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SCATTERING FOR SYSTEMS OF N WEAKLY COUPLED NLS EQUATIONS ON $\mathbb{R}^D \times \mathcal{M}^2$ IN THE ENERGY SPACE

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We study scattering properties in the energy space of the solution to the following system of N nonlinear Schrödinger equations (NLS), with $N \geq 2$, posed on product spaces $\mathbb{R}^n \times \mathcal{M}^2$, for $d \geq 1$ and $\mathcal{M}^2$ any 2-dimensional compact Riemaniann manifold:

$$\begin{cases} i\partial_t u_\mu + (\Delta_x + \Delta_y)u_\mu + \displaystyle\sum_{\mu,\nu=1}^{N} G_{\mu\nu}(u_\mu, u_\nu)u_\mu = 0, \qquad \mu = 1, \ldots, N, \\ (u_\mu(0,\cdot,\cdot))_{\mu=1}^N = (u_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^d \times \mathcal{M}^2)^N. \end{cases}$$

Here, for all $\mu, \nu = 1, \ldots, N$, $u_\mu = u_\mu(t,x,y) : \mathbb{R} \times \mathbb{R}^d \times \mathcal{M}^2 \to \mathbb{C}$, $(u_\mu)_{\mu=1}^N = (u_1, \ldots, u_N)$, moreover we require that each function $G_{\mu\nu} = G_{\mu\nu}(\cdot,\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is measurable and such that

$$|G_{\mu\nu}(x,x)| \leq \beta_{\mu\nu}|x|^{\frac{4}{d}},$$

for any $x \in \mathbb{C}$ and with $\beta_{\mu\nu} \geq 0$ being coupling parameters, for any $\mu, \nu = 1, \ldots, N$.

1. Introduction

Consider the Cauchy problem associated the system of N nonlinear Schrödinger equations (NLS) on product spaces $\mathbb{R}^d \times \mathcal{M}^k$, $d \geq 1$ and $\mathcal{M}^k$ is a compact manifold

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and $k \geq 1$:

$$\begin{cases} i\partial_t u_\mu + \Delta_{x,y} u_\mu + \sum_{\mu,\nu=1}^{N} G_{\mu\nu}(u_\mu, u_\nu) u_\mu = 0, \\ (u_\mu(0,\cdot,\cdot))_{\mu=1}^N = (u_{\mu,0})_{\mu=1}^N \in \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^k)^N, \end{cases} \tag{1}$$

with $N \geq 2$ and $\sigma > \dfrac{k}{2}$. Here, for $\mu = 1, \cdots, N$, $u_\mu = u_\mu(t,x,y) : \mathbb{R} \times \mathbb{R}^d \times \mathcal{M}^k \to \mathbb{C}$ and $(u_\mu)_{\mu=1}^N = (u_1, \ldots, u_N)$. Moreover

$$\Delta_{x,y} = \sum_{l=1}^{n} \partial_{x_l}^2 + \Delta_y,$$

with Δ_y the Laplace-Beltrami operator associated to the manifold $\mathcal{M}^k$, defined in local coordinates by

$$\frac{1}{\sqrt{|g(y)|}} \partial_{y_i} \sqrt{|g(y)|} g^{hi}(y) \partial_{y_i},$$

where $g_{hi}(y)$ is the metric tensor, $|g(y)| = \det(g_{hi}(y))$ and $g^{hi} = (g_{hi}(y))^{-1}$. We require that, for any $\mu, \nu = 1, \ldots, N$ each bilinear function $G_{\mu\nu} = G_{\mu\nu}(\cdot,\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is measurable and there exist $0 \leq \theta_{\mu\nu} \leq 4/d$ such that the inequality

$$|G_{\mu\nu}(x,y)| \leq \beta_{\mu\nu} |x|^{\theta_{\mu\nu}} |y|^{\frac{4}{d} - \theta_{\mu\nu}}, \tag{2}$$

is fulfilled for $(x,y) \in \mathbb{C}^2$ and with the coupling parameters such that $\beta_{\mu\nu} \geq 0$.

$$\begin{aligned} &|G_{\mu\nu}(x_1,y_1)x_1 - G_{\mu\nu}(x_2,y_2)x_2| \\ &\leq \overline{\beta}_{\mu\nu} \left(|x|^{\theta_{\mu\nu}} |y|^{\frac{4}{d} - \theta_{\mu\nu}} + |x|^{\theta_{\mu\nu}} |y|^{\frac{4}{d} - \theta_{\mu\nu}} \right) |x_1 - x_2|, \end{aligned} \tag{3}$$

for $(x_1,y_1), (x_2,y_2) \in \mathbb{C}^2$ and $\overline{\beta}_{\mu\nu} \geq 0$. For any $\sigma \in \mathbb{R}$, we denote the non-isotropic fractional Sobolev spaces by

