PLISKA
STUDIA MATHEMATICA

ПЛИСКА
МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. **26** (2016), 203–214

PLISKA
STUDIA MATHEMATICA

DYNAMICS OF HYSTERESIS CNN WITH MEMRISTOR SYNAPSES*

Angela Slavova, Ronald Tetzlaff

In this paper a new hysteresis Cellular Nonlinear Networks (CNN) model will be studied in which we shall introduce memristor in the synapses. Dynamics of such model will be investigated. Local activity theory will be applied in order to determine the edge of chaos domain of the parameter set in which the model under consideration can exhibit complexity. Simulations and applications will be provided.

1. Introduction

Hysteresis is defined in the literature [7] as a rate independent memory effect. We shall use the following definition:

Definition 1. *Hysteresis = Rate Independence Memory Effect.*

Actually, even in most typical hysteresis phenomena, like ferromagnetism, ferroelectricity, plasticity, memory effects are not purely rate independent, since hysteresis is coupled with viscous-type effects. However, in several cases the rate independent component prevails, provided that evolution is not too fast.

Several physical phenomena exhibit hystersis. In classical continuum mechanics, hysteresis behavior is inherent in many constitutive laws. In systems and control applications, hysteresis regularly appears via mechanical play and

2010 *Mathematics Subject Classification:* 92B20, 34C55, 34C28.
Key words: hysteresis Cellular Nonlinear Networks (CNN), dynamic behaviour, memristor.
*The first author is supported by Fellowship at Technical University of Dresden.
This paper is partially supported by the grant No. DFNI-I 02/12.

friction, or in the form of a relay or thermostat, often deliberately built into the system. If the hysteretic behavior is described using a hysteresis operator, then the mathematical model for the dynamical system consists of differential equations coupled with one or several hysteresis operators, which is complemented by initial and boundary conditions. The oscillator with hystresis restoring force,

$$x''(t) + \mathcal{F}[x](t) = f(t),$$

$\mathcal{F}$ being a hysteresis operator is a basic example.

The coupling of rate independent hysteretic nonlinearities with ordinary differential equations leads to interesting mathematical problems in the theory of nonlinear oscillations. Hysteretic constitutive laws in continuum mechanics formulated in terms of hysteresis operators lead in a natural way to partial differential equations coupled with hysteresis operators, where the former represent the balance laws for mass, momentum and internal energy.

In this paper we propose a hysteresis CNN (HCNN) model [6] which can perform applications in image processing. Although the typical CNN does not oscillate and becomes chaotic, one can expect interesting phenomena – bifurcations and complex dynamics to occur. Moreover, because of the applications of CNN, it will be interesting to consider a special type of memory-based relation between an input signal and an output signal in this circuit. The main goal of this paper is to model and investigate such relation, hysteresis [7] for a CNN. The HCNN is derived from standard CNN which is made of first-order cells with hysteresis switches. Such cell can operate in two modes – bistable multivibrator mode and relaxation oscillator mode. In the case of relaxation oscillator mode, the HCNN can generate various patterns and nonlinear waves. Moreover, HCNN can function as both associate and dynamic memories.

We shall introduce memristor [3] in HCNN model in order to improve the resolution in static and dynamic image analysis. Since the memristor exhibits nonlinear current-voltage characteristic with locally negative differential resistance, the memristor is also considered to replace the original linear resistor in a HCNN model. Thus, a HCNN with memristor synapses (M-HCNN), equipped with nonvolatile and programmable synapse circuits, is more versatile and compact and saves the traditional complex output function realization circuits. In Section 2 we shall introduce our HCNN with memristor synapses. Section 3 will deal with the dynamics of the proposed model by means of local activity theory. An algorithm for determination of edge of chaos regime in the parameter set of our M-HCNN model will be derived in this section. Simulations and applications will be presented in Section 4.

