.13%	-2.04%	-1.87%	-1.77%	-1.57%	-1.29%	-1.10%	-0.87%	-1.06%	-1.20%	-1.22%	-1.18%	-1.12%	-1.01%
GR	-1.71%	-1.66%	-1.62%	-1.58%	-1.51%	-1.36%	-1.34%	-1.14%	-1.11%	-1.16%	-1.33%	-1.45%	-1.35%	-1.16%	-0.94%
IE		-0.11%	-0.14%	-0.18%	-0.29%	-0.18%	-0.04%	0.03%	-0.01%	-0.02%	-0.07%	-0.11%	-0.08%	0.00%	0.09%
IT	-1.67%	-1.61%	-1.54%	-1.41%	-1.44%	-1.35%	-1.27%	-1.16%	-1.16%	-1.15%	-1.17%	-1.19%	-1.10%	-0.96%	-0.79%
PT	-1.23%	-1.20%	-1.26%	-1.34%	-1.32%	-1.41%	-1.47%	-1.36%	-1.48%	-1.37%	-1.46%	-1.63%	-1.70%	-1.83%	-1.93%
ES	-1.64%	-1.60%	-1.49%	-1.45%	-1.41%	-1.33%	-1.29%	-1.16%	-1.32%	-1.34%	-1.40%	-1.46%	-1.49%	-1.55%	-1.58%
Changes	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2015**	**2020**	**2025**
BG	-0.28%	0.14%	0.08%	0.17%	0.10%	0.19%	0.29%	0.18%	0.23%	-0.19%	-0.14%	-0.02%	0.04%	0.07%	0.10%
GR	-0.05%	0.05%	0.04%	0.05%	0.06%	0.16%	0.01%	0.20%	0.03%	-0.04%	-0.17%	-0.13%	0.11%	0.18%	0.22%
IE			-0.03%	-0.04%	-0.11%	0.11%	0.14%	0.06%	-0.04%	-0.01%	-0.05%	-0.04%	0.04%	0.08%	0.09%
IT	-0.01%	0.06%	0.07%	0.13%	-0.02%	0.09%	0.08%	0.10%	0.00%	0.01%	-0.02%	-0.02%	0.09%	0.14%	0.17%
PT	-0.22%	0.04%	-0.06%	-0.08%	0.03%	-0.09%	-0.06%	0.11%	-0.12%	0.12%	-0.10%	-0.17%	-0.07%	-0.13%	-0.10%
ES	0.02%	0.04%	0.11%	0.04%	0.04%	0.08%	0.03%	0.14%	-0.16%	-0.02%	-0.06%	-0.06%	-0.03%	-0.06%	-0.03%

Table 3: Total population count by sex. History and forecast

Women	2006	2007	2008	2009	2010	2011	2012	2015	2020	2025
BG	3,886,094	3,857,635	3,831,264	3,807,808	3,780,505	3,760,457	3,739,479	3,659,826	3,521,568	3,377,739
GR	5,661,793	5,683,242	5,691,258	5,694,716	5,669,948	5,673,231	5,649,218	5,596,096	5,501,951	5,401,372
IE	2,167,328	2,231,011	2,268,664	2,288,454	2,301,455	2,312,918	2,318,297	2,375,325	2,463,508	2,536,963
IT	30,011,567	30,241,728	30,430,674	30,540,773	30,649,483	30,667,647	30,795,688	30,473,528	29,933,663	29,376,774
PT	5,468,576	5,483,431	5,497,204	5,510,009	5,518,814	5,511,961	5,491,592	5,432,487	5,316,788	5,181,516
ES	22,665,740	23,077,549	23,358,784	23,504,415	23,617,764	23,719,302	23,710,189	23,652,159	23,440,155	23,092,671
Men	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2015**	**2020**	**2025**
BG	3,686,579	3,660,367	3,635,855	3,613,958	3,588,926	3,566,767	3,545,073	3,462,705	3,319,218	3,168,029
GR	5,481,987	5,498,982	5,499,396	5,488,800	5,453,444	5,449,803	5,413,290	5,360,935	5,264,564	5,161,529
IE	2,172,790	2,226,754	2,252,658	2,260,974	2,269,426	2,269,789	2,272,790	2,333,298	2,424,850	2,500,022
IT	28,212,271	28,411,288	28,570,035	28,649,404	28,715,311	28,726,634	28,889,654	28,700,322	28,336,112	27,932,304
PT	5,064,395	5,069,747	5,066,239	5,063,745	5,053,543	5,030,437	4,995,697	4,943,694	4,837,389	4,712,178
ES	22,119,008	22,591,563	22,880,570	22,982,326	23,049,535	23,099,061	23,017,822	22,980,804	22,771,542	22,398,983

Table 4: Percentage of population on working age

	1970	1980	1990	1995	2000	2001	2002	2003	2004	2005
BG	61.7%	61.3%	60.6%	61.5%	62.9%	62.9%	63.2%	63.5%	63.9%	64.1%
GR			62.5%	63.3%	63.6%	63.7%	63.6%	63.6%	63.6%	63.5%
IE						63.6%	64.0%	64.3%	64.7%	65.2%
IT			62.8%	63.7%	62.9%	62.7%	62.4%	62.4%	62.2%	61.9%
PT		57.7%	61.2%	62.9%	63.6%	63.6%	63.7%	63.6%	63.5%	63.5%
ES		57.8%	61.6%	63.6%	64.8%	65.0%	65.3%	65.5%	65.8%	65.8%
	2006	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2015**	**2020**	**2025**
BG	64.1%	64.3%	64.3%	64.1%	63.9%	63.5%	63.1%	61.9%	60.5%	59.9%
GR	63.6%	63.6%	63.6%	63.4%	63.1%	62.7%	62.3%	61.9%	61.3%	60.3%
IE	65.6%	65.7%	65.4%	64.9%	64.4%	63.7%	63.0%	62.0%	60.9%	60.1%
IT	61.6%	61.6%	61.4%	61.2%	61.0%	61.0%	60.7%	60.0%	59.0%	57.4%
PT	63.4%	63.4%	63.3%	63.2%	63.1%	62.9%	62.7%	62.4%	62.1%	61.3%
ES	65.9%	66.0%	65.7%	65.4%	65.1%	64.8%	64.4%	63.7%	63.0%	62.2%

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