$$\mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^k) = \mathscr{H}^{0,\sigma}_{x,y} = (1 - \Delta_y)^{-\frac{\sigma}{2}} L^2_{x,y}, \tag{4}$$

where $L^2_{x,y} = L^2(\mathbb{R}^d \times \mathcal{M}^k)$, indicating by $f \in L^q(\mathbb{R}^d \times \mathcal{M}^k)$, for $1 \leq q < \infty$, if

$$\|f\|^q_{L^q(\mathbb{R}^d \times \mathcal{M}^k)} = \int_{\mathbb{R}^d \times \mathcal{M}^k} |f(x,y)|^q \, dx d\mathrm{v}_g < +\infty,$$

with dv_g the volume element of $\mathcal{M}^k$ which reads in local coordinates as $\sqrt{|g(y)|}dy$. Furthermore the h-th component of the gradient operator ∇_y is given in local coordinates by $g^{hi}(y)\partial_{y_i}$. We also denote by

$$H^1(\mathbb{R}^d \times \mathcal{M}^k) = H^1_{x,y} = (1 - \Delta_x - \Delta_y)^{-\frac{1}{2}} L^2(\mathbb{R}^d \times \mathcal{M}^k).$$

There is a consistent literature regarding the global well-posedness theory as well as the bound state theory for the problem (1), on the other hand the systems of Schrödinger equations have an important role in many models of mathematical physics: it gives a description of the interactions of M–wave packets, the nonlinear waveguides, the optical pulse propagation in birefringent fibers, the propagation of polarized laser beam in Kerr-like photorefractive media and in the Bose-Einstein condensates theory, just to name a few. We remand to [2], [3], [12] and [10] in the case $N = 2$ and to [9] and [11] in the general case $N \geq 2$ for a general overview on references both on mathematical and on physical setting and applications.

Motivated by this, it is possible to investigate some relevant questions as:

- Local and global existence as well as the persistence of regularity for the map data-solution $(u_{\mu,0})_{\mu=1}^N \to (u_\mu(t,\cdot,\cdot))_{\mu=1}^N$, assuming the initial data in the space $H^1(\mathbb{R}^d \times \mathcal{M}^k)^N$ (or eventually in slightly different non-isotropic Sobolev spaces).
- The long-time behavior of the solutions to (1) in the space $H^1(\mathbb{R}^d \times \mathcal{M}^k)^N$.

The study of the Schrödinger equation posed on product spaces was recently initiated by considering general problems involving global well-posedness (see for example [5] and references therin), long time asymptotics (see [15], [5], [6] and references therein) and ground states (see for example [13] and references therein). In the above papers it arises, with some few exceptions, not only that to earn informations on well-posedness and scattering for the NLS it is required an appropriate geometry for the compact manifolds, but also that the asymptotic behavior of the solutions is poorly understood when the nonlinearity is pure power and the nonlinearity parameter is a fractional number in the interval $\left[\frac{4}{d}, \frac{4}{d+k-2}\right]$. Moreover, as well as we know, it seems that there is a lack of literature in the case of N-systems of NLS having interacting nonlinearities such as we considered in (1).

According to these obsevations, our main contribution is the transposition of the well-posedness and scattering analysis to the framework of (1) with coupled nonlinearities behaving like pure fractional power, emphasizing that the only assumption we impose to the manifolds $\mathcal{M}^k$ is the compactness. In fact the topological structure of the manifold is not relevant in our study and the small data theory does not need any specific property on $\mathcal{M}^k$. The main tools to prove scattering are a particular version of Strichartz estimates. We will use the informations and numerology available for the flat part to deduce Strichartz estimates on the whole product space.

We can state now main contributions of this paper. The first results deals with the mass NLS.

Theorem 1. *For every $d \geq 1$, $k \geq 2$ and $\frac{k}{2} < \sigma < \frac{4}{d}+1$ there exists a positive number $\varepsilon = \varepsilon(\sigma)$ such that the problem (1) enjoys a unique global solution*

$$(u_\mu(t,x,y))_{\mu=1}^N \in L^\ell(\mathbb{R}; L_x^\ell(\mathbb{R}^d; H_y^\sigma(\mathcal{M}^k))^N, \tag{5}$$

where $\ell = \dfrac{2d+4}{d}$, for any initial data $(u_{\mu,0})_{\mu=1}^N \in \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^k)^N$ such that $\sup_{\{1\leq\mu\leq N\}} \|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}(\mathbb{R}^d\times\mathcal{M}^k)} < \varepsilon$. In addition $(u_\mu(t,x,y))_{\mu=1}^N \in L^\infty(\mathbb{R}; \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^k))^N$ and there exist $(\varphi_{\pm,\mu})_{\mu=1}^N \in \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^k)^N$ so that

$$\lim_{t\to\pm\infty} \|(u_\mu(t,x,y))_{\mu=1}^N - e^{-it\Delta_{x,y}}(\varphi_{\pm,\mu})_{\mu=1}^N\|_{\mathscr{H}^{0,\sigma}(\mathbb{R}^d\times\mathcal{M}^k)^N} = 0. \tag{6}$$