2. Hysteresis CNN with memristor synapses (M-HCNN)

Let us consider the following HCNN model:

$$\text{(1)}\quad \frac{dx_{ij}}{dt} = -m(x_{ij}) + \sum_{k,l\in N_{ij}} (a_{k-i,l-j} f(x_{kl}) + b_{k-i,l-j} u_{kl}) + z_{ij}, 1 \le i, j \le M,$$

where x_{ij} denotes the state of the cell $C(i,j)$, $y_{ij} = f(x_{ij})$ is the output and u_{kl} is the input of the CNN cell. We consider the state equation (1) in the case when $y_{ij} = h(x_{ij})$, $h(x_{ij})$ is dynamic hysteresis function defined by:

$$\text{(2)}\qquad h(x(t)) = \begin{cases} 1, & \text{for } x(t) > -1, \quad h(x(t_-) = 1 \\ -1, & \text{for } x(t) = -1, \\ -1, & \text{for } x(t) < 1, \quad h(x(t_-)) = -1 \\ 1, & \text{for } x(t) = 1, \end{cases}$$

$t_- = \lim_{\varepsilon\to 0}(t-\varepsilon)$, $\varepsilon > 0$. The term $m(\cdot)$ is the current flowing through the memristor (M). Memristor is a 2-terminal electronic device, which was postulated by L.Chua [3] in 1971. The memristor is described by the relation between the charge q and the flux φ. Its terminal voltage v and the the terminal current i are described by: $v = M(q)i$ or $i = V(\varphi)v$, where $v = \frac{d\varphi}{dt}$ and $i = \frac{dq}{dt}$. The two nonlinear functions $M(q)$ and $V(\varphi)$ are called the memristance and memductance, respectively and are defined by $M(q) = \frac{d\varphi(q)}{dq}$ and $V(\varphi) = \frac{dg(\varphi)}{d\varphi}$ representing the slope of a scalar function $\varphi = \varphi(q)$ and $q = g(\varphi)$ called the memristor constitutive relations. In our case we shall consider $m(\cdot)$ in the equation (1) to be in the form

$$\text{(3)}\qquad m(x_{ij}(t)) = \frac{v_M}{M(t)} = \frac{x_{ij}(t)}{M(t)},$$

where $M(t)$ is the memristance of the memristor state resistor.

In the case when the template is space invariant, each cell is described by simple identical cloning templates defined by two real matrices A and B. Continuous input(output) signal values are presented by values in the range $[-1,1]$ or $[0,1]$. For instance the following two square matrices are used for a CNN with neighbourhood radius $r = 1$ [2]:

$$A = \begin{pmatrix} A(i,j;i-1,j-1) & A(i,j;i-1,j) & A(i,j;i-1,j+1) \\ A(i,j;i,j-1) & A(i,j;i,j) & A(i,j;i,j+1) \\ A(i,j;i+1,j-1) & A(i,j;i+1,j) & A(i,j;i+1,j+1) \end{pmatrix}$$

$$B = \begin{pmatrix} B(i,j;i-1,j-1) & B(i,j;i-1,j) & B(i,j;i-1,j+1) \\ B(i,j;i,j-1) & B(i,j;i,j) & B(i,j;i,j+1) \\ B(i,j;i+1,j-1) & B(i,j;i+1,j) & B(i,j;i+1,j+1) \end{pmatrix}$$

This form enables us to rewrite state equation (1) in a more compact form by means of the two-dimensional convolution operator defined below.

Definition 2. *For any cloning template A which defines the dynamic rule of the cell circuit, we define the convolution operator $*$ by*

(4) $$A * y_{ij} = \sum_{C(k,l) \in N_r(i,j)} A(k-i, l-j) y_{kl},$$

where $A(m,n)$ denotes the entry in the mth row and nth column of the cloning template, $m = -1.0, 1$, and $n = -1, 0, 1$, respectively.

In this way we can rewrite the HCNN model (1) in the following way:

(5) $$\frac{dx_{ij}}{dt} = -m(x_{ij}) + A * h(x_{ij}) + B * u_{kl} + z_{ij}.$$

We can transform equation (5) into the following form:

(6) $$\frac{dx_{ij}}{dt} = -m(x_{ij}) + h(u_{ij}) + (a_c - 1)h(x_{ij}) + \tilde{A} * h(x_{ij}) + B * u_{kl} + z_{ij},$$

where $\tilde{A}$ is the template for the case $(k,l) \neq (i,j)$. Then the dynamics of an isolated cell of equation (6) (without control $B * u_{kl}$ and threshold z_{ij} parameters) is given by:

(7) $$\frac{dx_{ij}}{dt} = -m(x_{ij}) + a_c h(x_{ij}).$$

Thus, the isolated cell has the following properties:

– If $a_c = -2$, equation (7) reduces to a relaxation oscillator:

(8) $$\frac{dx_{ij}}{dt} = -m(x_{ij}) - 2h(x_{ij}),$$

because x_{ij} oscillates between upper and lower transition levels (see Figure 1).

