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THE INFLUENCE OF INHOMOGENEITY ON THE DYNAMIC BEHAVIOR OF FUNCTIONALLY GRADED MAGNETO-ELECTRO-ELASTIC MATERIALS WITH CRACKS*

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Functionally graded materials are extensively used in modern industry. They are composite materials with continuously varying properties in one or more spacial dimensions, according to the specific purpose. In view of their applications, stress analysis of such materials is important for their structural integrity. In this study we will consider cracked functionally graded magneto-electro-elastic materials subjected to SH waves. We assume that the material properties vary in one and the same way, described by an inhomogeneity function. The boundary value problem is reduced to a system of integro-differential equations based on the existence of fundamental solutions. Different inhomogeneity classes are used to obtain a wave equation with constant coefficients. Radon transform is applied to derive the fundamental solution in a closed form. Program code in FORTRAN 77 is developed and validated using available examples from literature. Simulations show the dependence of stress on the frequency of the applied time-harmonic load for different types of material inhomogeneity.

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1. Introduction

Magneto-electro-elastic composites are advanced materials that possess large magneto-electric effect. This effect doesn't exist in piezoelectric or piezomagnetic phase and has variety of applications in modern smart structures. Due to different physical and chemical properties of the constituents in the composite a process called delamination can occur. To overcome this problem functionally graded materials (FGM) have been created. FGM were used for the first time in Japan, 1984, during the project SPACEPLANE, when researches confronted with the challenge to develop material that can create 1000°F temperature difference in 10mm thickness. FGM are composites in which two or more materials are combined to obtain the desired properties needed for the application of FGM. Their composition is in continuously graded form to avoid the abrupt change of material properties. The main application area of FGM are aerospace industry, communication field, energy sector, medical field, nuclear projects ([1]). Fracture analysis of FGM is important for their structural integrity and reliable service life. The smooth variation of material properties is described by a continuous function $h(x)$. In most published papers this function is exponential.Different types of inhomogeneity can be found in Li and Weng [2], Manolis et al [3], Sutradhar and Paolino [4], Rangelov et al [5].

In this research we will consider functionally graded magneto-electro-elastic materials (FGMEEM) with cracks subjected to time-harmonic SH wave. Different types of functions that describe the smooth variation of the material properties will be used in our model. The boundary integral equation method (BIEM) is used for the numerical solution. Program code in FORTRAN 77 is developed for the numerical calculations. Numerical results show the dependence of the stress intensity factor on the frequency of the external time-harmonic load.

2. Statement of the problem

We consider transversely-isotropic functionally graded magnetoelectroelastic material in the coordinate system $Ox_1x_2x_3$, where Ox_3 is poling direction and symmetry axis and Ox_1x_2 is the isotropic plane. MEEM is subjected to anti-plane mechanical impact on the Ox_3 axis, and in-plane electrical and magnetic impacts in the plane Ox_1x_2. The constitutive equations for this type of medium (see Soh and Liu [6]) are:

(1) $$\sigma_{iQ} = C_{iQJl} u_{J,l},$$

where $x = (x_1, x_2)$, and $\Gamma = \Gamma^+ \bigcup \Gamma^-$ is an internal crack - an open arc. Here and in what follows comma denotes partial differentiation, it is assumed summation in repeating indexes and small indexes vary $1, 2$, while capital indexes vary $3, 4, 5$; u_J is the generalized displacement vector $u_J = (u_3, \phi, \varphi)$, where u_3 is out of plane elastic displacement, ϕ is the electric potential, φ is the magnetic potential; $\sigma_{iJ} = (\sigma_{i3}, D_i, B_i)$ is the generalized stress tensor, where σ_{iJ} is the stress, D_i and B_i are the components of the electric and magnetic induction respectively along Ox_i axis; C_{iQJl} is the generalized elasticity tensor defined as: $C_{i33l}(x) = \begin{cases} c_{44}(x), i = l \\ 0, \quad i \neq l \end{cases}$, $C_{i34l}(x) = C_{i43l}(x) = \begin{cases} e_{15}(x), i = l \\ 0, \quad i \neq l \end{cases}$, $C_{i35l}(x) = C_{i53l}(x) = \begin{cases} q_{15}(x), i = l \\ 0, \quad i \neq l \end{cases}$, $C_{i44l}(x) = \begin{cases} -\varepsilon_{11}(x), i = l \\ 0, \qquad i \neq l \end{cases}$, $C_{i45l}(x) = C_{i54l}(x) = \begin{cases} -d_{11}(x), i = l \\ 0, \qquad i \neq l \end{cases}$, $C_{i55l}(x) = \begin{cases} -\mu_{11}(x), i = l \\ 0, \qquad i \neq l \end{cases}$. Functions $c_{44}(x)$, $e_{15}(x)$, $\varepsilon_{11}(x)$ are: elastic stiffness, piezoelectric coefficient and dielectric permittivity, while $q_{15}(x)$, $d_{11}(x)$, $\mu_{11}(x)$ are piezo-magnetic and magneto-elastic coefficients and magnetic permeability correspondingly.