Remark 1. In the above Theorem 1 are presented peculiar properties of the solutions to (1). Specifically if we select in the Cauchy problem (1), for suitable $\epsilon > 0$, initial data $(u_{\mu,0})_{\mu=1}^N \in (H_{x,y}^{0,\frac{k}{2}+\epsilon})^N$ with size ε, one gets scattering in non-isotropic Sobolev spaces characterized as in (4) with regularity strictly connected with the spatial dimensions of the manifold $\mathcal{M}^k$ and the Euclidean part $\mathbb{R}^d$. Such an asymptotic behavior of the solutions $(u_\mu(t,x,y))_{\mu=1}^N$ is completely independent from the geometry of $\mathcal{M}^k$, which is required only to be a compact Riemannian manifold. Moreover, in the case $k = 2$, we observe that if we take a sufficiently small $\epsilon > 0$, then the space $\mathscr{H}_{x,y}^{0,1+\epsilon}$ is slightly stronger than $H_{x,y}^1$ only w.r.t. the y-variable.

Then we have the second result in the special case $k = 2$, that is the case of mass-energy system of NLS. More precisely, assuming now that, for any $\mu, \nu = 1, \ldots, N$ each function $G_{\mu\nu} = G_{\mu\nu}(\cdot,\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ enjoys

$$G_{\mu\nu}(x,y) = \beta_{\mu\nu}|x|^{\theta_{\mu\nu}}|y|^{\frac{4}{d}-\theta_{\mu\nu}}, \tag{7}$$

with $0 \le \theta_{\mu\nu} \le 4/d$, for $(x, y) \in \mathbb{R}^2$ and $\beta_{\mu\nu} \in \mathbb{R}$, one achieves:

Theorem 2. *For every* $d \ge 1$ *and* $1 < \sigma < 1 + \frac{4}{d}$ *there exists a positive number* $\varepsilon = \varepsilon(\sigma)$ *such that the problem* (1) *with* $G_{\mu\nu}$ *as in* (7) *enjoys a unique global solution*

$$(u_\mu(t, x, y))_{\mu=1}^N \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d \times \mathcal{M}^2))^N \cap L^\ell(\mathbb{R}; L^\ell_x(\mathbb{R}^d; H^\sigma_y(\mathcal{M}^2))^N, \tag{8}$$

where $\ell = \frac{2d+4}{d}$, *in the following cases:*

1. *if* $\beta_{\mu\nu} < 0$ *(i.e.* (1) *is defocusing), for initial data* $(u_{\mu,0})_{\mu=1}^N \in (H^1(\mathbb{R}^d \times \mathcal{M}^2) \cap \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^2))^N$ *such that* $\sup_{\{1 \le \mu \le N\}} \|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^2)} < \varepsilon$;
2. *if* $\beta_{\mu\nu} > 0$, *for initial data* $(u_{\mu,0})_{\mu=1}^N \in (H^1(\mathbb{R}^d \times \mathcal{M}^2) \cap \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^2))^N$ *such that such that* $\sup_{\{1 \le \mu \le N\}} \|u_{\mu,0}\|_{H^1(\mathbb{R}^d \times \mathcal{M}^2) \cap \mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^2)} < \varepsilon$ *(i.e.* (1) *is focusing and the initial data are small).*

In addition there exist $(\varphi_{\pm,\mu})_{\mu=1}^N \in\in H^1(\mathbb{R}^d \times \mathcal{M}^2)^N$ *so that*

$$\lim_{t \to \pm\infty} \|(u_\mu(t, x, y))_{\mu=1}^N - e^{-it\Delta_{x,y}}(\varphi_{\pm,\mu})_{\mu=1}^N\|_{H^1(\mathbb{R}^d \times \mathcal{M}^2)^N} = 0. \tag{9}$$

Remark 2. As we noticed in Theorem 2, by pickung up $k = 2$, the N-system of NLS in (1) becomes both mass and energy critical. In such a fashion the technicalities developed along the proof of the Theorem 1 can be improved guaranteeing also well-posedness and scattering in the energy space (other than $\mathscr{H}^{0,\sigma}(\mathbb{R}^d \times \mathcal{M}^2)^N$) if we consider additionally that initial data are $H^1(\mathbb{R}^d \times \mathcal{M}^2)^N$-bounded. This phenomenon is completely new, because we neither require particular geometries for the manifold $\mathcal{M}^2$, nor make an use of multiplier techniques such as Morawetz identities and inequalities (as used for example in the work [16]).