– If $a_c = 2$, then we have a bistable multivibrator:

(9) $$\frac{dx_{ij}}{dt} = -m(x_{ij}) + 2h(x_{ij}),$$

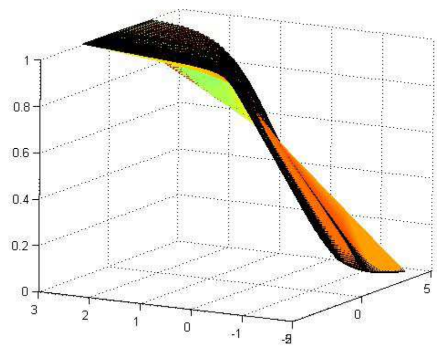

Figure 1: Relaxation oscillator

because it has two possible stable states: $x_{ij} = \pm 2$.

– If $a_c = -0.5$, the equation (7) is reduced to a bistable multivibrator:

$$\frac{dx_{ij}}{dt} = -m(x_{ij}) - 0.5h(x_{ij}), \tag{10}$$

since it has two possible stable states: $x_{ij} = \pm 0.5$.

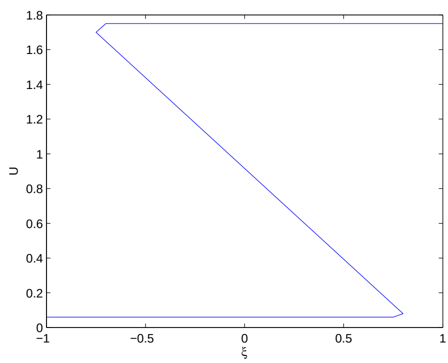

Figure 2: Bistable multivibrator

3. Dynamics of M-HCNN model

Without loss of generality let us consider the following M-HCNN model with zero control and threshold template which is obtained simply from (1):

$$\frac{dx_{ij}}{dt} = -m(x_{ij}) + a_c h(x_{ij}) + bf(x_{ij}) + \sum_{k,l \in N_{ij}} a_{k-i,l-j} f(x_{kl}), \tag{11}$$

$$1 \leq i, j \leq M.$$

In general, a spatially continuous or discrete medium made of identical cells interacting with all cells located within a neighborhood is said to manifest complexity if the homogeneous medium can exhibit a non-homogeneous static or spatio-temporal pattern under homogeneous initial and boundary conditions. In this section we shall apply theory of local activity in order to study the dynamics of M-HCNN model (11). Since complexity can occur only if the active parameter region is not an empty set, it follows that local activity is indeed the origin of complexity, such as limit cycles or chaos. Mathematically, the signal must be infinitesimally small in order to model the cell by only the linear terms in its Taylor series expansion. This proves that complexity originates from infinitesimally small perturbations, notwithstanding the fact that the complete system is typically highly nonlinear.

Constructive and explicit mathematical inequalities can be obtained for identifying the region in the CNN parameter space where complexity phenomena may emerge, as well as for localizing in further into a relatively small parameter domain called edge of chaos where the potential for emergency is maximized. By restricting the cell parameter space to the local activity domain, a major reduction in the computing time required by the parameter search algorithms is achieved [4].

We develop the following constructive algorithm for determination of the edge of chaos domain:

1. Map the M-HCNN model (11) into its discrete-space version by choosing the Laplace template $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ as the feedback template A. Then the dynamics of an isolated cell working in relaxation oscillator mode is given by:

$$\frac{dx_{ij}}{dt} = -m(x_{ij}) + a_c h(x_{ij}) + bf(x_{ij}) = F(x_{ij}). \tag{12}$$

2. Find the equilibrium points E_k of (12). According to the theory of dynamical systems equilibrium points are these for which $F(x_{ij}) = 0$. In general, this system may has four real roots and these roots are functions of the cell parameters a_c, b.

3. Calculate now the four cell coefficients of the Jacobian matrix a_{11}, a_{12}, a_{21}, a_{22}.

4. Calculate the trace $Tr(E_k)$ and the determinant $\Delta(E_k)$, $k = 1, 2, 3, 4$ of the Jacobian matrix for each equilibrium point E_k.

Remark. The importance of the circuit model is not only in the fact that we have a convenient physical implementation, but also in the fact that well-known results from classic circuit theory can be used to justify the cells' local activity. In this sense, if there is at least one equilibrium point for which the circuit model of the cell acts like a source of "small signal", i.e. if the cell is capable of injecting a net small-signal average power into the passive resistive grids, then the cell is said to be locally active.