Assuming quasi-static approximation in the Maxwell equation the governing equation in the frequency domain in absence of body force, electric charge and magnetic current is the following:

$$\sigma_{iQ,i} + \rho_{QJ}\omega^2 u_J = 0 \tag{2}$$

where $\rho_{QJ} = \begin{cases} \rho, Q = J = 3 \\ 0, Q, J = 4 \text{ or } 5 \end{cases}$, $\rho(x)$ is the mass density, $\omega > 0$ is the frequency.

We assume that all material properties depend on x in one and the same way and describe this by an inhomogeneity function $h(x)$: $c_{44}(x) = h(x)c_{44}$, $e_{15}(x) = h(x)e_{15}$, $\varepsilon_{11}(x) = h(x)\varepsilon_{11}$, $q_{15}(x) = h(x)q_{15}$, $d_{11}(x) = h(x)d_{11}$, $\mu_{11}(x) = h(x)\mu_{11}$. The inhomogeneity function has the following form: $h(x) = F\langle k, x\rangle$, where $\langle k, x\rangle = k_1x_1 + k_2x_2$.

When the incident SH-wave interacts with the cracks a scattered wave is produced. The total displacement and traction at any point of the plane can be calculated by the superposition principle:

$$u_J = u_J^{in} + u_J^{sc}, t_J = t_J^{in} + t_J^{sc}, \tag{3}$$

where $t_J = \sigma_{iJ}n_i$ and $n_i = (n_1, n_2)$ is the outer normal vector.u_J^{in} and t_J^{in} are displacement and traction of the incident wave fields u_J^{sc} and t_J^{sc} are the scattered by the cracks wave fields.We impose the following boundary conditions: $t_J = 0$, $t_J^{in} = -t_J^{sc}$

$$t_J = 0 \;\; or \;\; t_J^{in} = -t_J^{sc}, \quad x \in \Gamma \tag{4}$$

$$u_J^{sc} \to 0 \;\; when \;\; (x_1^2 + x_2^2)^{1/2} \to \infty \tag{5}$$

The boundary condition (4) means that the cracks are free of mechanical traction and also they are magnetoelectrically impermeable. This boundary condition is widely used in literature, see Sladek et al. [7]. Permeable cracks are also used in literature. Impermeable and permeable crack models are discussed in details in Dineva et al. [8]. We will reduce the boundary value problem (2), (4) and (5) to an equivalent system of integro-differential equations along the cracks and then solved this system numerically.

3. Boundary integral equation method

The fundamental solution u^*_{KM} of (2) is the solution of the equation:
$\sigma_{iQ,i} + \rho_{QJ}\omega^2 u_J = \delta_{JM}\delta(x,\xi)$,
where δ_{JM} is the Kronecker symbol and $\delta(x,\xi)$ is the Dirak's delta function. Following [9], [10]T the fundamental solution can be represented in the following way
$u^*_{KM} = h^{-1/2}U^*_{KM}$,
where U^*_{KM} is solution of:

$$C_{iJKi}U^*_{KM,ii} + [\rho_{JK}\omega^2 - C_{iJKi}k_i^2]U^*_{KM} = h^{-1/2}(\xi)\delta_{JM}\delta(x,\xi). \quad (6)$$

Equation (6) is with constant coefficients if $p_{JK} = C_{iJKi}h^{-1/2}(h^{1/2})_{,ii} = \text{const}$. We will consider the following types of functions for which this condition is satisfied.