2. Preliminaries

We introduce the following further notations: for any Banach space X we define, for $q \ge 1$,

$$\|f\|_{L^q_t X} = \left(\int_{\mathbb{R}} \|f(x)\|^q_X \, dt \right)^{1/q},$$

(for its version local in time we adopt the symbol $L^q_{(t_1,t_2)}X$, with preassigned $t_1, t_2 \in \mathbb{R}$). We denote, from now on, by $H^\sigma_y = W^{\sigma,2}_y$ with

$$L^r_x W^{\sigma,l}_y = (1-\Delta_y)^{-\frac{\sigma}{2}} L^r_x L^l_y, \tag{10}$$

for $\sigma \in \mathbb{R}$, where $L^r_x L^l_y$ is the space $L^r(\mathbb{R}^d; L^l(\mathcal{M}^k))$ for $l, r \geq 1$.

We need to recall the following useful facts concerning Strichartz inequalities:

Definition 1. *An exponent pair (ℓ, p) is Schrödinger-admissible if $2 \leq \ell, p \leq \infty$, $(\ell, p, d) \neq (2, \infty, 2)$, and*

$$\frac{2}{\ell} + \frac{d}{p} = \frac{d}{2}. \tag{11}$$

Then we earn, for the free systems of N Schrödinger equations, that is (1) with general initial data $(f_\kappa)_{\kappa=1}^N$ and forcing terms $(F_\kappa)_{\kappa=1}^N$ the following:

Proposition 1. (Strichartz estimates) *Let be $d \geq 1$ and indicate by $\mathcal{D}_i$ the operators $\mathcal{D}_1 = \nabla_x$ and $\mathcal{D}_2 = \nabla_y$, then we have for any $\kappa = 1, \ldots, N$ and $\gamma = 0, 1$ the following estimates*

$$\begin{aligned} \|\mathcal{D}_i^\gamma e^{it\Delta_{x,y}} f_\kappa\|_{L^\ell_t L^p_x H^\sigma_y} &+ \left\| \mathcal{D}_i^\gamma \int_0^t e^{-i(t-\tau)\Delta} F_\kappa(\tau, \cdot) d\tau \right\|_{L^\ell_t L^p_x H^\sigma_y} \\ &\leq C \|\mathcal{D}_i^\gamma f_\kappa\|_{L^2_x H^\sigma_y} + \|\mathcal{D}_i^\gamma F_\kappa\|_{L^{\tilde{\ell}'}_t L^{\tilde{p}'}_x H^\sigma_y}, \end{aligned} \tag{12}$$

if for the pairs (ℓ, p) and $(\tilde{\ell}, \tilde{p})$ the condition (11) *is satisfied for $\ell, \tilde{\ell} \geq 2$, if $d \geq 3$, $\ell, \tilde{\ell} > 2$ if $d = 2$, and $\ell, \tilde{\ell} \geq 4$ if $d = 1$.*
Moreover we have also the following estimates satisfied

$$\begin{aligned} \|\mathcal{D}_i^\gamma e^{it\Delta_{x,y}} f_\kappa\|_{L^\infty_t L^2_x H^\sigma_y} &+ \left\| \mathcal{D}_i^\gamma \int_0^t e^{-i(t-\tau)\Delta} F_\kappa(s, \cdot) d\tau \right\|_{L^\infty_t L^2_x H^\sigma_y} \\ &\leq C \|\mathcal{D}_i^\gamma f_\kappa\|_{L^2_x H^\sigma_y} + C\, \|\mathcal{D}_i^\gamma F_\kappa\|_{L^{\tilde{\ell}'}_t L^{\tilde{p}'}_x H^\sigma_y}, \end{aligned} \tag{13}$$

with (ℓ, p) and γ as above.

3. Proof of Theorems 1 and 2

In this section we give the proof of the two our main results. In the former we show that to any choice of $n \geq 1$, $k \geq 2$ and small initial data in $\mathscr{H}^{0,\sigma}_{x,y}$ it is possible to prove that $(u_\mu(t,x,y))_{\mu=1}^N \in (L^\ell_t L^p_x H^\sigma_y)^N \cap (L^\infty_t \mathscr{H}^{0,\sigma}_{x,y})^N$ for any Schrödinger admissible pair (ℓ, p) by using only the modified Strichartz estimates presented in the previous section. Moreover, we get scattering in $\mathscr{H}^{0,\sigma}_{x,y}$. In the latter we exhibit in which way the previous analysis leads to the well-posedness in the spaces $L^\infty(\mathbb{R}; H^1_{x,y})^N$ in the case $k = 2$ for the solutions to the Cauchy problem (1). Finally we get, in this framework, scattering in $H^1_{x,y}$.