Definition 3. *Stable and locally active region $SLAR(E_k)$ at the equilibrium point E_k for the M-HCNN model* (12) *is such that*

$$a_{22} > 0 \quad or \quad 4a_{11}a_{22} < (a_{12} + a_{21})^2$$

and

$$Tr < 0 \quad and \quad \Delta > 0.$$

5. Edge of chaos. In the literature, the so-called edge of chaos (EC) means a region in the parameter space of a dynamical system where complex phenomena and information processing can emerge. We shall try to define more precisely this phenomena till now known only via empirical examples.

Definition 4. *M-HCNN* (12) *is said to be operating on the edge of chaos EC iff there is at least one equilibrium point which is both locally active and stable.*

Following the above algorithm we have proved the following theorem:

Theorem 1. *M-HCNN model* (12) *is operating in the EC regime iff the following conditions for the parameters are satisfied: $a_c \geq 2$ and $-1 < b < 3$. In this parameter set at least on equilibrium point E_k is both locally active and stable.*

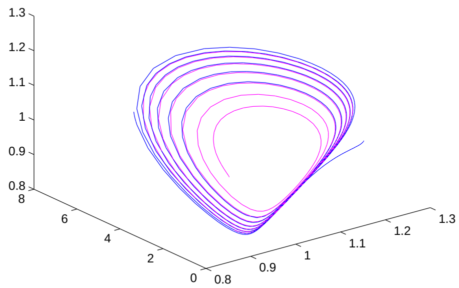

Figure 3: EC region for M-HCNN model (12)

After simulations we obtain the following edge of chaos region for the M-HCNN model (12):

Remark. In this paper, a forward Euler algorithm with a time step size $\Delta t = 0.0.1$ is applied to all computer simulations. The dynamic hysteresis function $h(x)$ is programmed as follows:

$$h(x(t_n)) = \begin{cases} 1, & \text{for } x(t_n) > -1, \quad h(x(t_{n-1}) = 1, \\ -1, & \text{for } x(t_n) = -1, \\ -1, & \text{for } x(t_n) < 1, \quad h(x(t_{n-1})) = -1, \\ 1, & \text{for } x(t_n) = 1, \end{cases}$$

where $t_n = n\Delta t$, $n = 1, 2, \ldots$.

4. Simulations and applications

In this section we shall present simulations of the dynamic nonhomogeneous patterns which have emerged from the M-HCNN for several cell parameter points chosen arbitrarily within, or nearby, the edge of chaos domain. After extensive computer simulations we obtain the following results (see Fig. 4).

As one possible application of our M-HCNN we shall present the dynamic memories function. In the case of relaxation oscillator mode (8), the M-HCNN has the function of dynamic memories. It is known that actual human being's association is not always static, but dynamic. It sometimes wonders from a certain memory to another memory, one after another. Furthermore, a flash of inspiration (new pattern) sometimes appears which is relevant to known memories. They are called spurious memories. The generation of spurious memories can be

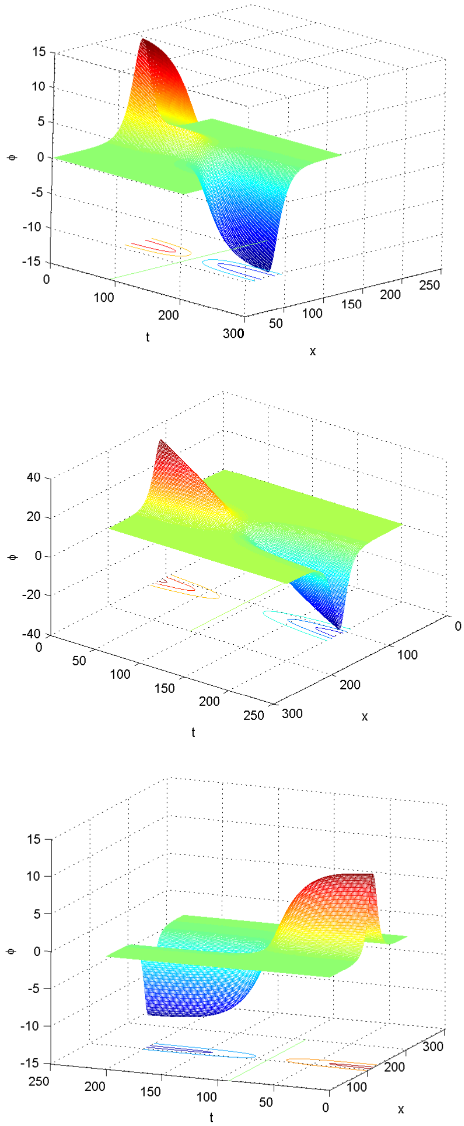

Figure 4: Simulations of M-HCNN model (12)

