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ON AN IMPLEMENTATION OF α-SUBORDINATED BROWNIAN MOTION AND OPTION PRICING WITH AND WITHOUT TRANSACTION COSTS VIA *CAS MATHEMATICA*

Angela Slavova, Nikolay Kyrkchiev

In this we suppose that the underlying of the option contract is driven by a subordinated geometric Brownian motion. Firstly, we investigate the case when there is no transaction cost during trading. We derive the pricing formula for a European option in this case. Then, we study the case with transaction costs. We apply the mean self-financing delta-hedging strategy. We develop α-subordinated Brownian motion and option pricing without transaction costs module via CAS MATHEMATICA. We obtain bounds for call and put options for various values of . Then we propose -subordinated Brownian motion and option pricing with and without transaction costs modules.

1. Introduction

The classical Black-Scholes model is based on the diffusion process called geometric Brownian motion [1]–[4]:

$$dS_t = \mu S_t dt + \sigma S_t dB(t), \tag{1}$$

where μ, σ are constants, and $B(\tau)$ is the standard Brownian motion.

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In [5], Magdziarz applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion (SGBM) $S_t = X(S_\alpha(t))$ as the model of asset prices exhibiting subdiffusuve dynamics. Here the present process $X(\tau)$ is the geometric Brownian motion (GBM) given by equation (1).

Here $S_\alpha(t)$ is the inverse time α-stable subordinator with the parameter $\alpha \in (0, 1)$.

This model can also be expressed as

$$dS_t = \mu S_t dS_\alpha(t) + \sigma S_t dB(S_\alpha(t)). \tag{2}$$

We suppose that the underying of the option contract is driven by a subordinated geometric Brownian motion, i.e. the price of underlying S_t follows the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB(S_\alpha(t)), \tag{3}$$

where $S_\alpha(t)$ is the inverse α-stable subordinator, defined by

$$S_\alpha(t) = inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{4}$$

where $U_\alpha(t)$ is a strictly increasing α-stable Levy process with Laplace transform given by $E(e^{-kU_\alpha(\tau)}) = e^{-rk^\alpha},\ 0 < \alpha < 1$ and $S_\alpha(t)$ is independent of $B(\tau)$.

Remark. The Black-Scholes PDE has a fundamental probabilistic interpretation. The correspondence between PDEs and probabilities via the Fokker-Plank formalism yields

$$C(S_t, t) = \mathbf{E}\{\sum_{i:t<T_i} e^{-r(T_i-t)} F(S_{T_i}|I_t\},$$

where $\mathbf{E}\{.|I_t\}$ represents the conditional expectation, $T_1 < T_2 < \ldots < T_N$ are different dates for the series of cash-flows represented by $F_i(S_{T_i})$, $i = 1, 2, \ldots, N$. S_t is the diffusion process governed by the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma dY_t + rdt.$$

Here we discuss the approach of Longjin Lv, Jianbin Xiao, Fu-Yao ren [4].

2. Option pricing model

1. In the case – option pricing model without transaction costs, the call and put can be calculated by the following formulas:

$$C(t, S_t) = S_t N(d_1) - Ke^{-rT} N(d_2), \tag{5}$$

```
Print["α-Subordinated Brownian motion and option pricing without transaction costs"];

St = Input["Input St - Current stock price"];
Print["Current stock price St=", St];

X = Input["Input X - Strike price"];
Print["Strike price X=", X];

r = Input["Input r - Current continuously compounded risk-free interest rate"];
Print["Current continuously compounded risk-free interest rate r=", r];

T = Input["Input T - Time to option's maturity (in years)"];
Print["Time to option's maturity (in years) T=", T];

σ = Input["Input σ - Standard deviation of the annualized continuously compounded rate of
return on the stock"];
Print["Standard deviation of the annualized continuously compounded rate of return on the stock σ=", σ]

t0 = Input["Input Date for which expected values of call and put options"];
Print["Date for which expected values of call and put options t0=", t0];

α = Input["Input α - defined the strictly increasing α-stable process"];
Print["Parameter, defined the strictly increasing α-stable Levy process α=", α];

d1 := (Log[St/X] + r (T - t0) + σ^2/(2*Gamma[α+1]) (T^α - (t0)^α)) / (σ √((T^α - (t0)^α) / Gamma[α + 1]));

d2 := d1 - σ √((T^α - (t0)^α) / Gamma[α + 1]);

Print["Current value of the call option: "];
Print["C0 = ", St*CDF[NormalDistribution[], d1] - X*e^{-r*T}*CDF[NormalDistribution[], d2]];

Print["Current value of the put option: "];
Print["P0 = ", X*e^{-r*T}*CDF[NormalDistribution[], -d2] - St*CDF[NormalDistribution[], -d1]];
```

Figure 1: "α-subordinated Brownian motion and option pricing without transaction costs" module.