P r o o f o f T h e o r e m 1. Let be defined the integral operator associated to the Cauchy problem (1) ,

$$\mathcal{T}(u_\mu)_{\mu=1}^N = e^{it\Delta_{x,y}}(u_{\mu,0})_{\mu=1}^N + I \sum_{\mu,\nu=1}^N \int_0^t e^{i(t-\tau)\Delta_{x,y}} G_{\mu\nu}(u_\mu(\tau), u_\nu(\tau)) u_\mu(\tau) d\tau, \tag{14}$$

with I the $N \times N$ identity matrix. One needs to show that for $\overline{\ell} = \overline{p} = \dfrac{4}{d} + 2$, any $\mu = 1, \dots, N$,

$$\forall u_{\mu,0} \in \mathscr{H}^{0,\sigma}_{x,y} \text{ s.t. } \|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}_{x,y}} < \varepsilon, \quad \exists!\ (u_\mu(t,x,y))_{\mu=1}^N \in (L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N,$$

satisfying the property

$$\mathcal{T}(u_\mu(t))_{\mu=1}^N = (u_\mu(t))_{\mu=1}^N. \tag{15}$$

As well as we require

$$(u_\mu(t,x,y))_{\mu=1}^N \in (L^\infty(\mathbb{R}; \mathscr{H}^{0,\sigma}_{x,y}))^N \cap (L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N.$$

For simplicity, we split the proof in four further different steps

Step One:
For any $\sigma > \dfrac{k}{2}$, $\exists\, \varepsilon = \varepsilon(\sigma) > 0$ *and an* $R = R(\sigma) > 0$, *such that*

$$\mathcal{T} B_{(L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N}(0,R)) \subset B_{(L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N}(0,R), \tag{16}$$

for any $u_{\mu,0} \in \mathscr{H}^{0,\sigma}_{x,y}$ *so that* $\|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}_{x,y}} < \varepsilon$.

It is sufficient to deal directly with the case $(t_1, t_2) = \mathbb{R}$, specifying a different domain for the t-variable when it is required. We need to show (16), to this end by (2) we start first by recalling the following result (we remand to [15] for more details).

Proposition 2. *Assume $\mathcal{M}^k$ a compact manifold with dimension $k \geq 1$. Then for any $f \in H^\sigma(\mathcal{M}^k) \cap L^\infty(\mathcal{M}^k)$ let $G(f) = f|f|^\mu$ be a real function with $\mu > 0$. Then one has*

$$\|f|f|^\mu\|_{H^\sigma(\mathcal{M}^k)} \leq C \, \|f\|^{\mu+1}_{H^\sigma(\mathcal{M}^k)}, \tag{17}$$

with $C > 0$, provided that $0 < \sigma < 1 + \mu$.

Thus, by applying for fixed μ the elementary inequality (see for example [4])

$$|u_\nu|^{\frac{4}{d}-\theta_{\nu\mu}} |u_\mu|^{\theta_{\nu\mu}} \leq C \left(|u_\mu|^{4d} + |u_\nu|^{4d} \right), \tag{18}$$

in combination with the above Sobolev inequality (17), which reads in our setting as

$$\| |u_\nu|^{\frac{4}{d}+1}(t,x,\cdot) \|_{H^\sigma_y} \leq \|u_\nu(t,x,\cdot)\|^{\frac{4}{d}+1}_{H^\sigma_y}, \tag{19}$$

satisfied if $\dfrac{k}{2} < \sigma < 1 + \dfrac{4}{d}$, we are able to estimate the nonlinear term in the $L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y$-norm by using the inhomogeneous Strichartz estimates in (12) with $\gamma = 0$, that is

$$\| |u_\nu|^{\frac{4}{d}+1}(t,x,\cdot) \|_{L^{\tilde{\ell}'}_t L^{\tilde{p}'}_x H^\sigma_y} \leq \| \|u_\nu(t,x,\cdot)\|^{\frac{4}{d}+1}_{H^\sigma_y} \|_{L^{\tilde{\ell}'}_t L^{\tilde{p}'}_x}. \tag{20}$$

Let us select

$$\frac{1}{\tilde{\ell}'} = \frac{4+d}{d\overline{\ell}} = \frac{4+d}{4+2d}, \quad \frac{1}{\tilde{p}'} = \frac{(4+d)}{d\overline{p}} = \frac{4+d}{4+2d}, \tag{21}$$

then the r.h.s. of inequality (20) can be controlled by

$$\| \|u_\nu(t,\cdot,\cdot)\|^{\frac{4}{d}+1}_{L^{\overline{p}}_x H^\sigma_y} \|_{L^{\tilde{\ell}'}_t} \leq \|u_\nu\|^{\frac{4}{d}+1}_{L^{(\frac{4}{d}+1)\tilde{\ell}'}_t L^{\overline{p}}_x H^\sigma_y} \leq \|u_\nu\|^{\frac{4}{d}+1}_{L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y}. \tag{22}$$

Then, summing twice over μ, we arrive at the following

$$\|\mathcal{T}(u_\mu)^N_{\mu=1}\|_{(L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N} \leq C \|(u_{\mu,0})^N_{\mu=1}\|_{(\mathscr{H}^{0,\sigma}_{x,y})^N} + C \|(u_\mu)^N_{\mu=1}\|^{\frac{4}{d}+1}_{(L^{\overline{\ell}}_t L^{\overline{p}}_x H^\sigma_y)^N}$$

and by a standard bootstrap argument (see for example Theorem 6.2.1 in [1]) the previous estimate guarantees the existence of an $\varepsilon > 0$ and $R(\varepsilon) > 0$ such that $\lim_{\varepsilon \to 0} R(\varepsilon) = 0$, provided that $\|(u_{\mu,0})_{\mu=1}^N\|_{\mathscr{H}^{0,\sigma}_{x,y}} < \varepsilon$, for any $\mu = 1, \dots, N$. Thus we achieve the proof.