(i) $h(x) = (< a, x > +b)^2$ – quadratic type, $p_{JK} = 0$;

(ii) $h(x) = e^{2<a,x>+b}$, $h(x) = \cos h^2(2 < a, x > +b)$– exponential type, $p_{JK} = C_{iJKi}a_i^2$;

(iii) $h(x) = \sin^2(2 < a, x > +b)$– trigonometric type, $p_{JK} = -C_{iJKi}a_i^2$;

When a system with constant coefficients is obtained we find the fundamental solutions in a closed form using direct and inverse Radon transform and calculus with generalized functions (see [8], [11])

Following Wang and Zhang [10] and Gross et al. [12]the following representation formulae are valid:

$$\begin{array}{l} t^{sc}_J(x,\omega) = C_{iJKl}(x)n_i(x)\int_{Cr}\{[\sigma^*_{\eta PK}(x,y,\omega)\triangle u_{P,\eta}(y,\omega) \\ -\rho_{QP}(y)\omega^2 u^*_{QK}(x,y,\omega)\triangle u_P(y)]\delta_{\lambda l} \\ -\sigma^*_{\lambda PK}(x,y,\omega)\triangle u_{P,l}(y,\omega)]n_\lambda(y)\}d\Gamma(y), \end{array} \quad (7)$$

where $\Delta u_J = u_J|_{Cr^+} - u_J|_{Cr^-}$ are jumps of the displacement along the crack or crack opening displacement (COD),$Cr = Cr_1 \bigcup Cr_2$, Cr^+ and Cr^- are the upper and lower bounds of the cracks correspondingly, u^*_{KM} is the fundamental solution and $\sigma^*_{iJK} = C_{iJMl}u^*_{KM,l}$,

Since the fundamental solution and incident wave field are known, using (7) we obtain integro-differential equation along the crack, where unknowns are COD:

$$
\begin{aligned}
t^{in}_J(x,\omega) &= -C_{iJKl}(x)n_i(x)\int_{Cr}\{[\sigma^*_{\eta PK}(x,y,\omega)\triangle u_{P,\eta}(y,\omega) \\
&-\rho_{QP}(y)\omega^2 u^*_{QK}(x,y,\omega)\triangle u_P(y)]\delta_{\lambda l} \\
&-\sigma^*_{\lambda PK}(x,y,\omega)\triangle u_{P,l}(y,\omega)]n_\lambda(y)\}d\Gamma(y), x\in Cr,
\end{aligned}
\tag{8}
$$

The equation (8) is solved numerically. Once COD are found we can calculate the scattered field at every point on the plane using (7)

4. Numerical realization and results

The two cracks are discretisized using 7BE for each crack. The unknown COD are approximated by parabolic shape functions. The singular integrals are solved analitically using assymptotic behavior of the fundamental solution for small arguments. The 2D integrals are solved numerically by the Monte-Carlo method. Program code in Fortran 77 is developed.

The generalized stress intensity factors (SIF) are computed as follows:
$K_{III} = \lim_{x_1\to\pm a} t_3\sqrt{2\pi(x_1\mp a)}$,
$K_E = \lim_{x_1\to\pm a} E_2\sqrt{2\pi(x_1\mp a)}$,
$K_H = \lim_{x_1\to\pm a} H_2\sqrt{2\pi(x_1\mp a)}$

The half-length of the cracks is $c = 5mm$, the MEEM that we use is the piezoelectric/piezomagnetic composite $BaTiO_3/CoFe_2O_4$ and the material constants for this composite can be found in Song and Sih [13], Li [14]. The components of the inhomogeneity function are presented in the following way: $k = (k_1, k_2) = \frac{\beta}{2c}(\cos\alpha, \sin\alpha)$, where β is the magnitude of inhomogeneity, α is the inhomogeneity angle. We compare the following material gradations: $h(x) = (k_1x_1+k_2x_2+1)^2$-quadratic type, $h(x) = e^{2(k_1x_1+k_2x_2)}$, $h(x) = \cos h^2(k_1x_1+k_2x_2)$-exponential type, $h(x) = \sin^2(k_1x_1 + k_2x_2 + 1)$-trigonometric type.