where the function $N(x)$ is the cumulative probability distribution function for standard normal distribution, and

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{S_t}{K} + r(T-t) + \frac{\sigma^2}{2\Gamma(\alpha+1)}(T^\alpha - t^\alpha)}{\sigma\sqrt{(T^\alpha - t^\alpha)/\Gamma(\alpha+1)}} \\
d_2 &= d_1 - \sigma\sqrt{(T^\alpha - t^\alpha)/\Gamma(\alpha+1)}.
\end{aligned} \tag{6}
$$

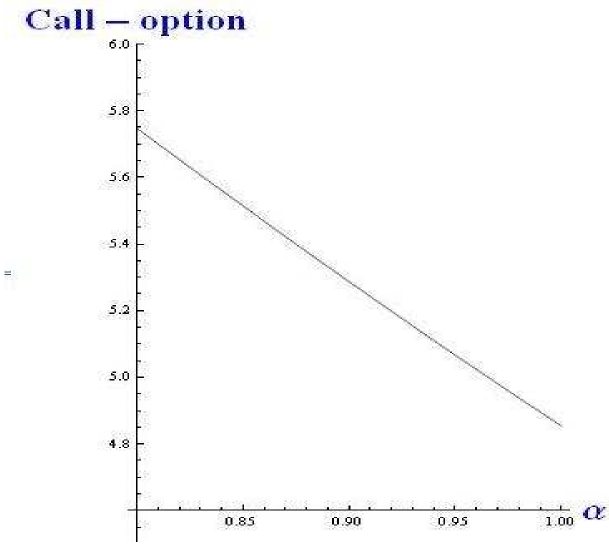

Figure 2: Bounds for call-option for various α

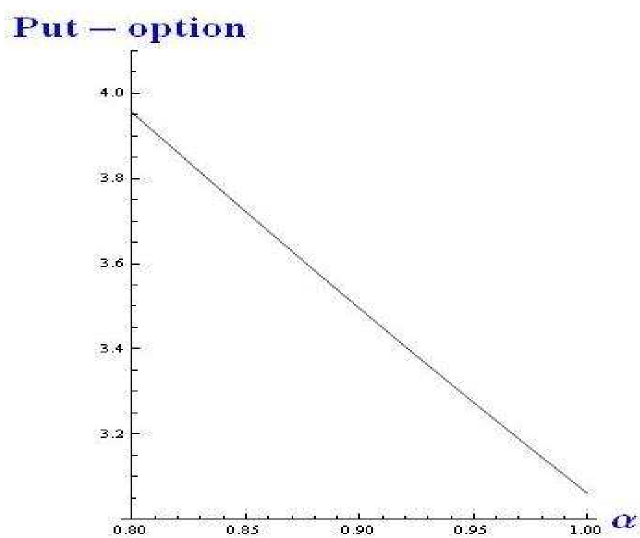

Figure 3: Bounds for put-option for various α

Following the same procedures, we can get the evaluation formula $P(t, S_t)$ for the European put option on the same underlying

$$P(t, S_t) = Ke^{-rT}N(-d_2) - S_tN(-d_1). \tag{7}$$

```
Print["II. Subordinated Brownian motion and option pricing with transaction costs"];

St = Input["Input St - Current stock price"];
Print["Current stock price St=", St];

X = Input["Input X - Strike price"];
Print["Strike price X=", X];

r = Input["Input r - Current continuously compounded risk-free interest rate"];
Print["Current continuously compounded risk-free interest rate r=", r];

T = Input["Input T - Time to option's maturity (in years)"];
Print["Time to option's maturity (in years) T=", T];

k = Input["Input k - the round trip transaction cost"];
Print["k - the round trip transaction cost k=", k];

σ = Input["Input σ - Standard deviation of the annualized continuously compounded rate of return on the :
Print["Standard deviation of the annualized continuously compounded rate of return on the stock σ=", σ];

t0 = Input["Input Date for which expected values of call and put options"];
Print["Date for which expected values of call and put options t0=", t0];

α = Input["Input α - defined the strictly increasing α-stable Levy process"];
Print["Parameter, defined the strictly increasing α-stable Levy process α=", α];

        Log[St/X] + r (T - t0) + 0.5 * NIntegrate[(x^{α - 1} / Gamma[α]) * σ^2 + Sqrt[2 / Pi] * Gamma[1.5] / Gamma[α / 2 + 1]
d1 := -----------------------------------------------------------------------------------------------------------------
                 Sqrt[NIntegrate[(x^{α - 1} / Gamma[α]) * σ^2 + Sqrt[2 / Pi] * Gamma[1.5] / Gamma[α / 2 + 1] * k * σ * (T -

d2 := d1 - Sqrt[NIntegrate[(x^{α - 1} / Gamma[α]) * σ^2 + Sqrt[2 / Pi] * Gamma[1.5] / Gamma[α / 2 + 1] * k * σ * (T - t0)^{

Print["Current value of the call option: "];
Print["C0 = ", St * CDF[NormalDistribution[], d1] - X * e^{-r * T} * CDF[NormalDistribution[], d2]];
```

Figure 4: "α-subordinated Brownian motion and option pricing with transaction costs" module

Thus, the put-call party holds, i.e.

$$C(t, S_t) - P(t, S_t) = S_t - Ke^{-r(T-t)}, \ \ t \in [0, T]. \tag{8}$$

We should also mention that all the results obtained here are consistent with that got by classical Black-Scholes formula when $\alpha \to 1$.