Step Two:

$\mathcal{T}$ is a contraction on $B_{(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N}(0, R)$, equipped with the norm $\|.\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N}$.

Given $(v_{\mu,1})_{\mu=1}^N, (v_{\mu,2})_{\mu=1}^N \in B_{(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N}$ we get, by (3), (18) and the inhomogeneous estimate in (12), the chain of bounds

$$
\begin{aligned}
(23) \qquad & \|\mathcal{T}(v_{\mu,1})_{\mu=1}^N - \mathcal{T}(v_{\mu,2})_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N} \\
&\leq C \left\| \|(v_{\mu,1})_{\mu=1}^N - (v_{\mu,2})_{\mu=1}^N\|_{(L_y^2)^N} \sup_{i=1,2} \left\{ \|(v_{\mu,i})_{\mu=1}^N\|_{(L_y^\infty)^N}^{\frac{4}{d}} \right\} \right\|_{(L_t^{\tilde{\ell}'} L_x^{\tilde{p}'})^N} \\
&\leq C \left\| \|(v_{\mu,1})_{\mu=1}^N - (v_{\mu,2})_{\mu=1}^N\|_{(L_x^{\overline{p}} L_y^2)^N} \sup_{i=1,2} \left\{ \|(v_{\mu,i})_{\mu=1}^N\|_{(L_x^{\overline{p}} H_y^\sigma)^N}^{\frac{4}{d}} \right\} \right\|_{(L_t^{\tilde{\ell}'})^N},
\end{aligned}
$$

where in the last inequality we used the second of the identities in (21),Minkowski and Hölder inequalities and the Sobolev embedding $H_y^\sigma \subset L_y^\infty$. By a further application of the Hölder inequality the term in the third line of the previous (23) can be controlled as follows

$$
\begin{aligned}
& C \left\| \|(v_{\mu,1})_{\mu=1}^N - (v_{\mu,2})_{\mu=1}^N\|_{L_x^{\overline{p}} L_y^2} \sup_{i=1,2} \left\{ \|(v_{\mu,i})_{\mu=1}^N\|_{(L_x^{\overline{p}} H_y^\sigma)^N}^{\frac{4}{d}} \right\} \right\|_{(L_t^{\tilde{\ell}'})^N} \\
(24) \qquad & \leq C \|(v_{\mu,1})_{\mu=1}^N - (v_{\mu,2})_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N} \sup_{i=1,2} \left\{ \|(v_{\mu,i})_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N}^{\frac{4}{d}} \right\}.
\end{aligned}
$$

The last inequality of the above (24) is a consequence now of the first of the identities in (21). Thus, by combining (23) and (24) and what we got in the previous step, we arrive at

$$
\begin{aligned}
(25) \qquad & \|\mathcal{T}(v_{\mu,1})_{\mu=1}^N - \mathcal{T}(v_{\mu,2})_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N} \\
& \leq C(R(\varepsilon))^\mu \|(v_{\mu,1})_{\mu=1}^N - (v_{\mu,2})_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N}.
\end{aligned}
$$

Then $\mathcal{T}$ is a contraction provided that $\varepsilon > 0$ is suitable small.

Step Three:
The solution exists and it is unique in $(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N$.

We apply the contraction principle to the map $\mathcal{T}$ defined on the complete metric space $B_{(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N}(0,R)$ and equipped with the topology induced by $\|.\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} L_y^2)^N}$.

Step Four:
Regularity of the solution: proof of $(u_\mu(t,x,y))_{\mu=1}^N \in (L^\infty(\mathbb{R};\mathscr{H}_{x,y}^{0,\sigma}))^N$.