4.1. Validation

We validate our results with available examples from the literature. Comparison is made with the results in [5] for the piezoelectric material PZT-6B, sinusoidal inhomogeneity function, inhomogeneity magnitude $\beta = 0.4$ and inhomogeneity angle $\alpha = \pi/2$. The comparison is given in Figure 1 The normalized SIF is $K^*_{III} = \frac{K_{III}}{t^{in}_3\sqrt(\pi c)}$ and the normalized frequency is $\Omega = c\sqrt(\rho\omega^2 a_0^{-1} + a_1^2 + A_2^2)$, where $a_0 = c_{44} + \frac{e^2_{15}}{\varepsilon_{11}}$. We see good coincidence with maximum difference of no more than

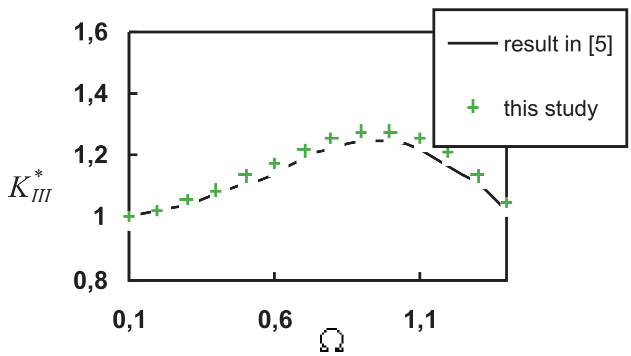

Figure 1: Normalized SIF versus the normalized frequency for sinusoidal type of inhomogeneity $\alpha = \pi/2$

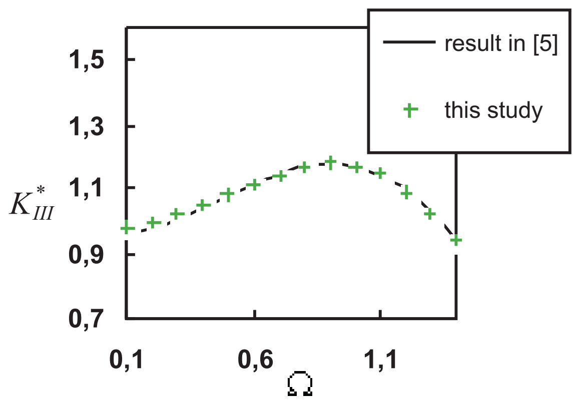

Figure 2: Normalized SIF versus the normalized frequency for sinusoidal type of inhomogeneity $\alpha = 0.0$

5 percents. In Figure 2 we present comparison for the same function, but for $\alpha = 0.0$. The coincidence is very good, the difference is no more than 2 percents. In Figure 3 we compare our results with the results in [5] for quadratic inhomogeneity function, inhomogeneity magnitude $\beta = 0.4$ the normalized frequency is $\Omega = c\sqrt(\rho\omega^2 a_0^{-1})$. The results show good coincidence.

4.2. Parametric studies

The aim of the simulations is to show the sensitivity of the SIF to the different inhomogeneity types for functionally graded magneto-electro-elastic com-