2. Now let us come to the case with transaction costs.

From the practical point of view. we assume that the trading occurs at t and $t + \Delta t$, but not in between. Then, from t to $t + \Delta t$, the change in the value of

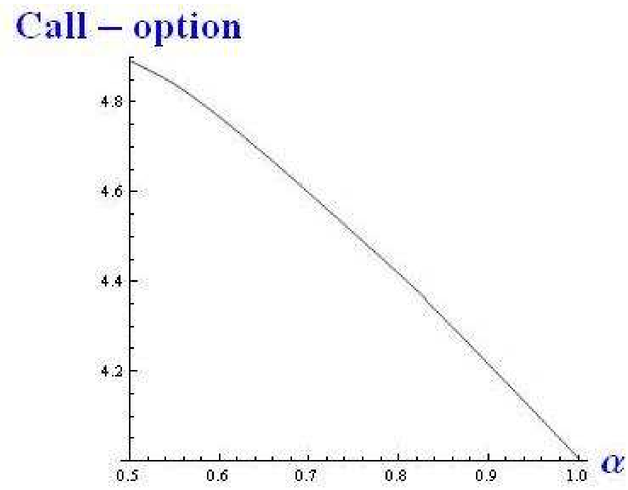

Figure 5: Bounds for call-option for various α

the portfolio is

$$\Delta\Pi_t = \Delta C(t, S_t) - \frac{\partial C}{\partial S_t}\Delta S_t + \frac{k}{2}\left|\Delta\frac{\partial C}{\partial S_t}\right| S_t, \tag{9}$$

where $\Delta\frac{\partial C}{\partial S_t}$ is the change in the number of units of underlying asset held in the portfolio, and k represent the round trip transaction cost, measured as a fraction of the volume of transaction. We also can check that $\frac{\partial^2 C}{\partial S_t \partial t}$, $\frac{\partial^2 C}{\partial S_t^2}$ and $\frac{\partial^3 C}{\partial S_t^3}$ is $o(\Delta t^{\frac{1}{2}})$. Since $\Delta S_\alpha(t) = o(\Delta t^{\alpha-\epsilon})$ for arbitrary $\epsilon \in (0, \alpha)$. So, if $\alpha > \frac{1}{2}$, we have

$$\begin{aligned} \Delta\frac{\partial C}{\partial S_t} &= \frac{\partial^2 C}{\partial S_t \partial t}\Delta t + \frac{\partial^2 C}{\partial S_t^2}\Delta S_t + \frac{1}{2}\frac{\partial^3 C}{\partial S_t^3}\Delta S_t^2 + o(\Delta t) \\ &= \sigma S_t \left|\frac{\partial^2 C}{\partial S_t^2}\right| |\Delta B(S_\alpha(t))| + o(\Delta t). \end{aligned} \tag{10}$$

Here, we also use the mean self-financing delta-hedging strategy.

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\hat{\sigma}^2(t)S^2\frac{\partial^2 C}{\partial S^2} = rC, \tag{11}$$

where

$$\begin{aligned} \hat{\sigma}^2(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sigma^2 + \mathrm{sing}(\Gamma)k\sigma E[|\Delta B(S_\alpha(t))|].(\Delta t)^{-1} \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sigma^2 + \sqrt{\frac{2}{\pi}}\frac{\Gamma(3/2)}{\Gamma(\alpha/2+1)}\,\mathrm{sing}(\Gamma)k\sigma.(\Delta t)^{\frac{\alpha}{2}-1}, \end{aligned} \tag{12}$$

here, sing(Γ) is the sing of $\frac{\partial^2 C}{\partial S_t^2}$. Following the same procedure above, we can get the price formula of an European call option with transaction costs, given by

$$C(t, S_t) = S_t N(d_1^*) - Ke^{-rT} N(d_2^*), \tag{13}$$

where

$$\begin{aligned} d_1^* &= \frac{\ln \frac{S_t}{K} + r(T-t) + \frac{1}{2}\int_t^T \hat{\sigma}^2(s)ds}{\sqrt{\int_t^T \hat{\sigma}^2(s)ds}} \\ d_2^* &= d_1^* - \sqrt{\int_t^T \hat{\sigma}^2(s)ds}, \end{aligned} \tag{14}$$

which is dependent on the time length Δt.

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