It is enough to argue as in the previous steps just exploiting estimates (13) instead of (12) in the proof of *Step One*. This observation enhances to

$$\|\mathcal{T}(u_\mu)_{\mu=1}^N\|_{L_t^\infty \mathscr{H}_{x,y}^{0,\sigma}} \leq C\|(u_{\mu,0})_{\mu=1}^N\|_{(\mathscr{H}_{x,y}^{0,\sigma})^N} + C\|(u_\mu)_{\mu=1}^N\|_{(L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N}^{\frac{4}{d}+1}. \tag{26}$$

The above inequality with $\|u_{\mu,0}\|_{\mathscr{H}_{x,y}^{0,\sigma}} < \varepsilon$ guarantees the fact that $(u_\mu(t,x,y))_{\mu=1}^N \in (L^\infty(\mathbb{R};\mathscr{H}_{x,y}^{0,\sigma}))^N$. The proof of the part of Theorem 1 concerning the global well-posedness is obtained. The remaining asymptotic completeness property (6) follows easily by standard arguments (we remand for instance to [1] and [14]). $\square$

Remark 3. We underline here that the unique global solution $(u_\mu(t,x,y))_{\mu=1}^N \in L_t^{\frac{4}{d}+2}(\mathbb{R};L_x^{\frac{4}{d}+2}(\mathbb{R}^d \times H_y^\sigma(\mathcal{M}^k))^N$ earned in Theorems 1 fulfills also

$$(u_\mu(t,x,y))_{\mu=1}^N \in L_t^\ell(\mathbb{R};L_x^p(\mathbb{R}^d;H_y^\sigma(\mathcal{M}^k))^N$$

for the full set of Strichartz exponents (ℓ,p) as in Definition 1 because of

$$\|(u_\mu)_{\mu=1}^N\|_{(L_t^\ell L_x^p H_y^\sigma)^N} \leq C\varepsilon < 1. \tag{27}$$

Proof of Theorem 2. From the proof of the Theorem (1) we know that there exists a unique solution $(u_\mu)_{\mu=1}^N \in (L_t^{\overline{\ell}} L_x^{\overline{p}} H_y^\sigma)^N$ to the problem (1), once $k > \dfrac{\sigma}{2}$ and $\sup\limits_{1\leq\mu\leq N} \|u_{\mu,0}\|_{\mathscr{H}_{x,y}^{0,\sigma}} < \varepsilon$. Consider now the auxiliary norms

$$\|(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}_{t,x,y}^{(1)}(\ell,p))^N} = \sum_{k=0,1} \left\|\nabla_x^k (u_\mu(t,x,y))_{\mu=1}^N\right\|_{(L_t^\ell L_x^p L_y^2)^N}, \tag{28}$$

$$\|(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}_{t,x,y}^{(2)}(\ell,p))^N} = \sum_{k=0,1} \left\|\nabla_y^k (u_\mu(t,x,y))_{\mu=1}^N\right\|_{(L_t^\ell L_x^p L_y^2)^N}, \tag{29}$$

where (ℓ, p) are Schrödinger-admissible pairs. We can start by proving the following.

Step One:
Let $u_\mu(t,x,y))_{\mu=1}^N$ *be the unique solution to* (1) *with initial data* $(u_{\mu,0})_{\mu=1}^N \in (H^1_{x,y} \cap \mathscr{H}^{0,\sigma}_{x,y})^N$ *such that* $\|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}_{x,y}} < \varepsilon$, *for any* $\mu = 1, \ldots, N$. *Then*

$$\|(u_\mu(t,x,y))_{\mu=1}^N\|_{(\mathscr{X}^{(1)}_{t,x,y}(\overline{\ell},\overline{p}))^N} + \|(u_\mu(t,x,y))_{\mu=1}^N\|_{(\mathscr{X}^{(2)}_{t,x,y}(\overline{\ell},\overline{p}))^N} < \infty, \tag{30}$$

with $\overline{\ell} = \overline{p} = \dfrac{4}{d} + 2.$

The classical Strichartz estimates (12) in connection with the bound (18) and the Hölder inequality, yield for any $i = 1, 2$ and $(\tilde{\ell}', \tilde{p}')$ as in (21),

$$\begin{aligned}(31)\qquad & \|\mathcal{T}(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}^{(i)}_{t,x,y}(\ell,p))^N} \\ &\leq \|\mathcal{D}_i^k(u_{\mu,0})_{\mu=1}^N\|_{(L^2_{x,y})^N} + C\sum_{\mu,\nu=1}^N \|\mathcal{D}_i^k(u_\mu|u_\mu|^{\frac{4}{d}-\theta_{\mu\nu}}|u_\nu|^{\theta_{\mu\nu}})\|_{(L_t^{\tilde{\ell}'}L_x^{\tilde{p}'}L_y^2)^N} \\ &\leq C\|\mathcal{D}_i^k(u_{\mu,0})_{\mu=1}^N\|_{(L^2_{x,y})^N} + C\Big\|\|\mathcal{D}_i^k(u_\mu)_{\mu=1}^N)\|_{(L^2_y)^N}\|(u_\mu)_{\mu=1}^N\|^{\mu}_{(H^\sigma_y)^N}\Big\|_{(L_t^{\tilde{\ell}'}L_x^{\tilde{p}'})^N} \\ &\leq C\|\mathcal{D}_i^k(u_{\mu,0})_{\mu=1}^N\|_{(L^2_{x,y})^N} + C\|\mathcal{D}_i^k(u_\mu)_{\mu=1}^N\|_{(L_t^{\overline{\ell}}L_x^{\overline{p}}L_y^2)^N}\|(u_\mu)_{\mu=1}^N\|^{\frac{4}{d}}_{(L_t^{\overline{\ell}}L_x^{\overline{p}}H_y^\sigma)^N}.\end{aligned}$$