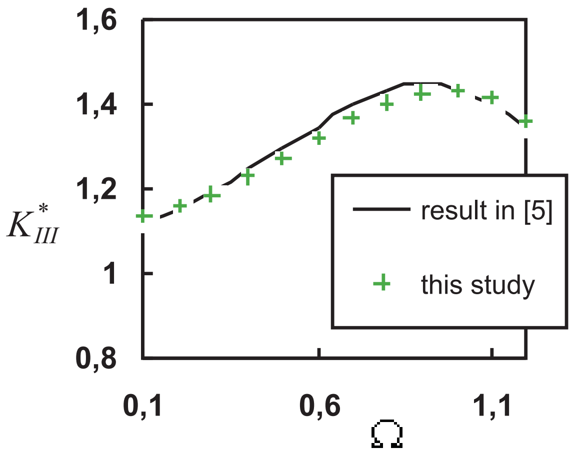

Figure 3: Normalized SIF versus the normalized frequency for quadratic type of inhomogeneity $\alpha = 0.0$

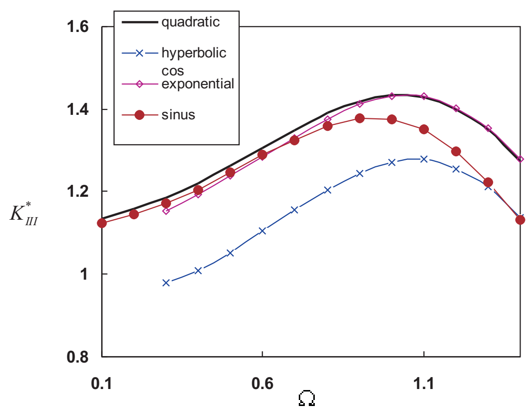

Figure 4: Normalized stress intensity factor versus the normalized frequency for different inhomogeneity types, left crack tip

posite material. The material used is piezoelectric/piezomagnetic composite $BaTiO_3/CoFeO_4$. The following inhomogeneity functions are used: $h(x) = (k_1x_1 + k_2x_2 + 1)^2$, $h(x) = e^{2(k_1x_1+k_2x_2)}$, $h(x) = \cos h^2(k_1x_1 + k_2x_2)$ $h(x) = \sin^2(k_1x_1 + k_2x_2 + 1)$. The inhomogeneity magnitude is $\beta = 0.4$ and the inhomogeneity angle is $\alpha = 0.0$. The normalized frequency is $\Omega = c\omega\sqrt{(}\rho c_{44}^{-1})$. The results for left and right crack tip are given in Figures 2 and 3. The results for

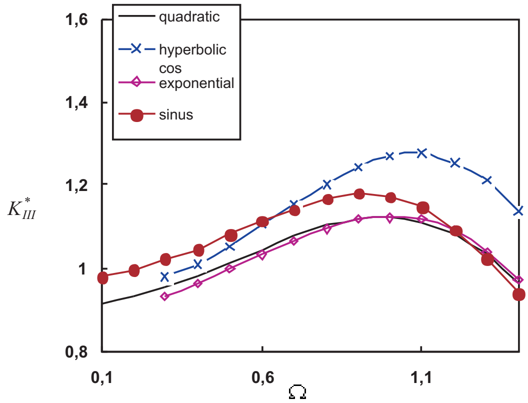

Figure 5: Normalized stress intensity factor versus the normalized frequency for different inhomogeneity types, right crack tip

$h(x) = (k_1x_1 + k_2x_2 + 1)^2$ and $h(x) = e^{2(k_1x_1+k_2x_2)}$ are only slightly different while hyperbolic cosine inhomogeneity and sinusoidal inhomogeneity show considerable decreasing of the normalized SIF. We see that generally the stress concentration at right crack tip is lower than at the left crack tip. The stress intensity factor is strongly dependant on inhomogeneity types.

5. Conclusion

Functionally graded materials are advanced materials with continuously varying material properties. They have wide range of applications in modern technology. In this study we presented numerical solution of integrodifferential equations for functionally graded magneto-electro-elastic materials with different material gradation using the BIEM. The fundamental solution is obtained using Radon transform. FORTRAN 77 code, based on the BIEM is developed. The validation studies show the correctness of the developed numerical tool. The numerical examples reveal the significant differences that occur for different types of material gradation. They also demonstrate that SIF depends strongly on the inhomogeneity function. This software can be further developed to solve problems in materials with nano-heterogeneities, laminar structures, thermo-elastic problems, direct and inverse problems in finite solids and has application in non-destructive material testing.

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