interpreted as the brain's creative activity. Let us consider the dynamic memories described by

$$\frac{dx_{ij}}{dt} = -m(x_{ij} - 2h(x_{ij}) = d(p_{ij}x_0 - x_{ij}), 1 \leq i, j \leq M, \tag{13}$$

where d is a sufficiently large constant, p_{ij} is defined by

$$p_{ij} = sgn(\sum_{j=1}^{M} s_{ij}w_j), \tag{14}$$

$w_j = \text{sng}(h(x_{ij})h(x_0))$. Given M binary patterns σ^1, σ^2, $\ldots$, σ^M, each pattern σ contains N bit of information σ_i^q, then the coupling coefficients s_{ij} are defined as follows:

$$s_{ij} = \frac{1}{N}\sum_{m=1}^{M} \sigma_i^q \sigma_j^q.$$

x_0 is obtained from the master relaxation oscillator

$$\frac{dx_0}{dt} = -x_0 - 2h(x_0). \tag{15}$$

For the time period satisfying $w_j = \text{sng}(h(x_0(t))h(x_{ij}(t))) = \sigma_j^1$, we get the following relation

$$p_{ij} = \sum_{j=1}^{N} s_{ij}w_j = \sum_{j=1}^{N} s_{ij}\sigma_j^1 = \sum_{j=1}^{N}(\frac{1}{N}\sum_{m=1}^{M} \sigma_i^q \sigma_J^q)\sigma_j^1 = \tag{16}$$

$$\sigma_i^1 + \frac{1}{N}\sum_{q \neq 1}^{M} \sigma_i^q \sum_{j=1}^{M} \sigma_j^q \sigma_j^1.$$

If we assume that $M \ll N$ (i.e. the number of stored patterns is much smaller than the total number of cells), then we have

$$p_{ij} \approx \sigma_i^1, \tag{17}$$

$$d(p_{ij}x_0 - x_{ij}) \approx d(\sigma_{ij}^1 x_0 - x_{ij}).$$

Thus the master relaxation oscillator and ith cell may not synchronize completely. That is, the output pattern oscillates in the phase relation corresponding to the pattern σ^1 for a while, but does not converge to the pattern σ^1. Then, the output pattern may occasionally travel around the stored patterns and their reverse patterns or may travel around the new patterns which are related to the stored patterns. Furthermore, there may be the case where the output pattern converges to an unexpected pattern or oscillates between some of the stored and unstored patterns. Therefore, it is necessary to carry out numerical simulations of dynamic memories.

Remark. Prigogine's perspective of complexity stems mainly from his deep insights of nonequilibrium thermodynamics. In his classic book [6], Prigogine addresses a much broader class of homogeneous systems exhibiting what he called "dissipative structures". In one broad stroke, he had correctly identified the fundamental mechanism of complexity as an innate property of a homogeneous medium to destabilize from the homogeneous "thermodynamic" branch and to bifurcate into various spatio-temporal dissipative structures, as some relevant parameters cross over a bifurcation boundary. Prigogine's vision is correct but only a qualitative one at that. All complexity related examples, problems such as pattern generation, wave propagation and oscillations can be analyzed and explained via local activity.

References

[1] N. F. Britton. Reaction-Diffusion Equations and Their Applications to Biology. New York, Academic, 1986.

[2] L. O. Chua, L. Yang. Cellular Neural Network: Theory and Applications. *IEEE Trans. CAS* **35** (1988), 1257.

[3] L. O. Chua. Memristor: the missing circuit element. *IEEE Trans. On Circuit Theory* **18**, *5* (1971), 507.

[4] L. O. Chua. Local activity is the origin of complexity. *International Journal of Bifurcation and Chaos* **15**, *11* (2005), 3435.

[5] I. Prigogine. From Being to Becoming. Freeman, San Francisco, 1980.

[6] A. Slavova. Cellular Neural Networks: Dynamics and Modeling. Kluwer Academic Publishers, 2003.

[7] A. Visintin. Differential Models of Hysteresis. Springer-Verlag, 1994.

Angela Slavova
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
e-mail: slavova@math.bas.bg

Ronald Tetzlaff
Institute of Fundamentals of Electrical Engineering and Electronics
Technical University Dresden
D-01069 Dresden, Germany
e-mail: Ronald.Tetzlaff@tu-dresden.de