In that way we must have

$$\begin{aligned} & \|\mathcal{T}(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}^{(i)}_{t,x,y}(\ell,p))^N} \\ &\leq C\|(u_{\mu,0})_{\mu=1}^N\|_{(H^1_{x,y})^N} + C\|(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}^{(i)}_{t,x,y}(\overline{\ell},\overline{p}))^N}\|(u_\mu)_{\mu=1}^N\|^{\frac{4}{d}}_{(L_t^{\overline{\ell}}L_x^{\overline{p}}H_y^\sigma)^N} \\ &\leq C\|(u_{\mu,0})_{\mu=1}^N\|_{(H^1_{x,y})^N} + C(R(\varepsilon))^{\frac{4}{d}}\|(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}^{(i)}_{t,x,y}(\overline{\ell},\overline{p}))^N},\end{aligned} \tag{32}$$

where in the third line we have used the bound (27). The proof of (30) is thus complete.

Step Two:
Regularity of the solution: proof of $(u_\mu(t,x,y))_{\mu=1}^N \in L^\infty(\mathbb{R}; H^1_{x,y})^N$.

It is enough to argue as in the previous step just using estimates (13) instead of (12) in the proof of (31). This fact gives

$$\|\mathcal{T}(u_\mu)_{\mu=1}^N\|_{(L^\infty_{(t_1,t_2)}L^2_{x,y})^N} + \sum_{i=1,2}\|\mathcal{D}_i\mathcal{T}(u_\mu)_{\mu=1}^N\|_{(L^\infty_{(t_1,t_2)}L^2_{x,y})^N} \tag{33}$$

$$\leq C\|(u_{\mu,0})_{\mu=1}^N\|_{(H^1_{x,y})^N}$$
$$+C\sum_{i=1,2}\|(u_\mu)_{\mu=1}^N\|_{(\mathscr{X}^{(i)}_{(t_1,t_2),x,y}(\bar{\ell},\bar{p}))^N}\|(u_\mu)_{\mu=1}^N\|^{\frac{4}{d}}_{(L^{\bar{\ell}}_{(t_1,t_2)}L^{\bar{p}}_x H^\sigma_y)^N},$$

An use of (27) and (30) provided that $\|u_{\mu,0}\|_{\mathscr{H}^{0,\sigma}_{x,y}} < \varepsilon$ allows to take $(t_1,t_2)=\mathbb{R}$ and it leads to $(u_\mu(t,x,y))_{\mu=1}^N \in L^\infty(\mathbb{R};H^1_{x,y})^N$.

As a direct consequence we have (8) in the case of (2). In the focusing case we are forced to necessitate in (2) also that $\|u_{\mu,0}\|_{H^1_{x,y}} < \varepsilon$, for any $\mu=1,\dots,N$, in order to avoid some blow-up phenomena, as noticed in [8].

It remains to show the asymptotic completeness property (9). By the integral equation associated with (1) it is sufficient to prove that, for any $\mu,\nu=1,\dots,N$

$$\lim_{t_1,t_2\to\infty}\|\int_{t_1}^{t_2} e^{-is\Delta_{x,y}}G_{\mu\nu}(u_\mu,u_\nu)u_\mu ds\|_{H^1_{x,y}}=0. \tag{34}$$

The dual estimate to the homogeneous inequality in (12) gives

$$\|\int_{t_1}^{t_2} e^{-is\Delta_{x,y}}F(s)ds\|_{L^2_{x,y}} \leq C\|F\|_{L^{\tilde{\ell}'}_{(t_1,t_2)}L^{\tilde{p}'}_x L^2_y}, \tag{35}$$

where $(\tilde{\ell}',\tilde{p}')$ are as in (21). Hence (34) follows if one earns

$$\begin{aligned}\lim_{t_1,t_2\to\infty}\Big(\|G_{\mu\nu}(u_\mu,u_\nu)u_\mu\|_{L^{\tilde{\ell}'}_{(t_1,t_2)}L^{\tilde{p}'}_x L^2_y}+\|\nabla_y G_{\mu\nu}(u_\mu,u_\nu)u_\mu\|_{L^{\tilde{\ell}'}_{(t_1,t_2)}L^{\tilde{p}'}_x L^2_y}\Big)\\ +\lim_{t_1,t_2\to\infty}\|\nabla_x G_{\mu\nu}(u_\mu,u_\nu)u_\mu\|_{L^{\tilde{\ell}'}_{(t_1,t_2)}L^{\tilde{p}'}_x L^2_y}=0,\end{aligned} \tag{36}$$

which is given by the argument used along the proof of the previous steps in conjunction with (30). □

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